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q-ANALOGUES OF SOME OPERATIONAL FORMULAS

THOMAS ERNST

Abstract A q-analogue of the Hörmander [35] vector notation for partial derivatives is invented and the corresponding Ward- and Jackson q-Taylor formulas are found and extended in a natural way. The Ward [65] q-addition is extended to q-shifted factorials and the corresponding q-Taylor formula is found. The Cigler [13] ϵ operator plays a crusial rôle in the process. An attempt to present a q-analogue of holomorphic functions in n variables is made.

We find q-analogues of some operational representations for polynonials from W.A. Al-Salam [3]. Whenever possible, for every q-analogue we will make a reference to the corresponding equation there. By coincidence, some of our formulas appeared with different notation in [1], and we will mention the corresponding equation there too. We will find multiple q-analogues of many formulas in Carlitz [11] and a few examples of commutative q-difference operators in the process. Then we try to generalize further to q-analogues of Manocha and Sharma [49] formulas for Jacobi polynomial. A field of fractions [29, p. 183] for Cigler's [13] multiplication operator is used in the computations.
Keywords: vector notation, Al-Salam operator, Gould–Hopper formula, Ward q-Taylor formula, Jackson q-Taylor formula, Ward–AlSalam q-addition, Jackson-Hahn q-addition, field of fractions, Carlitz Rodriguez operator, Manocha and Sharma Jacobi polynomial formula

1. Introduction and notation

Definition 1. The power function is defined by $q^a \equiv e^{alog(q)}$. We always use the principal branch of the logarithm.

The variables $a, b, c, a_1, a_2, \ldots, b_1, b_2, \ldots \in \mathbb{C}$ denote parameters in hypergeometric series or q-hypergeometric series. The variables i, j, k, l, m, n, p, r will denote natural numbers except for certain cases where it will be clear from the context that i will denote the imaginary unit. In the whole paper, the symbol $\equiv$ will denote definitions, except when we work with congruences.

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The q-analogues of a complex number a and of the factorial function are defined by:

$$\{a\}_q \equiv \frac{1-q^a}{1-q},\ q \in \mathbb{C}\backslash\{1\}, \tag{1}$$

$$\{n\}_q! \equiv \prod_{k=1}^{n}\{k\}_q,\ \{0\}_q! = 1,\ q \in \mathbb{C}, \tag{2}$$

The q-shifted factorial (compare [28, p.38]) is defined by

$$\langle a;q\rangle_n \equiv \begin{cases} 1, & n=0; \\ \prod_{m=0}^{n-1}(1-q^{a+m}) & n=1,2,\dots, \end{cases} \tag{3}$$

The Watson notation [26] will also be used

$$(a;q)_n \equiv \begin{cases} 1, & n=0; \\ \prod_{m=0}^{n-1}(1-aq^m), & n=1,2,\dots \end{cases} \tag{4}$$

If we work with operators, the definition will be changed to

$$(a;q)_n \equiv \prod_{m=0}^{n-1}(I-aq^m), \tag{5}$$

where I denotes the identity operator.

Furthermore,

$$(a;q)_\infty \equiv \prod_{m=0}^{\infty}(1-aq^m),\ 0<|q|<1. \tag{6}$$

$$(a;q)_\alpha \equiv \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty},\ a \neq q^{-m-\alpha}, m=0,1,\dots. \tag{7}$$

We will use the following abbreviation

$$\langle (a);q\rangle_n \equiv \langle a_1,\dots,a_A;q\rangle_n \equiv \prod_{j=1}^{A}\langle a_j;q\rangle_n. \tag{8}$$

The following notation will be convenient.

Definition 2.

$$\mathrm{QE}(x) \equiv q^x. \tag{9}$$

Definition 3. In the following, $\frac{\mathbb{C}}{\mathbb{Z}}$ will denote the space of complex numbers mod$\frac{2\pi i}{\log q}$. This is isomorphic to the cylinder $\mathbb{R} \times e^{2\pi i\theta}$, $\theta \in \mathbb{R}$. The operator

$$\sim\colon \frac{\mathbb{C}}{\mathbb{Z}} \mapsto \frac{\mathbb{C}}{\mathbb{Z}}$$

is defined by

$$a \mapsto a + \frac{\pi i}{\log q}. \tag{10}$$

Furthermore we define

$$\widetilde{\langle a; q\rangle_n} \equiv \langle \widetilde{a}; q\rangle_n. \tag{11}$$

By (10) it follows that

$$\widetilde{\langle a; q\rangle_n} = \prod_{m=0}^{n-1}(1 + q^{a+m}), \tag{12}$$

where this time the tilde denotes an involution which changes a minus sign to a plus sign in all the n factors of $\langle a; q\rangle_n$.

Furthermore we make the convention that the $\widetilde{\ }$operator is always related to the base q as will be important in (32). Since products of q-shifted factorials occur so often, to simplify them we shall frequently use the more compact notation

$$\langle a_1, \ldots, a_m; q\rangle_n \equiv \prod_{j=1}^{m}\langle a_j; q\rangle_n. \tag{13}$$

The q-hypergeometric series was developed by Heine 1846 [34] as a generalization of the hypergeometric series.

Definition 4. Generalizing Heine's series, we shall define a q-hypergeometric series by (compare [26, p.4]):

$$\begin{aligned} {}_p\phi_r(\hat{a_1}, \ldots, \hat{a_p}; \hat{b_1}, \ldots, \hat{b_r}|q, z) \equiv\ {}_p\phi_r\left[\begin{array}{c} \hat{a_1}, \ldots, \hat{a_p} \\ \hat{b_1}, \ldots, \hat{b_r} \end{array} |q, z\right] \equiv \\ \sum_{n=0}^{\infty} \frac{\langle \hat{a}_1, \ldots, \hat{a}_p; q\rangle_n}{\langle 1, \hat{b_1}, \ldots, \hat{b_r}; q\rangle_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+r-p} z^n, \end{aligned} \tag{14}$$

where $q \neq 0$ when $p > r + 1$, and

$$\widehat{a} = \begin{cases} a \\ \widetilde{a} \end{cases} \tag{15}$$

Remark 1. In a few cases the parameter $\hat{a}$ in (14) will be the real plus infinity
$(0 < |q| < 1)$. They correspond to multiplication by 1.

Remark 2. The parameters $\hat{a_i}$ and $\hat{b_i}$ to the left of | in (14) are thought to be exponents, they are periodic $\mod \frac{2\pi i}{\log q}$.

The following simple rules follow from (10). Clearly the first two equations are applicable to q-exponents. Compare [61, p. 110].

$$\widetilde{a} \pm b \equiv \widetilde{a \pm b} \mod \frac{2\pi i}{\log q}, \tag{16}$$

$$\widetilde{a} \pm \widetilde{b} \equiv a \pm b \mod \frac{2\pi i}{\log q}, \tag{17}$$

$$q^{\widetilde{a}} = -q^a, \tag{18}$$

where the second equation is a consequence of the fact that we work $\mod \frac{2\pi i}{\log q}$.

Definition 5. Further generalizing (14), we shall define a q-hypergeometric series by

$$\begin{aligned} &{}_{p+p'}\phi_{r+r'}(\hat{a_1}, \dots, \hat{a_p}; \hat{b_1}, \dots, \hat{b_r}|q, z||s_1, \dots, s_{p'}; t_1, \dots, t_{r'}) \equiv \\ &{}_{p+p'}\phi_{r+r'}\left[\begin{array}{c} \hat{a_1}, \dots, \hat{a_p} \\ \hat{b_1}, \dots, \hat{b_r} \end{array} |q, z|| \begin{array}{c} s_1, \dots, s_{p'} \\ t_1, \dots, t_{r'} \end{array} \right] = \\ &= \sum_{n=0}^{\infty} \frac{\langle \hat{a}_1; q\rangle_n \dots \langle \hat{a}_p; q\rangle_n}{\langle 1; q\rangle_n \langle \hat{b}_1; q\rangle_n \dots \langle \hat{b}_r; q\rangle_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+r+r'-p-p'} \times \\ &z^n \prod_{k=1}^{p'} (s_k; q)_n \prod_{k=1}^{r'} (t_k; q)_n^{-1}, \end{aligned} \tag{19}$$

where $q \neq 0$ when $p + p' > r + r' + 1$.

Remark 3. Equation (19) is used in certain special cases when we need factors $(t; q)_n$ in the q-series. One example is the q-analogue of a bilinear generating formula for Laguerre polynomials.

Example 1. The following equation is a q-analogue of [31, (2), p. 98], originally due to Gauss.

$${}_2\phi_1(-\frac{n}{2}, \frac{1-n}{2}; \mu + \frac{1}{2}|q^2, q^{2(\mu+n)}) = \frac{\langle \widetilde{\mu}, \widetilde{\mu + \frac{n}{2}}; q\rangle_{\frac{n}{2}} \langle \mu; q\rangle_n}{\langle 2\mu; q\rangle_n}. \tag{20}$$

Definition 6. Euler found the following two q-analogues of the exponential function:

(21)

$$e_q(z) \equiv {}_1\phi_0(\infty; -|q, z) \equiv \sum_{n=0}^{\infty} \frac{z^n}{\langle 1; q\rangle_n} = \frac{1}{(z; q)_\infty},\ |z| < 1,\ 0 < |q| < 1.$$

(22)

$$e_{\frac{1}{q}}(z) \equiv {}_0\phi_0(-; -|q, -z) \equiv$$

$$\equiv \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{\langle 1; q\rangle_n} z^n = (-z; q)_\infty,\ 0 < |q| < 1.$$

If $|q| > 1$, or $0 < |q| < 1$ and $|z| < |1-q|^{-1}$, the q-exponential function $E_q(z)$ was defined by Jackson [36] 1904, and by Exton [22]

$$E_q(z) \equiv \sum_{k=0}^{\infty} \frac{1}{\{k\}_q!} z^k. \tag{23}$$

By the Euler equation (21), we can replace $E_q(z)$ by

$$\frac{1}{(z(1-q); q)_\infty},\ |z(1-q)| < 1,\ 0 < |q| < 1.$$

So by meromorphic continuation, the meromorphic function $\frac{1}{(z(1-q);q)_\infty}$, with simple poles at $\frac{q^{-k}}{1-q},\ k \in \mathbb{N}$, is a good substitute for $E_q(z)$ in the whole complex plane. We shall however continue to call this function $E_q(z)$, since it plays an important role in the operator theory.

Definition 7. There is another q-exponential function which is entire when
$0 < |q| < 1$ and which converges when $|z| < |1-q|^{-1}$ if $|q| > 1$. To obtain it we must invert the base in (23), i.e. $q \to \frac{1}{q}$.

$$E_{\frac{1}{q}}(z) \equiv \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}}}{\{k\}_q!} z^k. \tag{24}$$

The following equations obtain:

$$E_q(-z) E_{\frac{1}{q}}(z) = 1. \tag{25}$$

$$D_q^k E_q(az) = a^k E_q(az). \tag{26}$$

$$D_q^k E_{\frac{1}{q}}(az) = a^k q^{\binom{k}{2}} E_{\frac{1}{q}}(q^k az). \tag{27}$$

Because of the last two equations, the function $E_q(z)$ is easier to handle than $E_{\frac{1}{q}}(z)$.

Definition 8. The operator

$$\widetilde{\frac{m}{n}} : \frac{\mathbb{C}}{\mathbb{Z}} \mapsto \frac{\mathbb{C}}{\mathbb{Z}}$$

is defined by

$$a \mapsto a + \frac{2\pi i m}{n \log q}. \tag{28}$$

The following simple rules follow from (28).

$$\widetilde{\frac{m}{n}} a \pm b \equiv \widetilde{\frac{m}{n}} (a \pm b) \bmod \frac{2\pi i}{\log q}, \tag{29}$$

$$\sum_{k=1}^{n} \widetilde{\frac{1}{n}} \pm a_k \equiv \sum_{k=1}^{n} \pm a_k \bmod \frac{2\pi i}{\log q}, \tag{30}$$

$$\mathrm{QE}(\widetilde{\frac{m}{n}} a) = \mathrm{QE}(a) e^{\frac{2\pi i m}{n}}, \tag{31}$$

where the second equation is a consequence of the fact that we work $\bmod \frac{2\pi i}{\log q}$.

As before, we make the convention that the $\widetilde{\frac{m}{n}}$ operator is always related to the base q.

Furthermore,

Theorem 1.1.

$$\langle \widetilde{a}; q^2 \rangle_n = \langle \widetilde{\frac{1}{4}} a, \widetilde{\frac{3}{4}} a; q \rangle_n. \tag{32}$$

Definition 9. Let the q-Pochhammer symbol $\{a\}_{n,q}$ be defined by

$$\{a\}_{n,q} \equiv \prod_{m=0}^{n-1} \{a+m\}_q. \tag{33}$$

An equivalent symbol is defined in [22, p.18] and is used throughout that book. See also [4, p.138].

It turns out that q-addition is the natural way to work with addition for the function argument in a q-hypergeometric series.

The Ward–AlSalam q-addition was invented by Ward 1936 [65, p. 256] and Al-Salam 1959 [2, p. 240]

$$(a \oplus_q b)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q a^k b^{n-k}, \ n = 0, 1, 2, \ldots. \tag{34}$$

The Jackson–Hahn–Cigler (JHC) q-addition, compare [32, p. 362] , is given by

(35)
$$[x \pm y]_q^n \equiv \sum_{k=0}^{n} (\pm)^k \binom{n}{k}_q q^{\binom{k}{2}}\ y^k x^{n-k} = x^n (\mp \frac{y}{x}; q)_n,\ \ n = 0, 1, 2, \ldots .$$

Theorem 1.2. *The Ward–AlSalam q-addition (34) has the following properties, $a, b, c \in \mathbb{C}$:*

$$\begin{aligned} &(a \oplus_q b) \oplus_q c = a \oplus_q (b \oplus_q c) \\ &a \oplus_q b = b \oplus_q a \\ &a \oplus_q 0 = 0 \oplus_q a = a \\ &ca \oplus_q cb = c(a \oplus_q b). \end{aligned} \tag{36}$$

Proof. The first property (associativity) is proved as follows: We must prove that

$$[(a \oplus_q b) \oplus_q c]^n = [a \oplus_q (b \oplus_q c)]^n. \tag{37}$$

But this is equivalent to

$$\begin{aligned} &\sum_{k=0}^{n} \binom{n}{k}_q \sum_{l=0}^{k} \binom{k}{l}_q a^l b^{k-l} c^{n-k} = \\ &= \sum_{k'=0}^{n} \binom{n}{k'}_q a^{k'} \sum_{l'=0}^{n-k'} \binom{n-k'}{l'}_q b^{l'} c^{n-k'-l'}. \end{aligned} \tag{38}$$

Now put $l = k'$ and $l' = k - l$ to conclude the proof.

The proof of the distributive law is obvious. □

Definition 10. [65] If $F(x)$ is the power series $\sum_{n=0}^{\infty} a_n x^n$,

$$F(x \oplus_q y) \equiv \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \binom{n}{k}_q x^k y^{n-k}, \tag{39}$$

$$F[x \pm y]_q \equiv \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} (\pm)^k \binom{n}{k}_q q^{\binom{k}{2}}\ y^k x^{n-k}. \tag{40}$$

Example 2. A q-analogue of Gauss [27, p. 127, (2)], [8, p. 23].

$$(1 \oplus_q x)^n + (1 \ominus_q x)^n = 2\ {}_4\phi_1(-\frac{n}{2}, \frac{1-n}{2}, \infty, \infty; \frac{1}{2} | q^2, x^2 q^{2n-1}). \tag{41}$$

2. q-Taylor expansions and q-derivatives

In this chapter we will study q-derivatives in n dimensions and the corresponding q-Taylor expansions both for functions defined by power series and by sums of inverse q-shifted factorials. Future applications for q-functions of many variabels [20] are likely. The Hörmander [35] multindex notation will be used. Because the $\mathbf{x}$ [35] notation will be used in the next chapter we will change it to $\vec{x}$ here. We shall conclude by presenting a q-Taylor formula with a remainder term expressed as q-integral [18] which generalizes a result of Jackson from 1909 [40]. In 2002 Rajković & Stanković & Marinković [54] generalized this to a formula with a Lagrange remainder term by using a q-form of the generalized mean value theorem for integrals [54]. We will generalize this further to n variables.

Definition 11. [14] Let ϵ_j denote the operator which maps $f(x_j)$ to $f(q_j x_j)$. If there is only one variable, we simply write ϵ like in the next chapter.

The following notation [14] is equivalent to the JHC q-addition.

$$P_{n,q}(x,a) \equiv \prod_{m=0}^{n-1}(x-aq^m) = [x \pm a]_q^n,\ n=1,2,\ldots. \tag{42}$$

Definition 12. In 1908 Jackson [39] reintroduced the Euler-Heine-Jackson q-difference operator

$$\left(D_q\varphi\right)(x) = \begin{cases} \frac{\varphi(x)-\varphi(qx)}{(1-q)x}, & \text{if } q\in\mathbb{C}\backslash\{1\},\ x\neq 0; \\ \frac{d\varphi}{dx}(x) & \text{if } q=1; \\ \frac{d\varphi}{dx}(0) & \text{if } x=0 \end{cases} \tag{43}$$

If we want to indicate the variable which the q-difference operator is applied to, we write $\left(D_{q,x}\varphi\right)(x,y)$ for the operator.

Remark 4. The definition (43) is more lucid than the one previously given, which was without the condition for $x=0$. It leads to new so-called q-constants, or solutions to $(D_q\varphi)(x)=0$.

Definition 13. In 1994 [15] Chung K. S. & Chung W. S. & Nam S. T. & Kang H. J. defined a new q-derivative as follows:

$$D_{\oplus}f(x) = \lim_{\delta x\to 0}\frac{f(x\oplus_q \delta x)-f(x)}{\delta x}. \tag{44}$$

Theorem 2.1. *This q-derivative $D_{\oplus}$ satisfies the following rules:*

$$D_{\oplus}(x\oplus_q a)^n = \{n\}_q(x\oplus_q a)^{n-1}. \tag{45}$$

$$D_{\oplus}\mathrm{E}_q(x) = \mathrm{E}_q(x). \tag{46}$$

$$D_{\oplus}\mathrm{e}_q(x) = \frac{\mathrm{e}_q(x)}{1-q}. \tag{47}$$

Proof. The first equation is proved as follows:

$$\lim_{\delta x\to 0}\frac{(x\oplus_q \delta x\oplus_q a)^n-(x\oplus_q a)^n}{\delta x} = \tag{48}$$
$$\lim_{\delta x\to 0}\frac{\sum_{k=0}^{n}\binom{n}{k}_q\ \sum_{l=0}^{k}\binom{k}{l}_q\ x^l\delta x^{k-l}a^{n-k}-\sum_{k'=0}^{n}\binom{n}{k'}_q\ x^{k'}a^{n-k'}}{\delta x} =$$
$$\{n\}_q(x\oplus_q a)^{n-1}.$$

□

Theorem 2.2.

$$D_{\oplus}x^{\alpha} = \{\alpha\}_q x^{\alpha-1}, \tag{49}$$

just as for the q-difference operator.

$$D_{\oplus}\frac{1}{(x;q)_{\alpha}} = \frac{\{\alpha\}_q}{(x;q)_{\alpha+1}}, \tag{50}$$

just as for the q-difference operator.

Proof. Use the q-binomial theorem. □

Definition 14. The notation $\sum_{\vec{m}}$ denotes a multiple summation with the indices $m_1,\dots,m_n$ running over all non-negative integer values. In this connection we put $|m|\equiv\sum_{j=1}^{n}m_j$.

If $\vec{m}$ and $\vec{k}$ are two arbitrary vectors with n elements, their q-binomial coefficient is defined as

$$\binom{\vec{m}}{\vec{k}}_{\vec{q}} \equiv \prod_{j=1}^{n}\binom{m_j}{k_j}_{q_j}. \tag{51}$$

If $\{x_j\}_{j=1}^{n}$ and $\{y_j\}_{j=1}^{n}$ are two arbitrary sequences of complex numbers, then their scalar product is defined by

$$\vec{x}\vec{y} \equiv \sum_{j=1}^{n}x_jy_j. \tag{52}$$

The partial q-derivative of a function of n variables is defined in the spirit of Hörmander [35, p. 12].

$$D^{\vec{l}}_{\vec{q},\vec{x}}F(\vec{x},\vec{q}) \equiv \prod_{j=1}^{n}(D^{l_j}_{q_j,x_j})F(\vec{x},\vec{q}). \tag{53}$$

$$(D_{\vec{q},\vec{x}}\vec{\epsilon}^{\,\vec{t}})^{\vec{l}}F(\vec{x},\vec{q}) \equiv \prod_{j=1}^{n}(D_{q_j,x_j}\epsilon_j^{t_j})^{l_j}F(\vec{x},\vec{q}). \tag{54}$$

In the same way we define vector versions of powers, q-shifted factorials, q-Pochhammer symbols and JHC q-additions.

$$\vec{x}^{\vec{\alpha}} \equiv \prod_{j=1}^{n} x_j^{\alpha_j}, \tag{55}$$

$$\frac{1}{(\vec{x};\vec{q})_{\vec{\beta}}} \equiv \prod_{j=1}^{n} \frac{1}{(x_j;q_j)_{\beta_j}}, \tag{56}$$

$$\langle \vec{\alpha};\vec{q}\rangle_{\vec{k}} \equiv \prod_{j=1}^{n} \langle \alpha_j;q_j\rangle_{k_j}, \tag{57}$$

$$\{\vec{l}\}_{\vec{q}}! \equiv \prod_{j=1}^{n} \{l_j\}_{q_j}!, \tag{58}$$

$$\{\vec{\alpha}\}_{\vec{m},\vec{q}} \equiv \prod_{j=1}^{n} \{\alpha_j\}_{m_j,q_j}, \tag{59}$$

$$\vec{q}^{\binom{\vec{k}}{2}} \equiv \prod_{j=1}^{n} q_j^{\binom{k_j}{2}}, \tag{60}$$

$$(-1)^{\vec{k}} \equiv (-1)^{|k|}, \tag{61}$$

$$P_{\vec{k},\vec{q}}(\vec{x},\vec{y}) \equiv \prod_{j=1}^{n} P_{k_j,q_j}(x_j,y_j). \tag{62}$$

Closed and open intervals are defined by

$$[\vec{a},\vec{b}] \equiv \prod_{j=1}^{n} [a_j,b_j]. \tag{63}$$

$$(\vec{a},\vec{b}) \equiv \prod_{j=1}^{n} (a_j,b_j). \tag{64}$$

Jackson [37, p. 145], [38], [41, p. 146] has shown certain connections between power series in x and series of the form $\sum_{k=0}^{\infty} \frac{a_k x^{f(k)}}{(x;q)_k}$, where $f(k)$ is an integer-valued function.

In 1921 Ryde [56] showed that certain linear homogeneous q-difference equations have series

$$\sum_{k=0}^{\infty} \frac{a_k}{(x;q^{-1})_k}$$

as solutions.

We need a q-analogue of holomorphic functions, which will be useful to characterize the functions to the far right in operator expressions.

Definition 15. $\vec{\mathrm{H}}_{\vec{q},n}$ denotes functions $F(\vec{x})$ of n variables which can be written as an infinite series

$$F(\vec{x}) \equiv \sum_{\vec{k}=\vec{0}}^{\vec{\infty}} \vec{x}^{\,\vec{\alpha}_{\vec{k}}} \frac{\vec{\gamma}_{\vec{k}}}{(\vec{x}\vec{q}^{\vec{\delta}_{\vec{k}}};\vec{q}^{\pm\vec{1}})_{\vec{\beta}_{\vec{k}}}}, \text{ where } \vec{\alpha}_{\vec{k}}, \vec{\beta}_{\vec{k}}, \vec{\gamma}_{\vec{k}}, \vec{\delta}_{\vec{k}} \in \mathbb{C}^n.$$

When $n = 1$, we just write H_q.

If the function is defined in an open region O, we write $\vec{\mathrm{H}}_{\vec{q},n}(\mathrm{O})$.

Definition 16. The generalized (noncommutative) Ward–AlSalam q-addition, is the function

$$(x \oplus_{q,t} y)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q x^{n-k} y^k q^{t\left(nk-\binom{k}{2}\right)}, \ n = 0, 1, 2, \ldots. \tag{65}$$

Definition 17. The generalized JHC q-addition, is the function

$$[x \pm y]_{q,t}^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q x^{n-k} (\pm y)^k q^{\binom{k}{2}+t\left(nk-\binom{k}{2}\right)}, \ n = 0, 1, 2, \ldots. \tag{66}$$

Following Hahn [32, p. 362], we will denote the power series $F([x+y]_{q,t})$ by $F[x+y]_{q,t}$.

Example 3.

$$(D_q \epsilon^t)^k \sum_{m=0}^{\infty} \frac{a_m x^m}{\{m\}_q!} = \sum_{m=0}^{\infty} \frac{a_{k+m} x^m}{\{m\}_q!} \mathrm{QE}\left(tk(m+1) + t\binom{k}{2}\right). \tag{67}$$

(68)

$$\prod_{j=1}^{2} (D_{q_j,x_j} \epsilon_j^{t_j})^{k_j} \Phi_4(a; b; c, c' | q; x_1, x_2) =$$

$$\frac{\langle a, b; q\rangle_N \langle c + N - k_2; q\rangle_{k_2} \langle c' + N - k_1; q\rangle_{k_1}}{\langle c, c'; q\rangle_N (1-q)^N}$$

$$\sum_{m_1,m_2=0}^{\infty} \frac{\langle a+N, b+N; q\rangle_{m_1+m_2}}{\langle 1, c+k_1; q\rangle_{m_1} \langle 1, c'+k_2; q\rangle_{m_2}} x_1^{m_1} x_2^{m_2} \mathrm{QE}\left(\vec{t}\vec{m}\vec{k} + \vec{t}\binom{\vec{k}}{2} + \vec{t}\vec{k}\right),$$

where $N \equiv k_1 + k_2, \vec{t}\vec{m}\vec{k} \equiv t_1 m_1 k_1 + t_2 m_2 k_2$, and (compare [20])
(69)

$$\Phi_4(a;b;c,c'|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a;q\rangle_{m_1+m_2}\langle b;q\rangle_{m_1+m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1}\langle c';q\rangle_{m_2}} x_1^{m_1} x_2^{m_2}.$$

Definition 18. The following notation for power series

$$F(\vec{x}) \equiv \sum_{\vec{k}=\vec{0}}^{\vec{\infty}} \vec{x}^{\vec{\alpha}_{\vec{k}}} \vec{\gamma}_{\vec{k}}$$

will also be useful.

$$F(\vec{x} \oplus_{\vec{q}} \vec{y}) \equiv F(x_1 \oplus_{q_1} y_1, \ldots, x_n \oplus_{q_n} y_n), \tag{70}$$

$$F[\vec{x} + \vec{y}]_{\vec{q}} \equiv F([x_1 + y_1]_{q_1}, \ldots, [x_n + y_n]_{q_n}), \tag{71}$$

$$F[\vec{x} + \vec{y}]_{\vec{q},\vec{t}} \equiv F([x_1 + y_1]_{q_1,t_1}, \ldots, [x_n + y_n]_{q_n,t_n}), \tag{72}$$

There are at least three q-analogues of the Taylor formula known from the literature. We list them here and give a few others, which are generalizations to n variables.

Theorem 2.3. *The Ward q-Taylor formula for formal power series.* [65, p. 259].

$$F(x \oplus_q y) = \sum_{n=0}^{\infty} \frac{y^n}{\{n\}_q!} D_q^n F(x). \tag{73}$$

The first Jackson q-Taylor formula [40, p. 63]

$$F(x) = \sum_{n=0}^{\infty} \frac{[x-y]_q^n}{\{n\}_q!} D_q^n F(y). \tag{74}$$

The second Jackson q-Taylor formula [42, (51, p.77)]

$$F[x+y]_q = \sum_{n=0}^{\infty} \frac{y^n}{\{n\}_q!} q^{\binom{n}{2}} D_q^n F(x). \tag{75}$$

Theorem 2.4. *The Ward q-Taylor formula for functions of n variables. Let $F(\vec{x})$ be a q-Kampé de Fériet function* [20]*, or more generally, a formal power series $\sum_{\vec{m}} \vec{a}_{\vec{m}} \vec{x}^{\vec{m}}$. Then (compare* [35, 1.1.7' p. 13]*)*

$$F(\vec{x} \oplus_{\vec{q}} \vec{y}) = \sum_{\vec{k}=\vec{0}}^{\vec{\infty}} \frac{\vec{y}^{\,\vec{k}}}{\{\vec{k}\}_{\vec{q}}!} D_{\vec{q},\vec{x}}^{\vec{k}} F(\vec{x}). \tag{76}$$

Theorem 2.5. *With the same prerequisits as above,*

$$F[\vec{x}+\vec{y}]_{\vec{q}} = \sum_{\vec{k}=0}^{\vec{\infty}} \frac{\vec{y}^{\,\vec{k}} \vec{q}^{\binom{\vec{k}}{2}}}{\{\vec{k}\}_{\vec{q}}!} D^{\vec{k}}_{\vec{q},\vec{x}} F(\vec{x}). \tag{77}$$

We will use a kind of umbral calculus.

Definition 19. The two q-additions for q-shifted factorials are defined in the following way. If $F(x)$ is of the form $F(x) = \sum_{n=0}^{\infty} \frac{a_n}{(x;q)_{-n}}$, then

$$F(x \oplus_q y) \equiv \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \binom{n}{k}_q \frac{1}{(x;q)_{k-n}(y;q)_{-k}}, n = 0, 1, 2, \ldots. \tag{78}$$

(79)

$$F[x \pm y]_q \equiv \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} (\pm)^k \binom{n}{k}_q q^{\binom{k}{2}} \frac{1}{(x;q)_{k-n}(y;q)_{-k}}, n = 0, 1, 2, \ldots.$$

The following two notations for q-shifted factorials will be used.

$$F(\vec{x} \oplus_{\vec{q}} \vec{y}) \equiv F(x_1 \oplus_{q_1} y_1, \ldots, x_n \oplus_{q_n} y_n), \tag{80}$$

$$F[\vec{x}+\vec{y}]_{\vec{q}} \equiv F([x_1+y_1]_{q_1}, \ldots, [x_n+y_n]_{q_n}), \tag{81}$$

Theorem 2.6. *The Ward q-addition (78) for q-shifted factorials is associative and commutative.*

Proof. Same as for power series. □

Theorem 2.7. *The Ward q-Taylor formula for q-shifted factorials. Let $F_k(x)$ be of the form*

$$F_k(x) \equiv \sum_{m=0}^{\infty} \frac{a_m q^{km}}{(x;q)_{-m}} \tag{82}$$

Then

$$F_0(x \oplus_q y) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{-\binom{k}{2}}}{\{k\}_q!(y;q)_{-k}} D_q^k F_k(x) \tag{83}$$

Proof.

$$\begin{aligned} LHS &= \sum_{m=0}^{\infty} a_m \sum_{k=0}^{m} \binom{m}{k}_q \frac{1}{(x;q)_{k-m}(y;q)_{-k}} = \\ &\sum_{k=0}^{\infty} \frac{(-1)^k q^{-\binom{k}{2}}}{(y;q)_{-k}\{k\}_q!} \sum_{m=k}^{\infty} \frac{a_m \{-m\}_{k,q} q^{km}}{(x;q)_{k-m}} = RHS. \end{aligned} \tag{84}$$

□

Theorem 2.8. *Two q-Taylor formulas for functions in q-shifted factorials of n variables. Let $F_{\vec{k}}(\vec{x})$ be of the form*

$$F_{\vec{k}}(\vec{x}) \equiv \sum_{\vec{m}=\vec{0}}^{\vec{\infty}} \vec{a}_{\vec{m}} \vec{q}^{\ \vec{m}\vec{k}} \frac{1}{(\vec{x};\vec{q})_{-\vec{m}}}. \tag{85}$$

Then

$$F_{\vec{0}}\left(\vec{x} \oplus_{\vec{q}} \vec{y}\right) = \sum_{\vec{k}=\vec{0}}^{\vec{\infty}} \frac{(-1)^{\vec{k}} \vec{q}^{\ -\binom{\vec{k}}{2}}}{\{\vec{k}\}_{\vec{q}}!(\vec{y};\vec{q})_{-\vec{k}}} D^{\vec{k}}_{\vec{q},\vec{x}} F_{\vec{k}}(\vec{x}). \tag{86}$$

$$F_{\vec{0}}\left[\vec{x} \pm \vec{y}\right]_{\vec{q}} = \sum_{\vec{k}=\vec{0}}^{\vec{\infty}} \frac{(\mp 1)^{\vec{k}}}{\{\vec{k}\}_{\vec{q}}!(\vec{y};\vec{q})_{-\vec{k}}} D^{\vec{k}}_{\vec{q},\vec{x}} F_{\vec{k}}(\vec{x}). \tag{87}$$

Proof. We prove the first equation.

$$\begin{aligned} LHS = \sum_{\vec{m}=\vec{0}}^{\vec{\infty}} \vec{a}_{\vec{m}} \sum_{\vec{k}=\vec{0}}^{\vec{m}} \binom{\vec{m}}{\vec{k}}_{\vec{q}} \frac{1}{(\vec{x};\vec{q})_{k-\vec{m}} (\vec{y};\vec{q})_{-\vec{k}}} = \\ \sum_{\vec{k}=\vec{0}}^{\vec{\infty}} \frac{(-1)^{\vec{k}} \vec{q}^{\ -\binom{\vec{k}}{2}}}{(\vec{y};\vec{q})_{-\vec{k}} \{\vec{k}\}_{\vec{q}}!} \sum_{\vec{m}=\vec{k}}^{\vec{\infty}} \frac{\vec{a}_{\vec{m}} \{-\vec{m}\}_{\vec{k},\vec{q}} \vec{q}^{\ k\vec{m}}}{(\vec{x};\vec{q})_{k-\vec{m}}} = RHS. \end{aligned} \tag{88}$$

□

Theorem 2.9. *The extended Ward q-Taylor formula for formal power series, where (39,) and (40) have been extended in a natural way.*

$$F(x \oplus_{q,t} y) = \sum_{n=0}^{\infty} \frac{y^n}{\{n\}_q!} (D_q \epsilon^t)^n F(x). \tag{89}$$

$$F[x \pm y]_{q,t} = \sum_{n=0}^{\infty} \frac{(\pm y)^n}{\{n\}_q!} (D_q \epsilon^t)^n F(x) q^{\binom{n}{2}}. \tag{90}$$

Theorem 2.10. *The extended Ward q-Taylor formula for functions of n variables. Let $F(\vec{x})$ be a q-Kampé de Fériet function* [20]*, or more generally, a formal power series $\sum_{\vec{m}} \vec{a}_{\vec{m}} \vec{x}^{\vec{m}}$. Then (compare* [35, 1.1.7' p. 13]*)*

$$F(\vec{x} \oplus_{\vec{q},t} \vec{y}) = \sum_{\vec{k}=0}^{\vec{\infty}} \frac{\vec{y}^{\ \vec{k}}}{\{\vec{k}\}_{\vec{q}}!} (D_{\vec{q},\vec{x}\ \vec{\epsilon}^{\vec{x}}})^{\vec{k}} F(\vec{x}). \tag{91}$$

Theorem 2.11. *With the same prerequisits as above,*

$$F[\vec{x} \pm \vec{y}]_{\vec{q},\vec{t}} = \sum_{\vec{k}=0}^{\vec{\infty}} \frac{(\pm\vec{y})^{\vec{k}} q^{\binom{\vec{k}}{2}}}{\{\vec{k}\}_{\vec{q}}!} (D_{\vec{q},\vec{x}\ \vec{\epsilon}})^{\vec{k}} F(\vec{x}). \tag{92}$$

In the rest of this chapter we assume that the function $f(t,q)$ and its vector version $f(\vec{t},\vec{q})$ satisfy certain restriction of growth given in Walliser [63] and [64], who improved a result of Gelfond from 1933.

Theorem 2.12. [18] *Let $0 < q < 1$ and let $f(t,q)$ be n times q-differentiable in the open interval $[a,x] \times (0,1)$. Then the following generalization of Jackson's formula holds for $n = 1,2,\ldots$:*

(93)

$$f(x,q) = \sum_{k=0}^{n-1} \frac{P_{k,q}(x,a)}{\{k\}_q!} (D_q^k f)(a,q) + \int_{t=a}^{x} \frac{P_{n-1,q}(x,qt)}{\{n-1\}_q!} (D_q^n f)(t,q)\, d_q(t).$$

Proof. Use q-integration by parts. □

Rajković & Stanković & Marinković generalized this to the Lagrange form

Theorem 2.13. [54, p. 176]. *There is a unique $q' \in (0,1)$, such that for the function $f(x,q)$ defined on $[b,c] \times (q',1)$, and $x, a \in (b,c)$, $\xi \in (b,c)$ can be found between x and a which satisfies*

$$f(x,q) = \sum_{k=0}^{n-1} \frac{P_{k,q}(x,a)}{\{k\}_q!} (D_q^k f)(a,q) + \frac{P_{n,q}(x,a)}{\{n\}_q!} (D_q^n f)(\xi,q) \tag{94}$$

This can be generalized to

Theorem 2.14. *There is a unique $\vec{q'} \in (\vec{0},\vec{1})$, such that for the function $f(\vec{x},\vec{q})$ defined on $[\vec{b},\vec{c}] \times (\vec{q'},\vec{1})$, and $\vec{x},\vec{a} \in (\vec{b},\vec{c})$, $\vec{\xi} \in (\vec{b},\vec{c})$ can be found between $\vec{x}$ and $\vec{a}$ which satisfies*

$$f(\vec{x},\vec{q}) = \sum_{\vec{k}=\vec{0}}^{\vec{n-1}} \frac{P_{\vec{k},\vec{q}}(\vec{x},\vec{a})}{\{\vec{k}\}_{\vec{q}}!} (D_{\vec{q}}^{\vec{k}}) f(\vec{a},\vec{q}) + \frac{P_{\vec{n},\vec{q}}(\vec{x},\vec{a})}{\{\vec{n}\}_{\vec{q}}!} D_{\vec{q}}^{\vec{n}}) f(\vec{\xi},\vec{q}). \tag{95}$$

Remark 5. For practical purposes it suffices to study functions in $\vec{\mathrm{H}}_{\vec{q},n}$, which can easily be q- differentiated.

3. q-Laguerre polynomials and Al–Salam operator expressions

Operational formulas were often used with big success in the theory of classical orthogonal polynomials and Bessel functions [30]. The results herewith obtained are both theoretically of a certain interest, and

also give important other formulas. The present paper is the first one in a series which tries to shed more light on the mysteries of so-called q-analogues of operational formulas. As Fujiwara [25] showed, the most important property of the Jacobi, Laguerre and Hermite polynomials (JLH) is the generalized Rodriguez formula. There are basically three different kinds of methods to treat orthogonality.

(1) Three term recurrence theory (Favard theorem)
(2) Rodriguez formula and integration by parts [21].
(3) Self-adjoint differential equations with boundary conditions (A. Krall and P. Lesky).

The classical JLH have many things in common as was beautifully explained by Feldheim [24]. If we have an equation for the Jacobi polynomials, we automatically get a corresponding formula for Laguerre and Hermite.

In this paper we will be working with two different q-Laguerre polynomials. The polynomial $L_{n,q,c}^{(\alpha)}(x)$ was used by Cigler [13].

Definition 20.

$$
\begin{aligned}
L_{n,q,c}^{(\alpha)}(x) &\equiv \sum_{k=0}^{n} \binom{n+\alpha}{n-k}_q \frac{\{n\}_q!}{\{k\}_q!} q^{k^2+\alpha k}(-1)^k x^k \\
&\equiv \sum_{k=0}^{n} \frac{\langle 1+\alpha;q\rangle_n}{\langle 1+\alpha;q\rangle_k} \frac{\langle -n;q\rangle_k}{\langle 1;q\rangle_k} \frac{q^{\frac{k^2+k}{2}+kn+\alpha k}(1-q)^k x^k}{(1-q)^n} \qquad (96) \\
&\equiv \frac{\langle \alpha+1;q\rangle_n}{(1-q)^n} {}_1\phi_1\left(-n;\alpha+1|q,-x(1-q)q^{n+\alpha+1}\right).
\end{aligned}
$$

The Al–Salam q-Laguerre polynomial [1, p. 4] $L_{n,q}^{(\alpha)}(x)$ is defined as follows. Except for the notation, this definition is equivalent to [50], [26] and [60].

$$
L_{n,q}^{(\alpha)}(x) = \frac{L_{n,q,c}^{(\alpha)}(x)}{\{n\}_q!} \qquad (97)
$$

Remark 6. In Koekoek & Swarttouw [47] the q-Laguerre polynomial is defined as

$$
\frac{\langle \alpha+1;q\rangle_n}{\langle 1;q\rangle_n} {}_1\phi_1\left(-n;\alpha+1|q,-xq^{n+\alpha+1}\right). \qquad (98)
$$

In the literature there are many definitions of q-Laguerre polynomials, but most of them are related to each other by some transformation.

The following formulas are useful.

Theorem 3.1.

$$D_q^l \frac{1}{(x;q)_\alpha} = \frac{\{\alpha\}_{l,q}}{(x;q)_{\alpha+l}}. \tag{99}$$

$$D_q^r L_{n,q}^{(\alpha)}(x) = (-1)^r q^{\alpha r + \binom{r+1}{2} + \binom{r}{2}} L_{n-r,q}^{(\alpha+r)}(xq^r). \tag{100}$$

$$(D_q \epsilon^{-1})^n L_{m,q}^{(\alpha)}(x) = (-1)^n L_{m-n,q}^{(\alpha+n)}(x) \mathrm{QE}\left(\binom{n}{2} + n\alpha\right). \tag{101}$$

$$D_q^k \frac{x}{1 - xq^{1+\alpha}} = \frac{q^{(1+\alpha)(k-1)}\{k\}_q!}{(xq^{1+\alpha};q)_{k+1}},\ k > 0. \tag{102}$$

Definition 21. Multiplication with x will be denoted by $\mathbf{x}$. Multiplication with $1 + x\gamma$ in numerator or denominator will be denoted by $\mathbf{1} + \mathbf{x}\boldsymbol{\gamma}$.

Remark 7. This is an extension of Cigler's [14, p. 24] definition for multiplication by powers of x, which can be transformed to the original definition by the q-binomial theorem.

We will use the following two operators operating on H_q as a basis for our calculations; the special case $\alpha = 0,\ q = 1$ was treated in [3, 1.1]. A related operator was used in [1, p. 4 (2.1)].

$$\theta_{q,\alpha} \equiv x(\{1+\alpha\}_q I + q^{1+\alpha} x D_{q,x}). \tag{103}$$

$$\Phi_{q,\alpha} \equiv y(\{1+\alpha\}_q I + q^{1+\alpha} y D_{q,y}). \tag{104}$$

From this we obtain [3, 1.2], [1, p. 4 (2.2)]

$$\theta_{q,\alpha}^n(x^\beta) = x^{\beta+n}\{1+\alpha+\beta\}_{n,q}. \tag{105}$$

We obtain by induction [3, 2.1], [1, p. 4 (2.5)]

$$\begin{aligned}\theta_{q,\alpha}^n = x^n \prod_{j=1}^n (\{j+\alpha\}_q I + q^{j+\alpha} x D_q) \equiv \frac{x^n}{(1-q)^n}(\epsilon_1 q^{1+\alpha};q)_n = \\ \frac{x^n}{(1-q)^n} \sum_{k=0}^n (-1)^k \binom{n}{k}_q q^{\binom{k}{2}+k(1+\alpha)} \epsilon_1^k.\end{aligned} \tag{106}$$

Theorem 3.2.

$$\begin{aligned}\theta_{q,\alpha}^n \frac{1}{(x;q)_\beta} = \frac{x^n}{(1-q)^n (x;q)_{\beta+n}} \sum_{k=0}^n (-1)^k \binom{n}{k}_q q^{\binom{k}{2}+k(1+\alpha)} \\ \sum_{l=0}^k (-1)^l \binom{k}{l}_q q^{\binom{l}{2}} x^l \sum_{m=0}^{n-k} (-1)^m \binom{n-k}{m}_q q^{\binom{m}{2}+m(k+\beta)} x^m.\end{aligned} \tag{107}$$

Proof.

(108)

$$LHS = \frac{x^n}{(1-q)^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}_q q^{\binom{k}{2}+k(1+\alpha)}\frac{1}{(xq^k;q)_\beta} =$$
$$\frac{x^n}{(1-q)^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}_q q^{\binom{k}{2}+k(1+\alpha)}\frac{(x;q)_k(xq^{k+\beta};q)_{n-k}}{(x;q)_{\beta+n}} = RHS.$$

□

The following special case applies.

Theorem 3.3.

$$\theta_{q,\alpha}^n\frac{1}{(x;q)_\alpha} = \frac{x^n}{(x;q)_{\alpha+n}}F_n(x), \tag{109}$$

where $F_n(x)$ *is a polynomial* $\sum_{k=0}^n a_{k,n}x^k$ *with*

(110)

$$a_{0,n} = \{1+\alpha\}_{n,q},\ a_{1,n} = -q^\alpha\{n\}_q\{2+\alpha\}_{n,q},\ a_{n,n} = (-1)^n q^{\binom{n}{2}+n\alpha}\{n\}_q!.$$

Proof. Assume that the theorem is true for $n-1$. Then

(111)

$$\theta_{q,\alpha}^n\frac{1}{(x;q)_\alpha} =$$
$$\frac{x^n}{(1-q)(x;q)_{\alpha+n-1}}\sum_{k=0}^{n-1}a_{k,n-1}x^k - \frac{x^nq^{\alpha+n}}{(1-q)(xq;q)_{\alpha+n-1}}\sum_{k=0}^{n-1}a_{k,n-1}x^kq^k.$$

We obtain

(112)

$$a_{1,n} = -\frac{q^{\alpha+n-1}}{1-q}\{1+\alpha\}_{n-1,q} - \frac{q^\alpha(1-q^{n-1})}{(1-q)^2}\{2+\alpha\}_{n-2,q} +$$
$$\frac{q^{\alpha+n}}{1-q}\{1+\alpha\}_{n-1,q} + \frac{q^{2\alpha+n+1}(1-q^{n-1})}{(1-q)^2}\{2+\alpha\}_{n-2,q} =$$
$$-\frac{\{2+\alpha\}_{n-2,q}q^\alpha}{(1-q)^2}(1-q^{\alpha+n}-q^n+q^{\alpha+2n}) = -q^\alpha\{n\}_q\{2+\alpha\}_{n,q}.$$

The formulas for $a_{0,n}$ and $a_{n,n}$ are proved in a similar way. □

If $F(x)$ and $f(x)$ are formal power series we obtain the following rule [3, 2.2][1, p. 4 (2.6)].

$$F(\theta_{q,0})x^\alpha f(x) = x^\alpha F(\theta_{q,\alpha})f(x). \tag{113}$$

The Leibniz rule is [3, 2.4], [1, p. 4 (2.3)]

Theorem 3.4.

(114)
$$\theta_{q,\alpha}^n(x^{1+\alpha}f(x)g(x)) = x^{1+\alpha}\sum_{j=0}^{n}\binom{n}{j}_q q^{(1+\alpha)(n-j)}(\theta_{q,\alpha}^j\epsilon^{n-j}g(x))(\theta_{q,\alpha}^{n-j}f(x)).$$

Proof. Put $f(x) = x^{-n-1-2\alpha}$, $g(x) = x^{n+\beta+\alpha}$. Then we must prove that

(115)
$$x^{n+\beta}\sum_{k=0}^{n}(-1)^k\binom{n}{k}_q \langle -n-\alpha;q\rangle_{n-k}\langle -\beta-2\alpha-n-k;q\rangle_k\times$$
$$\mathrm{QE}\left(\binom{k}{2}+(1+\alpha)(n-k)+(n-k)(n+\beta+\alpha)+k(1+n+\beta+2\alpha\right) =$$
$$x^{n+\beta}\langle 1+\beta+\alpha;q\rangle_m q^{m(x-j+m)}.$$

However, this follows from a simple change of varibles in the following result of Carlitz 1948 [10, p. 988], [18] :

$$\sum_{n=0}^{m}(-1)^n\binom{m}{n}_q q^{\binom{n}{2}}\langle x+1;q\rangle_{m-n}\langle x-j+1-n+m;q\rangle_n = \tag{116}$$
$$\langle j-m+1;q\rangle_m q^{m(x-j+m)},\ x\in\mathbb{C},\ j<m.$$

□

Remark 8. Equation (116) can be extended to arbitrary $j \in \mathbb{C}$ [51, p. 110].

The previous theorem implies the q-analogue of the corrected version of [3, 2.5]

(117)
$$E_q(t\theta_{q,\alpha})(x^{1+\alpha}f(x)g(x)) = x^{1+\alpha}\sum_{j,l=0}^{\infty}\frac{t^j q^{j(1+\alpha)}}{\{j\}_q!}\theta_{q,\alpha}^j(g(x))\frac{t^l}{\{l\}_q!}\theta_{q,\alpha}^l\epsilon^j(f(x))$$
$$= x^{1+\alpha}E_q(tq^{1+\alpha}\theta_{q,\alpha,g}\epsilon_f)g(x)E_q(tq^{1+\alpha}\theta_{q,\alpha,f})f(x).$$

Proof.

(118)

$$LHS = E_q(t\theta_{q,\alpha})(x^{1+\alpha}f(x)g(x)) = \sum_{k=0}^{\infty}\frac{t^k}{\{k\}_q!}\theta_{q,\alpha}^k(x^{1+\alpha}f(x)g(x))$$

$$= \sum_{k=0}^{\infty}\frac{t^k}{\{k\}_q!}x^{1+\alpha}\sum_{j=0}^{k}\frac{\{k\}_q!}{\{j\}_q!\{k-j\}_q!}q^{j(1+\alpha)}(\theta_{q,\alpha}^j g(x))(\theta_{q,\alpha}^{k-j}\epsilon^j f(x))$$

$$= x^{1+\alpha}\sum_{j=0}^{\infty}\frac{t^j q^{j(1+\alpha)}}{\{j\}_q!}\theta_{q,\alpha}^j g(x)\sum_{l=0}^{\infty}\frac{t^l}{\{l\}_q!}\theta_{q,\alpha}^l\epsilon^j f(x) = RHS$$

□

Definition 22. Compare [46, p. 87]. If $v(x)$ is the power series $\sum_{j=0}^{\infty}a_j x^j$, we define

$$v_q\left(\frac{x}{[1-t]}\right) \equiv \sum_{j=0}^{\infty}\frac{a_j x^j}{(t;q)_j}. \tag{119}$$

We obtain

Lemma 3.5.

$$E_q(t\theta_{q,\alpha})(v(x)) = \frac{1}{(tx;q)_{1+\alpha}}v_q\left(\frac{x}{[1-txq^{\alpha+1}]}\right). \tag{120}$$

Proof.

$$LHS = \sum_{k=0}^{\infty}\frac{t^k}{\{k\}_q!}\theta_{q,\alpha}^k\sum_{l=0}^{\infty}a_l x^l = \sum_{k=0}^{\infty}t^k\sum_{l=0}^{\infty}\frac{a_l x^{l+k}\langle 1+\alpha+l;q\rangle_k}{\langle 1;q\rangle_k} =$$

$$\sum_{l=0}^{\infty}a_l x^l\sum_{k=0}^{\infty}\frac{x^k t^k\langle 1+\alpha+l;q\rangle_k}{\langle 1;q\rangle_k} =$$

$$\sum_{l=0}^{\infty}a_l x^l\frac{1}{(tx;q)_{l+1+\alpha}} = \frac{1}{(tx;q)_{1-tx}}\sum_{l=0}^{\infty}a_l x^l\frac{1}{(txq;q)_{l+\alpha}} = RHS$$

□

We obtain a q-analogue of the corrected version of[3, 2.6].

$$E_q(t\theta_{q,\alpha})x^\beta = \frac{x^\beta}{(tx;q)_{\beta+1+\alpha}}. \tag{121}$$

Proof.

$$
\begin{aligned}
LHS &= E_q(t\theta_{q,\alpha})x^{\beta} = \sum_{k=0}^{\infty} \frac{t^k}{\{k\}_q!}\theta_{q,\alpha}^k x^{\beta} \\
&= \sum_{k=0}^{\infty} \frac{t^k}{\langle 1;q\rangle_k} x^{\beta+k}\langle \beta+1+\alpha;q\rangle_k = \frac{x^{\beta}}{\langle tx;q\rangle_{\beta+1+\alpha}} = RHS
\end{aligned} \tag{122}
$$

□

By the lemma we get

Theorem 3.6. *A q-analogue of* [3, 2.7]

$$
E_q(t\theta_{q,\alpha})x^{\beta}v(x) = \frac{x^{\beta}}{(tx;q)_{\beta+\alpha+1}} v_q\left(\frac{x}{[1-xtq^{\beta+\alpha+1}]}\right). \tag{123}
$$

Proof.

$$
\begin{aligned}
LHS &= E_q(t\theta_{q,\alpha})x^{\beta}v(x) \\
&= x^{1+\alpha}\sum_{j=0}^{\infty}\frac{(tq^{1+\alpha})^j}{\{j\}_q!}\theta_{q,\alpha}^j v \sum_{l=0}^{\infty}\frac{t^l}{\{l\}_q!}\theta_{q,\alpha}^l q^{j(\beta-\alpha-1)}x^{\beta-\alpha-1} \\
&= x^{1+\alpha}\sum_{j=0}^{\infty}\frac{(tq^{\beta})^j}{\{j\}_q!}\theta_{q,\alpha}^j(v)E_q(t\theta_{q,\alpha})x^{\beta-\alpha-1} \\
&= x^{1+\alpha}E_q(tq^{\beta}\theta_{q,\alpha})vE_q(t\theta_{q,\alpha})x^{\beta-\alpha-1} \\
&= x^{1+\alpha}\frac{1}{(txq^{\beta};q)_{1+\alpha}} v_q\left(\frac{x}{[1-txq^{\beta+\alpha+1}]}\right)\frac{x^{\beta-\alpha-1}}{(tx;q)_{\beta}} \\
&= \frac{x^{\beta}}{(tx;q)_{\beta+\alpha+1}} v_q\left(\frac{x}{[1-txq^{\beta+\alpha+1}]}\right) = RHS
\end{aligned} \tag{124}
$$

□

Theorem 3.7. *A q-analogue of* [3, 2.8]

(125)

$$
{}_p\phi_r((a);(b)|q,t\theta_{q,\alpha})x^{\gamma} = x^{\gamma}{}_{p+1}\phi_{r+1}((a),\gamma+\alpha+1;(b),\infty|q,\frac{tx}{1-q}),
$$

Proof.

(126)

$$
LHS = \sum_k \frac{\langle (a);q\rangle_k}{\langle 1,(b);q\rangle_k} q^{\binom{k}{2}(1+r-p)(-1)^{1+r-p}}(-1)^{k(1+r-p)}t^k x^{\gamma+k}\frac{\langle 1+\alpha+\gamma;q\rangle_k}{(1-q)^k}
$$

□

Corollary 3.8. [3, 2.9]

$$
\begin{aligned}
&{}_p\phi_r((a);(b)|q,t\theta_{q,\alpha})x^\gamma E_q(-x) = \\
&x^\gamma \sum_{l=0}^{\infty} \frac{(-x)^l}{\{l\}_q!} {}_{p+1}\phi_{r+1}((a),\gamma+\alpha+l+1;(b),\infty|q,\frac{tx}{1-q}).
\end{aligned} \tag{127}
$$

Proof.

(128)

$$
\begin{aligned}
&LHS = {}_p\phi_r \sum_n \frac{x^{\gamma+n}(-1)^n}{\{n\}_q!} \\
&= \sum_n \frac{x^{\gamma+n}(-1)^n}{\{n\}_q!} {}_{p+1}\phi_{r+1}((a),\gamma+n+\alpha+1;(b),\infty|q,\frac{tx}{1-q}) = RHS
\end{aligned}
$$

□

There is an inverse operator [3, 2.11]

$$
\theta_{q,\alpha}^{-n}(x^{-\beta}) = x^{-\beta-n}\frac{(-1)^n q^{n(\beta-\alpha)+\binom{n}{2}}}{\{-\alpha+\beta\}_{n,q}}. \tag{129}
$$

Proof.

$$
\begin{aligned}
&LHS = \theta_{q,\alpha}^{-n}(x^{-\beta}) = x^{-\beta-n}\frac{\langle 1+\alpha-\beta;q\rangle_{-n}}{(1-q)^{-n}} \\
&= x^{-\beta-n}(1-q)^n \frac{(-1)^n q^{n(\beta-\alpha)+\binom{n}{2}}}{\langle \beta-\alpha;q\rangle_n} = RHS.
\end{aligned} \tag{130}
$$

□

Theorem 3.9. *A q-analogue of the corrected version of* [3, 2.12]

$$
\begin{aligned}
&\left(\frac{\Phi_{q,\alpha}}{\theta_{q,\alpha}}\right)^k \left(\frac{y^\delta}{x^{\beta+1}}\right) \\
&= \frac{y^{\delta+k}}{x^{\beta+k+1}} \cdot \frac{\langle 1+\alpha+\delta;q\rangle_k}{\langle 1+\beta-\alpha;q\rangle_k}(-1)^k QE\left(k(1+\beta-\alpha)+\binom{n}{2}\right),
\end{aligned} \tag{131}
$$

where $\beta-\alpha \neq 0,-1,-2\ldots$.

Corollary 3.10. *A q-analogue of the corrected version of* [3, 2.13]

(132)

$$
\begin{aligned}
&{}_p\phi_r((a);(b)|q,t\frac{\Phi_{q,\alpha}}{\theta_{q,\alpha}})\frac{y^\delta}{x^{\beta+1}} \\
&= \frac{y^\delta}{x^{\beta+1}} {}_{p+1}\phi_{r+2}((a),\alpha+\delta+1;(b),-\alpha+\beta+1,\infty|q,\frac{ty}{x}q^{1+\beta-\alpha}).
\end{aligned}
$$

Proof.

$$
\begin{aligned}
&{}_p\phi_r((a);(b)|q,t\frac{\Phi_{q,\alpha}}{\theta_{q,\alpha}})\left(\frac{y^\delta}{x^{\beta+1}}\right)\\
&=\sum_{k=0}^{\infty}\frac{\langle(a);q\rangle_k}{\langle 1,(b);q\rangle_k}t^k\frac{y^\delta}{x^{\beta+1}}(-1)^{1-r-p}q^{\binom{k}{2}(1+r-p)}\left(\frac{y}{x}\right)^k\times\\
&\frac{\langle 1+\alpha+\delta;q\rangle_k}{\langle 1+\beta-\alpha;q\rangle_k}(-1)^kQE(k(1+\beta-\alpha)+\binom{k}{2})\\
&=\frac{y^\delta}{x^{\beta+1}}{}_{p+1}\phi_{r+2}((a),\alpha+\delta+1;(b),-\alpha+\beta+1,\infty|q,\frac{ty}{x}q^{1+\beta-\alpha})
\end{aligned}
\tag{133}
$$

□

Theorem 3.11. *A q-analogue of the corrected version of* [3, 2.14]

$$
{}_1\phi_0(c;-|q,t\frac{\Phi_{q,\alpha}}{\theta_{q,\alpha}})\frac{y^\delta}{x^{\beta+1}}=\frac{y^\delta}{x^{\beta+1}}{}_2\phi_2(c,\alpha+\delta+1;-\alpha+\beta+1,\infty|q,\frac{ty}{x}q^{1+\beta-\alpha}).
\tag{134}
$$

Theorem 3.12. *A q-analogue of the corrected version of* [3, 2.16]

$$
{}_2\phi_0(\infty,1-\alpha+\beta|q,t\frac{\phi_{q,\alpha}}{\theta_{q,\alpha}})\left(\frac{y^\delta}{x^{\beta+1}}\right)=\frac{y^\delta}{x^{\beta+1}}\cdot\frac{1}{(\frac{ty}{x}q^{1+\beta-\alpha};q)_{1+\alpha+\delta}}
\tag{135}
$$

By (135) we obtain

Corollary 3.13. *a q-analogue of* [3, 2.17]. *If $F(x)\in \mathrm{H}_q$,*

$$
{}_2\phi_0(\infty,1-\alpha+\beta|q,t\frac{\phi_{q,\alpha}}{\theta_{q,\alpha}})x^{-\beta-1}F(y)=\frac{x^{-\beta-1}}{(\frac{ty}{x}q^{1+\beta-\alpha})_{1+\alpha}}F_q\left(\frac{y}{[1-\frac{ty}{x}q^{2+\beta}]}\right)
\tag{136}
$$

Theorem 3.14. *Compare* [3, 2.18]

$$
E_q\left(-\frac{t}{\theta_{q,\alpha}}\right)x^{-\beta-1}=x^{-\beta-1}{}_1\phi_1(\infty;-\alpha+\beta+1|q,\frac{-(1-q)^2t}{x}q^{1+\beta-\alpha}).
\tag{137}
$$

Proof.

$$
LHS=\sum_{n=0}^{\infty}\frac{(-t)^n}{\{n\}_q!}x^{-\beta-1-n}\frac{(1-q)^n(-1)^nq^{n(\beta+1-\alpha)+\binom{n}{2}}}{\langle\beta-\alpha+1;q\rangle_n}=RHS
\tag{138}
$$

□

The q-Gould-Hopper [45, p. 77, 2.4], [3, 3.4] formula looks as follows:

$$\frac{1}{x^n}\theta_{q,\alpha}^n = \prod_{k=1}^{n}(q^{k+\alpha}\mathbf{x}D_q + \{\alpha+k\}_q I) = \sum_{k=0}^{n}\binom{n+\alpha}{n-k}_q \frac{\langle 1;q\rangle_n}{\langle 1;q\rangle_k} q^{k(k+\alpha)} x^k D_q^k (1-q)^{k-n}. \tag{139}$$

As was pointed out in [1, p. 4], the operator $\theta_{q,\alpha}$ is particularly useful in dealing with q-Laguerre polynomials.

Theorem 3.15. *The following equation is a q-analogue of the corr. version of* [3, 3.9, 3.12, 3.13].

$$\theta_{q,\alpha}^n E_q(-x) = x^n E_q(-x) L_{n,q,c}^{(\alpha)}(x). \tag{140}$$

Definition 23. The Rodriguez operator for q-Laguerre polynomials

$$\Omega_{n,q}^{(\alpha)} f(x) \equiv x^{-\alpha} E_{\frac{1}{q}}(x) D_q^n (x^{\alpha+n} E_q(-x) f(x)),\ f(x) \in \mathrm{H}_q. \tag{141}$$

We will now give a q-analogue of Carlitz [11, p. 219] operator expression for Laguerre polynomials and an extension of Khan's q-analogue of this paper [45, p. 79]. It turns out that we obtain an equivalence class of six objects for each element in Carlitz' paper. In the proof the product begins with $k = n$ and ends with $k = 1$. Let's pick out one of the six finite q-products in $\Omega_{n,q}^{(\alpha)}$. Then a calculations shows that the linear q-products commute, compare [11, p. 219].

Theorem 3.16.

(142)

$$\begin{aligned}
\Omega_{n,q}^{(\alpha)} &= \prod_{k=1}^{n}(q^{k+\alpha}(\mathbf{1}+(\mathbf{1}-\mathbf{q})\mathbf{x})\mathbf{x}D_q + \{\alpha+k\}_q I - q^{k+\alpha}\mathbf{x}) \\
&= \prod_{k=1}^{n}(q^{k+\alpha}\mathbf{x}D_q + \{\alpha+k\}_q I - q^{k+\alpha}\mathbf{x}\epsilon) \\
&= \prod_{k=1}^{n}(\mathbf{x}D_q + \{\alpha+k\}_q\epsilon - q^{k+\alpha}\mathbf{x}\epsilon) \\
&= \prod_{k=1}^{n}(q^{k+\alpha}(\mathbf{1}+(\mathbf{1}-\mathbf{q})\mathbf{x})\mathbf{x}D_q + \{\alpha+k\}_q(\mathbf{1}+(\mathbf{1}-\mathbf{q})\mathbf{x}) - \mathbf{x}) \\
&= \prod_{k=1}^{n}((\mathbf{1}+(\mathbf{1}-\mathbf{q})\mathbf{x})\mathbf{x}D_q + \{\alpha+k\}_q(\mathbf{1}+(\mathbf{1}-\mathbf{q})\mathbf{x}) - \mathbf{x}) \\
&= \prod_{k=1}^{n}(\mathbf{x}D_q + \{\alpha+k\}_q(\mathbf{1}+(\mathbf{1}-\mathbf{q})\mathbf{x}) - \mathbf{x}\epsilon).
\end{aligned}$$

Proof. We only prove the first identity. The five others are proved in a similar way by permutation of the three functions involved in the q-differentiation. We will use [12, (13), p. 91] in the computations.

(143)

$$\begin{aligned}
&\Omega_{n+1,q}^{(\alpha)} f(x) = x^{-\alpha}E_{\frac{1}{q}}(x)D_q^n\left[(1+(1-q)x)q^{n+1+\alpha}E_q(-x)x^{n+1+\alpha}D_q + \right. \\
&\left. +\{\alpha+n+1\}_q x^{n+\alpha}E_q(-x) - (xq)^{n+1+\alpha}E_q(-x)\right] f(x) \\
&= \Omega_{n,q}^{(\alpha)}\left[\{\alpha+n+1\}_q - xq^{n+1+\alpha} + (1+(1-q)x)q^{n+1+\alpha}xD_q\right] f(x).
\end{aligned}$$

□

Remark 9. This was the first occasion where multiple q-analogues occurred because of the q-Leibniz theorem. We had three functions and got $\binom{3}{2}$ q-analogues.

Theorem 3.17. *A first q-analogue of* [11, (4), p. 219].

$$\Omega_{n,q}^{(\alpha)} f(x) = \{n\}_q! \sum_{k=0}^{n} \frac{x^k}{\{k\}_q!} L_{n-k,q}^{(\alpha+k)}(x)\epsilon^{n-k}D_q^k f(x). \tag{144}$$

Proof.

(145)

$$\Omega_{n,q}^{(\alpha)} f(x) = x^{-\alpha} E_{\frac{1}{q}}(x) \sum_{k=0}^{n} \binom{n}{k}_q D_q^{n-k}(x^{\alpha+n} E_q(-x)) \epsilon^{n-k} D_q^k f(x) =$$

$$x^{-\alpha} E_{\frac{1}{q}}(x) \sum_{k=0}^{n} \binom{n}{k}_q x^{\alpha+k} E_q(-x) L_{n-k,q,c}^{(\alpha+k)}(x) \epsilon^{n-k} D_q^k f(x) = RHS.$$

□

The following special case of (144) is a q-analogue of [11, (6) p. 220], see also [45, p. 79].

$$L_{n,q,c}^{(\alpha)}(x) = \Omega_{n,q}^{(\alpha)} 1. \tag{146}$$

The following formula is the first q-analogue of [11, (7), p. 221], the proof is the same.

(147)

$$\binom{m+n}{m}_q L_{m+n,q}^{(\alpha)}(x) = \sum_{k=0}^{min(m,n)} \frac{(-x)^k}{\{k\}_q!} L_{m-k,q}^{(\alpha+n+k)}(x) q^{\alpha k + \binom{k+1}{2} + \binom{k}{2}} L_{n-k,q}^{(\alpha+k)}(xq^m).$$

Theorem 3.18. *A second q-analogue of* [11, (4), p. 219].

(148)

$$\Omega_{n,q}^{(\alpha)} f(x) = \{n\}_q! \sum_{k=0}^{n} \frac{x^k}{\{k\}_q!} q^{k(\alpha+k)} P_{k,q}(1, -(1-q)x) L_{n-k,q}^{(\alpha+k)}(xq^k) D_q^k f(x).$$

Proof. We will use [12, (13), p. 91] in the computations.

(149)

$$\Omega_{n,q}^{(\alpha)} f(x) = x^{-\alpha} E_{\frac{1}{q}}(x) \sum_{k=0}^{n} \binom{n}{k}_q \epsilon^k D_q^{n-k}(x^{\alpha+n} E_q(-x)) D_q^k f(x) =$$

$$= x^{-\alpha} E_{\frac{1}{q}}(x) \sum_{k=0}^{n} \binom{n}{k}_q \epsilon^k \left[x^{\alpha+k} E_q(-x) L_{n-k,q,c}^{(\alpha+k)}(x) \right] D_q^k f(x) =$$

$$x^{-\alpha} E_{\frac{1}{q}}(x) \sum_{k=0}^{n} \binom{n}{k}_q (xq^k)^{\alpha+k} E_q(-xq^k) L_{n-k,q,c}^{(\alpha+k)}(xq^k) D_q^k f(x) = RHS.$$

□

The following formula is the second q-analogue of [11, (7), p. 221].

$$\begin{pmatrix} m+n \\ m \end{pmatrix}_q L^{(\alpha)}_{m+n,q}(x) = \sum_{k=0}^{min(m,n)} \frac{(-x)^k}{\{k\}_q!} P_{k,q}(1, -(1-q)x) \times \tag{150}$$
$$L^{(\alpha+n+k)}_{m-k,q}(xq^k) q^{k(2\alpha+n+k)+\binom{k+1}{2}+\binom{k}{2}} L^{(\alpha+k)}_{n-k,q}(xq^k).$$

An interesting consequence of (147) is the following q-analogue of [11, (10), p. 222].

Theorem 3.19.

(151)

$$\sum_{n=0}^{\infty} \begin{pmatrix} m+n \\ m \end{pmatrix}_q L^{(\alpha-n)}_{m+n,q}(x) t^n q^{\binom{n}{2}-n\alpha} = L^{(\alpha)}_{m,q}(x \oplus_{q,-1} xtq^{-\alpha}) \frac{E_{\underline{1}}(-xtq^m)}{(-t;q)_{-\alpha}}.$$

Proof.

$$LHS = \sum_{n=0}^{\infty} \sum_{k=0}^{min(m,n)} \frac{(-x)^k}{\{k\}_q!} L^{(\alpha+k)}_{m-k,q}(x)$$

$$\text{QE}\left((\alpha-n)k + \binom{k+1}{2} + \binom{k}{2}\right) L^{(\alpha-n+k)}_{n-k,q}(xq^m) t^n q^{\binom{n}{2}-n\alpha} = \tag{152}$$
$$\sum_{n=0}^{\infty} \sum_{k=0}^{m} \frac{(-x)^k}{\{k\}_q!} L^{(\alpha+k)}_{m-k,q}(x) q^{\frac{k^2-k}{2}} L^{(\alpha-n)}_{n,q}(xq^m) t^{n+k} q^{\binom{n}{2}-n\alpha} = RHS,$$

where we have used [19, 5.29 p. 28], (91) and (101) in the last step. □

4. q-Jacobi polynomials

We now come to the definition of q-Jacobi polynomials. In the literature there is a very similar so-called little q-Jacobi polynomial [5]. We will however use the original definition, because it leads to a nice Rodriguez formula with corresponding orthogonality. For the orthogonality, see [21] and Hahn [33].

Remark 10. The reason Jacobi's original notation for Jacobi polynomials was changed was to separate the parameters so that the weight function has two functions one of which depends on one of parameters and the other on the second parameter.

In the limit $q \to 1$ we get the original Jacobi polynomials [43, p. 192], [7], [6, p. 162],[44, p. 467], [23, p. 242 (1)].

Definition 24.

$$P_{n,q}^{(\alpha,\beta)}(x) \equiv \frac{\langle 1+\alpha;q\rangle_n}{\langle 1;q\rangle_n} {}_2\phi_1(-n,\beta+n;1+\alpha|q,xq^{\alpha+1-\beta}) \equiv \frac{\langle 1+\alpha;q\rangle_n}{\langle 1;q\rangle_n}\sum_{k=0}^{n}\binom{n}{k}_q \frac{\langle \beta+n;q\rangle_k}{\langle 1+\alpha;q\rangle_k}(-x)^k q^{\binom{k}{2}+(\alpha+1-\beta-n)k}. \tag{153}$$

Theorem 4.1.

$$\lim_{\beta\to-\infty} P_{n,q}^{(\alpha,\beta)}(-x(1-q)) = L_{n,q}^{(\alpha)}(x). \tag{154}$$

Remark 11. The following special case applies:

$$P_{n,q}^{(\alpha,1+\alpha-n)}(x) = \frac{\langle 1+\alpha;q\rangle_n}{\langle 1;q\rangle_n}(x;q)_n. \tag{155}$$

The following formula is a q-analogue of [43, p. 192 (7)], [23, p.242].

Theorem 4.2. *Let $x \in (0, |q^{\beta-\alpha-1}|)$. Then*

$$P_{n,q}^{(\alpha,\beta)}(x) = \frac{x^{-\alpha}}{\{n\}_q!(xq^{\alpha+1-\beta};q)_{\beta-\alpha-1}} D_q^n\left(\frac{x^{\alpha+n}}{(x;q)_{\alpha+1-\beta-n}}\right). \tag{156}$$

Proof. The q-Leibniz formula gives

$$RHS = \frac{x^{-\alpha}}{\{n\}_q!(xq^{-\beta+\alpha+1};q)_{\beta-\alpha-1}}\times$$

$$\sum_{k=0}^{n}\frac{\langle 1;q\rangle_n\{1+\alpha-\beta-n\}_{k,q}\{1+\alpha+k\}_{n-k,q}x^{\alpha+k}q^{k\alpha+k^2}}{\langle 1;q\rangle_k\langle 1;q\rangle_{n-k}(x;q)_{-\beta+\alpha+k+1-n}} =$$

$$\sum_{k=0}^{n}\frac{x^k\langle 1+\alpha-\beta-n;q\rangle_k\langle 1+\alpha;q\rangle_n q^{k\alpha+k^2}}{\langle 1+\alpha;q\rangle_k(xq^{-\beta+\alpha+1};q)_{k-n}\langle 1;q\rangle_k\langle 1;q\rangle_{n-k}} =$$

$$\sum_{k=0}^{n}\frac{x^k\langle 1+\alpha-\beta-n,-n;q\rangle_k\langle 1+\alpha;q\rangle_n q^{k\alpha+k^2-\binom{k}{2}+nk}(-1)^k}{\langle 1,1+\alpha;q\rangle_k(xq^{-\beta+\alpha+1};q)_{k-n}\langle 1;q\rangle_n} =$$

$$\frac{\langle 1+\alpha;q\rangle_n}{\langle 1;q\rangle_n(xq^{-\beta+\alpha+1};q)_{-n}}\times$$

$${}_2\phi_2\left(-n,-n-\beta+\alpha+1;\alpha+1|q,xq^{n+\alpha+1}||-;xq^{-n-\beta+\alpha+1}\right) = LHS.$$

The interval for x is chosen to make certain infinite products converge, compare [55, p. 300]. □

Corollary 4.3.

$$P_{n,q}^{(\alpha,\beta)}(xq^{\gamma}) = \frac{x^{-\alpha}}{\{n\}_q!(xq^{\alpha+\gamma+1-\beta};q)_{\beta-\alpha-1}} D_q^n\left(\frac{x^{\alpha+n}}{(xq^{\gamma};q)_{\alpha+1-\beta-n}}\right). \tag{157}$$

Proof. Same as above. □

Corollary 4.4. *A function* $F(x) \in \mathrm{H}_q$

$$F(x) = \sum_{k=0}^{\infty} \frac{\alpha_k x^{\beta_k}}{(x;q)_{\gamma_k}}$$

has n:th q-difference given by

$$D_q^n F(x) = \sum_{k=0}^{\infty} \alpha_k P_{n,q}^{(\beta_k - n, \beta_k + 1 - 2n - \gamma_k)}(x) \{n\}_q! \frac{x^{\beta_k - n}}{(x;q)_{\gamma_k + n}}, \tag{158}$$

$x \in (0, |q^{-n-\gamma_k}|),\ \forall k.$

Definition 25. The Rodriguez operator for q-Jacobi polynomials

$$\Omega_{n,q}^{(\alpha,\beta)} f(x) \equiv \frac{x^{-\alpha}}{\{n\}_q! (xq^{\alpha+1-\beta};q)_{\beta-\alpha-1}} D_q^n \left(\frac{x^{\alpha+n}}{(x;q)_{\alpha+1-\beta-n}} f(x) \right), \tag{159}$$

$f(x) \in \mathrm{H}_q, x \in (0, |q^{\beta-\alpha-1}|).$

$$\Omega_{0,q}^{(\alpha,\beta)} = I. \tag{160}$$

Theorem 4.5.

(161)

$$\Omega_{n,q}^{(\alpha,\beta-n+1)} f(x) = \prod_{k=2}^{n} \left(\frac{1 - xq^{\alpha-\beta+1}}{\{k\}_q} \right) \Omega_{1,q}^{(\alpha,\beta)} \prod_{k=2}^{n} \Theta_{k,q}^{(\alpha,\beta)} f(x),\ n \geq 1,$$

where $\Theta_{k,q}^{(\alpha,\beta)}$ *is given by one of the following six equivalent expressions.* $\Theta_{k,q}^{(\alpha,\beta)}$ *is a bilinear function of* D_q *and* ϵ *with coefficients in the field of fractions of* $\mathbb{C}[\mathbf{x}]$.

(162)

$$\begin{aligned}
\Theta_{k,q}^{(\alpha,\beta)} &= \frac{q^{k+\alpha}(\mathbf{1}-\mathbf{x})}{\mathbf{1}-\mathbf{xq^{2-k+\alpha-\beta}}} \mathbf{x} D_q + \{\alpha+k\}_q I + \frac{q^{k+\alpha}\{2-k+\alpha-\beta\}_q}{\mathbf{1}-\mathbf{xq^{2-k+\alpha-\beta}}} \mathbf{x} \\
&= \{\alpha+k\}_q I + \frac{q^{k+\alpha}\{2-k+\alpha-\beta\}_q}{\mathbf{1}-\mathbf{xq^{2-k+\alpha-\beta}}} \mathbf{x}\epsilon + q^{k+\alpha} \mathbf{x} D_q \\
&= \mathbf{x} D_q + \{\alpha+k\}_q \epsilon + \frac{q^{k+\alpha}\{2-k+\alpha-\beta\}_q}{\mathbf{1}-\mathbf{xq^{2-k+\alpha-\beta}}} \mathbf{x}\epsilon \\
&= \frac{\{2-k+\alpha-\beta\}_q}{\mathbf{1}-\mathbf{xq^{2-k+\alpha-\beta}}} \mathbf{x} + \frac{q^{k+\alpha}(\mathbf{1}-\mathbf{x})}{\mathbf{1}-\mathbf{xq^{2-k+\alpha-\beta}}} \mathbf{x} D_q + \frac{\{\alpha+k\}_q(\mathbf{1}-\mathbf{x})}{\mathbf{1}-\mathbf{xq^{2-k+\alpha-\beta}}} \\
&= \frac{\{2-k+\alpha-\beta\}_q}{\mathbf{1}-\mathbf{xq^{2-k+\alpha-\beta}}} \mathbf{x} + \frac{(\mathbf{1}-\mathbf{x})}{\mathbf{1}-\mathbf{xq^{2-k+\alpha-\beta}}} \mathbf{x} D_q + \frac{\{\alpha+k\}_q(\mathbf{1}-\mathbf{x})}{\mathbf{1}-\mathbf{xq^{2-k+\alpha-\beta}}} \epsilon \\
&= \mathbf{x} D_q + \frac{\{\alpha+k\}_q(\mathbf{1}-\mathbf{x})}{\mathbf{1}-\mathbf{xq^{2-k+\alpha-\beta}}} \epsilon + \frac{\{1+\alpha-\beta\}_q}{\mathbf{1}-\mathbf{xq^{1+\alpha-\beta}}} \mathbf{x}\epsilon.
\end{aligned}$$

Proof. We only prove the first identity for $\Theta_{k,q}^{(\alpha,\beta)}$. The five others are proved in a similar way by permutation of the three functions involved in the q-differentiation.

(163)

$$\Omega_{n+1,q}^{(\alpha,\beta-1)} f(x) = \frac{x^{-\alpha}}{\{n+1\}_q! (xq^{\alpha+2-\beta};q)_{\beta-\alpha-2}} D_q^n$$
$$\left[\left[\frac{\{\alpha+n+1\}_q x^{\alpha+n}}{(x;q)_{1+\alpha-\beta-n}} + \frac{(xq)^{\alpha+n+1} D_q}{(xq;q)_{1+\alpha-\beta-n}} + \frac{(xq)^{\alpha+n+1}\{\alpha+1-\beta-n\}_q}{(x;q)_{2+\alpha-\beta-n}}\right] f(x)\right]$$
$$= \frac{1-xq^{\alpha+1-\beta}}{\{n+1\}_q} \Omega_{n,q}^{(\alpha,\beta)}$$
$$\left[\left[\frac{(1-x)q^{n+1+\alpha}}{1-xq^{1+\alpha-\beta-n}} x D_q + \{\alpha+n+1\}_q + \frac{xq^{n+1+\alpha}\{1+\alpha-\beta-n\}_q}{1-xq^{1+\alpha-\beta-n}}\right] f(x)\right].$$

The assertion now follows by induction. □

The following generalization of (144) is a first q-analogue of [57, 2.3 p. 239], with the difference that in the present paper Jacobi's original polynomial definition is used.

Theorem 4.6.

(164)

$$\Omega_{n,q}^{(\alpha,\beta)} f(x) = \sum_{k=0}^{n} \frac{x^k}{\{k\}_q!} (xq^{\alpha+1-k-\beta};q)_k P_{n-k,q}^{(\alpha+k,\beta+2k)}(x) \epsilon^{n-k} D_q^k f(x).$$

Proof.

$$\Omega_{n,q}^{(\alpha,\beta)} f(x) = \frac{x^{-\alpha}}{\{n\}_q! (xq^{\alpha+1-\beta};q)_{\beta-\alpha-1}}$$
$$\sum_{k=0}^{n} \binom{n}{k}_q D_q^{n-k} \left[\frac{x^{\alpha+n}}{(x;q)_{-\beta+\alpha+1-n}}\right] \epsilon^{n-k} D_q^k f(x) = \tag{165}$$
$$\frac{1}{\{n\}_q! (xq^{\alpha+1-\beta};q)_{\beta-\alpha-1}} \sum_{k=0}^{n} \binom{n}{k}_q x^k P_{n-k,q}^{(\alpha+k,\beta+2k)}(x)$$
$$\{n-k\}_q! (xq^{\alpha-k+1-\beta};q)_{\beta-\alpha+k-1} \epsilon^{n-k} D_q^k f(x) = RHS.$$

□

Lemma 4.7.

$$\Omega_{n,q}^{(\alpha,\beta)} \frac{x}{1-xq^{1+\alpha-\beta-n}} = P_{n,q}^{(\alpha,\beta)}(x) \frac{xq^n}{1-xq^{1+\alpha-\beta}} + \tag{166}$$
$$\sum_{k=1}^{n} x^k P_{n-k,q}^{(\alpha+k,\beta+2k)}(x) \frac{q^{(\alpha+1-\beta-n)(k-1)}}{1-xq^{1+\alpha-\beta}}.$$

Proof. Use (164) and (102). □

Theorem 4.8.

(167)

$$P_{n+1,q}^{(\alpha,\beta-1)}(x) = \frac{1-xq^{\alpha+1-\beta}}{\{n+1\}_q}\{\alpha+n+1\}_q P_{n,q}^{(\alpha,\beta)}(x)+$$
$$+\frac{q^{\alpha+n+1}\{\alpha+1-\beta-n\}_q}{\{n+1\}_q}\left[\sum_{k=1}^{n} x^k P_{n-k,q}^{(\alpha+k,\beta+2k)}(x)q^{(\alpha+1-\beta-n)(k-1)}+\right.$$
$$\left.+P_{n,q}^{(\alpha,\beta)}(x)xq^n\right].$$

Proof. Apply (163) to 1 and use (166). □

Corollary 4.9.

(168)

$$L_{n+1,q}^{(\alpha)} =$$
$$\frac{1}{\{n+1\}_q}\times[\{\alpha+n+1\}_q L_{n,q}^{(\alpha)}(x) - q^{n+\alpha+1}[xL_{n-1,q}^{(\alpha+1)}(x)+q^n xL_{n,q}^{(\alpha)}(x)]].$$

Proof.

(169)

$$LHS = \lim_{\beta\to-\infty}\frac{1+x(1-q)q^{\alpha+1-\beta}}{\{n+1\}_q}\left[\{\alpha+n+1\}_q P_{n,q}^{(\alpha,\beta)}(-x(1-q))+\right.$$
$$+q^{\alpha+n+1}\{\alpha+1-\beta-n\}_q\left[\sum_{k=1}^{n}(-x(1-q))^k P_{n-k,q}^{(\alpha+k,\beta+2k)}(-x(1-q))\times\right.$$
$$\left.\left.\frac{q^{(\alpha+1-\beta-n)(k-1)}}{1+x(1-q)q^{1+\alpha-\beta}}+P_{n,q}^{(\alpha,\beta)}(-x(1-q))\frac{-x(1-q)q^n}{1+x(1-q)q^{1+\alpha-\beta}}\right]\right] = RHS.$$

□

The following generalization of (148) is the second q-analogue of [57, 2.3 p. 239].

Theorem 4.10.

$$\Omega_{n,q}^{(\alpha,\beta)}f(x) = \sum_{k=0}^{n}\frac{x^k}{\{k\}_q!}q^{k(\alpha+k)}(x;q)_k P_{n-k,q}^{(\alpha+k,\beta+2k)}(xq^k)D_q^k f(x). \tag{170}$$

Proof.

$$\Omega_{n,q}^{(\alpha,\beta)}f(x) = \frac{x^{-\alpha}}{\{n\}_q!(xq^{\alpha+1-\beta};q)_{\beta-\alpha-1}}$$
$$\sum_{k=0}^{n}\binom{n}{k}_q\epsilon^k\left[D_q^{n-k}\left[\frac{x^{\alpha+n}}{(x;q)_{-\beta+\alpha+1-n}}\right]\right]D_q^k f(x) = RHS. \tag{171}$$

□

Lemma 4.11.

$$\Omega_{n,q}^{(\alpha,\beta)}\frac{x}{1-xq^{1+\alpha-\beta-n}} = P_{n,q}^{(\alpha,\beta)}(x)\frac{x}{1-xq^{1+\alpha-\beta-n}}+ \sum_{k=1}^{n} x^k q^{k(\alpha+k)} P_{n-k,q}^{(\alpha+k,\beta+2k)}(xq^k)\frac{q^{(\alpha+1-\beta-n)(k-1)}(x;q)_k}{(xq^{\alpha+1-\beta-n});q)_{k+1}}. \tag{172}$$

Proof. Use (170) and (102). □

Theorem 4.12.

$$P_{n+1,q}^{(\alpha,\beta-1)}(x) = \frac{1-xq^{\alpha+1-\beta}}{\{n+1\}_q}\left(\{\alpha+n+1\}_q P_{n,q}^{(\alpha,\beta)}(x)+ +q^{\alpha+n+1}\{\alpha+1-\beta-n\}_q\left(P_{n,q}^{(\alpha,\beta)}(x)\frac{x}{1-xq^{\alpha+1-\beta-n}}+ +\sum_{k=1}^{n} x^k q^{k(\alpha+k)}(x;q)_k P_{n-k,q}^{(\alpha+k,\beta+2k)}(xq^k)\frac{q^{(\alpha+1-\beta-n)(k-1)}}{(xq^{\alpha+1-\beta-n};q)_{k+1}}\times (x;q)_k\right)\right). \tag{173}$$

Proof. Apply (163) to 1 and use (172). □

Theorem 4.13.

$$P_{n,q}^{(\alpha,\beta)}(x) = \frac{x^{-\alpha-n-1}}{\{n\}_q!(xq^{\alpha+1-\beta};q)_{\beta-\alpha-1}}(x^2D_q)^n\left(\frac{x^{\alpha+1}}{(x;q)_{\alpha+1-\beta-n}}\right). \tag{174}$$

Proof. This follows from a q-analogue of [62, p. 220]. □

The limit to q-Laguerre polynomials for the above equation leads to [19, 6.11 p. 31].

The second of the following equations shows that the operator $D_q\epsilon^{-1}$ keeps the same function argument, while D_q doesn't. This will be important in future applications.

(175)
$$D_q^m P_{n,q}^{(\alpha,\beta)}(x) = P_{n-m,q}^{(\alpha+m,\beta+m)}(xq^m)\frac{(-1)^m\langle\beta+n;q\rangle_m}{(1-q)^m}\mathrm{QE}\left(\binom{m}{2}+m(\alpha+1-\beta-n)\right).$$

(176)
$$(D_q\epsilon^{-1})^m P_{n,q}^{(\alpha,\beta)}(x) = P_{n-m,q}^{(\alpha+m,\beta+m)}(x)\frac{(-1)^m\langle\beta+n;q\rangle_m}{(1-q)^m}\mathrm{QE}(m(\alpha-\beta-n)).$$

By the Ward q-Taylor formula (73), the first Jackson q-Taylor formula (74), and by (91) we obtain the following three q-analogues of [49, (8), p. 460].

Theorem 4.14.

$$P_{n,q}^{(\alpha,\beta)}(x \oplus_q y) = \sum_{k=0}^{n} \frac{y^k}{\{k\}_q!} P_{n-k,q}^{(\alpha+k,\beta+k)}(xq^k) \frac{(-1)^k \langle \beta+n;q\rangle_k}{(1-q)^k} \tag{177}$$
$$\mathrm{QE}\left(\binom{k}{2} + k(\alpha+1-\beta-n)\right).$$

$$P_{n,q}^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \frac{[x-y]_q^k}{\{k\}_q!} P_{n-k,q}^{(\alpha+k,\beta+k)}(yq^k) \frac{(-1)^k \langle \beta+n;q\rangle_k}{(1-q)^k} \times \tag{178}$$
$$\mathrm{QE}\left(\binom{k}{2} + k(\alpha+1-\beta-n)\right).$$

$$P_{n,q}^{(\alpha,\beta)}(x \oplus_{q,-1} y) = \sum_{k=0}^{n} \frac{(-y)^k}{\{k\}_q!} P_{n-k,q}^{(\alpha+k,\beta+k)}(x) \frac{\langle \beta+n;q\rangle_k}{(1-q)^k} \tag{179}$$
$$q^{k(\alpha-\beta-n)}.$$

The following two formulas are q-analogues of [49, (6), p. 459]

Theorem 4.15.

(180)
$$\binom{m+n}{m}_q (xq^{\alpha+1+n-\beta};q)_{\beta-n-\alpha-1} P_{m+n,q}^{(\alpha,\beta-n)}(x) = \sum_{k=0}^{min(m,n)} \frac{P_{n-k,q}^{(\alpha+k,\beta+2k-n)}(x)x^k}{\{k\}_q!(x;q)_{\alpha+1+n-\beta-k}} \times$$
$$P_{m-k,q}^{(\alpha+n+k,\beta+n+k)}(xq^n)(-1)^k \frac{\langle \beta+n+m;q\rangle_k}{(1-q)^k} \mathrm{QE}\left(\binom{k}{2} + k(\alpha+1-\beta-m\right).$$

Proof.

(181)

$$P^{(\alpha,\beta-n)}_{m+n,q}(x) = \frac{x^{-\alpha}}{\{m+n\}_q!(xq^{\alpha+1-\beta+n};q)_{\beta-\alpha-1-n}} D_q^{m+n}\left(\frac{x^{\alpha+m+n}}{(x;q)_{\alpha+1-\beta-m}}\right) =$$

$$\frac{x^{-\alpha}\{m\}_q!}{\{m+n\}_q!(xq^{\alpha+1-\beta+n};q)_{\beta-\alpha-1-n}} D_q^n[x^{\alpha+n}P^{(\alpha+n,\beta+n)}_{m,q}(x)(xq^{\alpha+1-\beta};q)_{\beta-\alpha-1}] =$$

$$\frac{x^{-\alpha}\{m\}_q!}{\{m+n\}_q!(xq^{\alpha+1-\beta+n};q)_{\beta-\alpha-1-n}} \sum_{k=0}^{n} \binom{n}{k}_q \times$$

$$\frac{P^{(\alpha+k,\beta+2k-n)}_{n-k,q}(x)x^{\alpha+k}\{n-k\}_q!}{(x;q)_{\alpha+1-\beta-k+n}} \epsilon^{n-k} D_q^k P^{(\alpha+n,\beta+n)}_{m,q}(x) \overset{\text{by}(175)}{=}$$

$$\frac{x^{-\alpha}\{m\}_q!\{n\}_q!}{\{m+n\}_q!(xq^{\alpha+1-\beta+n};q)_{\beta-\alpha-1-n}} \sum_{k=0}^{min(m,n)} \frac{P^{(\alpha+k,\beta+2k-n)}_{n-k,q}(x)x^{\alpha+k}}{\{k\}_q!(x;q)_{\alpha+1-\beta-k+n}} \times$$

$$P^{(\alpha+n+k,\beta+n+k)}_{m-k,q}(xq^n)(-1)^k \frac{\langle\beta+n+m;q\rangle_k}{(1-q)^k} \mathrm{QE}\left(\binom{k}{2} + k(\alpha+1-\beta-m)\right).$$

□

Theorem 4.16.

$$\binom{m+n}{m}_q (xq^{\alpha+1+n-\beta};q)_{\beta-n-\alpha-1} P^{(\alpha,\beta-n)}_{m+n,q}(x) =$$

$$\sum_{k=0}^{min(m,n)} \frac{P^{(\alpha+k,\beta+2k-n)}_{n-k,q}(xq^k)x^k}{\{k\}_q!(xq^k;q)_{\alpha+1+n-\beta-k}} P^{(\alpha+n+k,\beta+n+k)}_{m-k,q}(xq^k)(-1)^k \times \tag{182}$$

$$\frac{\langle\beta+n+m;q\rangle_k}{(1-q)^k} \mathrm{QE}\left(\binom{k}{2} + k(\alpha+1-\beta-m) + k + \alpha\right).$$

In the limit we obtain the following two additional q-analogues of [11, (7), p. 221].

(183)

$$\binom{m+n}{m}_q L^{(\alpha)}_{m+n,q}(x) =$$

$$\sum_{k=0}^{min(m,n)} \frac{(-x)^k}{\{k\}_q!} L^{(\alpha+n+k)}_{m-k,q}(xq^n)\mathrm{QE}(k(n+\alpha+1) + 2\binom{k}{2})L^{(\alpha+k)}_{n-k,q}(x).$$

(184)
$$\binom{m+n}{m}_q L^{(\alpha)}_{m+n,q}(x) = \sum_{k=0}^{min(m,n)} \frac{(-x)^k}{\{k\}_q!} L^{(\alpha+n+k)}_{m-k,q}(xq^k)\times$$
$$P_{k,q}(1,-(1-q)x)\mathrm{QE}\left(k(n+\alpha+1)+2\binom{k}{2}+k+\alpha\right) L^{(\alpha+k)}_{n-k,q}(xq^k).$$

The following two formulas are q-analogues of [49, (9), p. 460].

Theorem 4.17.

$$P^{(\alpha+\gamma,\beta+\delta)}_{n,q}(x) = \sum_{k=0}^{n} q^{(n-k)(\gamma-k)}\times \tag{185}$$
$$P^{(\alpha+k,\beta+k)}_{n-k,q}(x) P^{(\gamma-k,1+\delta-k)}_{k,q}(xq^{\alpha+1-\beta}).$$

Proof.

(186)
$$LHS = \frac{x^{-\alpha-\gamma}}{\{n\}_q!(xq^{\alpha+1+\gamma-\beta-\delta};q)_{\beta+\delta-\alpha-\gamma-1}} D^n_q\left(\frac{x^{\alpha+\gamma+n}}{(x;q)_{\alpha+\gamma+1-\beta-\delta-n}}\right) =$$
$$\frac{x^{-\alpha-\gamma}}{\{n\}_q!(xq^{\alpha+1+\gamma-\beta-\delta};q)_{\beta+\delta-\alpha-\gamma-1}} \sum_{k=0}^{n}\binom{n}{k}_q D^{n-k}_q\left(\frac{x^{\alpha+n}}{(x;q)_{\alpha+1-\beta-n+k}}\right)\times$$
$$\epsilon^{n-k} D^k_q\left(\frac{x^\gamma}{(xq^{\alpha+1-\beta-n+k};q)_{\gamma-\delta-k}}\right) \overset{\text{by (157)}}{=} \frac{x^{-\alpha-\gamma}}{(xq^{\alpha+1+\gamma-\beta-\delta};q)_{\beta+\delta-\alpha-\gamma-1}}\times$$
$$\sum_{k=0}^{n} x^{\alpha+k}(xq^{\alpha+1-\beta};q)_{\beta-\alpha-1} P^{(\alpha+k,\beta+k)}_{n-k,q}(x)\times$$
$$\epsilon^{n-k}[x^{\gamma-k}(xq^{\gamma+\alpha-\beta-\delta+1-n+k};q)_{\delta-\gamma} P^{(\gamma-k,1+\delta-k)}_{k,q}(xq^{\alpha+1-\beta-n+k}) = RHS.$$

□

Theorem 4.18.

$$P^{(\alpha+\gamma,\beta+\delta)}_{n,q}(x) = \sum_{k=0}^{n} q^{k(\alpha+k)}\times \tag{187}$$
$$P^{(\alpha+k,\beta+k)}_{n-k,q}(xq^k) P^{(\gamma-k,1+\delta-k)}_{k,q}(xq^{\alpha+1-\beta-n+k})\times$$
$$\frac{(x;q)_k}{(xq^{\alpha+1-\beta+\gamma-\delta};q)_{-n+k}(xq^{\alpha+1-\beta-n+k};q)_n}.$$

Proof. A slight modification of the previous proof.

(188)

$$LHS = \frac{x^{-\alpha-\gamma}}{\{n\}_q!(xq^{\alpha+1+\gamma-\beta-\delta};q)_{\beta+\delta-\alpha-\gamma-1}} \sum_{k=0}^{n} \binom{n}{k}_q \epsilon^k D_q^{n-k} \left(\frac{x^{\alpha+n}}{(x;q)_{\alpha+1-\beta-n+k}} \right) \times$$

$$D_q^k \left(\frac{x^{\gamma}}{(xq^{\alpha+1-\beta-n+k};q)_{\gamma-\delta-k}} \right) \overset{\text{by}(157)}{=} \frac{x^{-\alpha-\gamma}}{(xq^{\alpha+1+\gamma-\beta-\delta};q)_{\beta+\delta-\alpha-\gamma-1}} \times$$

$$\sum_{k=0}^{n} \epsilon^k [x^{\alpha+k}(xq^{\alpha+1-\beta};q)_{\beta-\alpha-1} P_{n-k,q}^{(\alpha+k,\beta+k)}(x)] \times$$

$$x^{\gamma-k}(xq^{\gamma+\alpha-\beta-\delta+1-n+k};q)_{\delta-\gamma} P_{k,q}^{(\gamma-k,1+\delta-k)}(xq^{\alpha+1-\beta-n+k}) = RHS.$$

□

5. Generating functions

The connection between generating functions and recurrences is wellknown. We will give an example of how the operator technique in chapter 3 can be used to obtain a new generating function for the q-Laguerre polynomials, compare [3, p. 132].

Theorem 5.1.

$$\sum_{n=0}^{\infty} L_{n,q}^{(\alpha)}(x)) E_q(-x)(xt)^n = \frac{1}{(tx;q)_{\alpha+1}} E_q\left(\frac{-x}{[1-xtq^{\alpha+1}]} \right). \tag{189}$$

Proof. Operate with $E_q(t\theta_{q,\alpha})$ on $E_q(-x)$ in two different ways and use (123) and (140). □

A wellknown generating function for q-Laguerre polynomials is given by the following q-analogue of [24, p. 120 11′], [3, p. 132 4.2]. Compare Moak [50, p. 29 4.17], [19, p.511–512].

Theorem 5.2.

$$\sum_{n=0}^{\infty} L_{n,q}^{(\alpha)}(x) t^n = \sum_{n=0}^{\infty} \frac{q^{n^2+\alpha n}(-xt)^n}{\{n\}_q!(t;q)_{1+\alpha+n}} \equiv F(x,t,q,\alpha). \tag{190}$$

Theorem 5.3. *Three q-analogues of Lebedev* [48, p. 78, 4.18.1]*, which are all different from* [50, p. 26 3.2]

(191)

$$\{n+1\}_q L_{n+1,q}^{(\alpha)}(x) + xq^{1+\alpha+n} L_{n,q}^{(\alpha)}(qx) - \{n\}_q(1+q) L_{n,q}^{(\alpha)}(x)$$
$$- \{\alpha+1\}_q\ q^n L_{n,q}^{(\alpha)}(x) + \{\alpha+1\}_q\ q^n L_{n-1,q}^{(\alpha)}(x) + \{n-1\}_q\ q L_{n-1,q}^{(\alpha)}(x) -$$
$$- xq^{n+\alpha+1}(1-q^{n+\alpha}) L_{n-1,q}^{(\alpha)}(qx) = 0.$$

(192)
$$\{n+1\}_q L^{(\alpha)}_{n+1,q}(x) + xq^{1+\alpha+n} L^{(\alpha)}_{n,q}(qx) - \{n\}_q(1+q)L^{(\alpha)}_{n,q}(x)$$
$$- \{\alpha+1\}_q\ q^n L^{(\alpha)}_{n,q}(qx) + \{\alpha+1\}_q\ q^n L^{(\alpha)}_{n-1,q}(qx) + \{n-1\}_q\ qL^{(\alpha)}_{n-1,q}(x) -$$
$$- xq^{n+\alpha+1}(1-q^{n-1})L^{(\alpha)}_{n-1,q}(qx) = 0.$$

(193)
$$\{n+1\}_q L^{(\alpha)}_{n+1,q}(x) + xq^{1+\alpha+2n} L^{(\alpha)}_{n,q}(x) - \{n\}_q(1+q)L^{(\alpha)}_{n,q}(x)$$
$$- \{\alpha+1\}_q\ q^n L^{(\alpha)}_{n,q}(q^{-1}x) + \{\alpha+1\}_q\ q^n L^{(\alpha)}_{n-1,q}(q^{-1}x) + \{n-1\}_q\ qL^{(\alpha)}_{n-1,q}(x) -$$
$$- xq^{\alpha+2n-1}(1-q^{\alpha+1})L^{(\alpha)}_{n-1,q}(x) = 0.$$

Proof. Compute the q-difference of $F(x,t,q,\alpha)$ with respect to t in 2 different ways. We start with (191).

(194)
$$D_{q,t}F = \sum_{n=0}^{\infty} \frac{q^{n^2+n\alpha}(-x)(-xt)^{n-1}\{n\}_q}{(t;q)_{\alpha+1+n}\{n\}_q!} + \sum_{n=0}^{\infty} \frac{q^{n^2+n(\alpha+1)}(-xt)^n\{\alpha+1+n\}_q}{(t;q)_{\alpha+2+n}\{n\}_q!} =$$
$$\sum_{n=0}^{\infty} \frac{q^{(n+1)^2+(n+1)\alpha}(-x)(-xt)^n}{(tq;q)_{\alpha+1+n}(1-t)\{n\}_q!} + \sum_{n=0}^{\infty} \frac{q^{n^2+n(\alpha+1)}(-xt)^n\{\alpha+1\}_q}{(tq;q)_{\alpha+1+n}(1-t)\{n\}_q!} +$$
$$\sum_{n=0}^{\infty} \frac{q^{(n+1)^2+(n+2)(\alpha+1)}(-xt)(-xt)^n}{(tq^2;q)_{\alpha+1+n}(1-t)(1-tq)\{n\}_q!}.$$

Then

(195)
$$\sum_{n=0}^{\infty} \frac{-xL^{(\alpha)}_{n,q}(qx)(tq)^n q^{\alpha+1}}{1-t} + \sum_{n=0}^{\infty} \frac{\{1+\alpha\}_q L^{(\alpha)}_{n,q}(x)(tq)^n}{1-t} +$$
$$\sum_{n=0}^{\infty} \frac{-xtL^{(\alpha)}_{n,q}(qx)(tq^2)^n q^{2\alpha+3}}{(1-t)(1-tq)} = \sum_{n=0}^{\infty} \{1+n\}_q L^{(\alpha)}_{n+1,q}(x)t^n.$$

Multiply with $(1-t)(1-tq)$ and equate the coefficients of t^n.

The proof of (193) goes as follows.

(196)
$$D_{q,t}F = \sum_{n=0}^{\infty} \frac{q^{n^2+n\alpha}(-x)^n((t;q)_{\alpha+1+n}\{n\}_q t^{n-1} - t^n D_q(t;q)_{\alpha+1+n})}{(tq;q)_{\alpha+1+n}(t;q)_{\alpha+1+n}\{n\}_q!} =$$
$$\sum_{n=0}^{\infty} \frac{q^{n^2+n\alpha}(-x)^n t^n((t;q)_{\alpha+1+n}\{n\}_q t^{-1} + \{\alpha+1+n\}_q(tq;q)_{\alpha+n})}{(tq;q)_{\alpha+1+n}(t;q)_{\alpha+1+n}\{n\}_q!} =$$
$$\sum_{n=0}^{\infty} \frac{q^{(n+1)^2+(n+1)\alpha}(-x)(-xt)^n}{(tq^2;q)_{\alpha+1+n}(1-tq)\{n\}_q!} + \sum_{n=0}^{\infty} \frac{q^{n^2+n\alpha}(-xt)^n\{\alpha+1\}_q}{(tq;q)_{\alpha+1+n}(1-t)\{n\}_q!} +$$
$$\sum_{n=0}^{\infty} \frac{q^{(n+1)^2+(n+1)\alpha}(-xt)(-xt)^n q^{\alpha+1}}{(tq^2;q)_{\alpha+1+n}(1-t)(1-tq)\{n\}_q!}.$$

Then

(197)
$$\sum_{n=0}^{\infty} \frac{-xL_{n,q}^{(\alpha)}(x)(tq^2)^n q^{\alpha+1}}{1-tq} + \sum_{n=0}^{\infty} \frac{\{1+\alpha\}_q L_{n,q}^{(\alpha)}(q^{-1}x)(tq)^n}{1-t} +$$
$$\sum_{n=0}^{\infty} \frac{-xtL_{n,q}^{(\alpha)}(x)(tq^2)^n q^{2\alpha+2}}{(1-t)(1-tq)} = \sum_{n=0}^{\infty} \{1+n\}_q L_{n+1,q}^{(\alpha)}(x)t^n.$$

Now multiply with $(1-t)(1-tq)$ and equate the coefficients of t^n. □

Remark 12. This proof is an example of a general method in q-calculus to obtain multiple q-analogues. In our case, $\{1+\alpha+n\}_q$ can be expressed as $\{1+\alpha\}_q + q^{1+\alpha}\{n\}_q$ or $\{1+\alpha\}_q q^n + \{n\}_q$. Compare the similar reasoning using the q-Leibniz theorem.

Remark 13. The three formulas in Theorem 5.3. have the polynomials with both x and qx or $\frac{x}{q}$ as variables. These are analogues of different results for Laguerre polynomials, formulas which involve both Laguerre polynomials and their derivatives. These formulas for Laguerre polynomials have some uses but are of secondary importance while the three term recurrence relation is of primary importance.

Theorem 5.4. *Yet another generating function for the q-Laguerre polynomials.*

(198)
$$\sum_{n=0}^{\infty} \frac{L_{n,q}^{(\alpha)}(xq^{-n})t^n q^{\binom{n}{2}}}{\{1+\alpha\}_{n,q}} = E_{\frac{1}{q}}(t)\,{}_0\phi_2(-;1+\alpha,\infty|q,q^{1+\alpha}(1-q)^2(xt)).$$

Proof.

(199)

$$\sum_{n=0}^{\infty}\frac{L_{n,q}^{(\alpha)}(xq^{-n})}{\{1+\alpha\}_{n,q}}t^nq^{\binom{n}{2}}=\sum_{k,n=0}^{\infty}\frac{t^nq^{\binom{n}{2}}(1-q)^{n+k}}{\langle 1+\alpha;q\rangle_k}\frac{\langle -n\rangle_k}{\langle 1\rangle_k}\frac{q^{\frac{k^2+k}{2}}}{\langle 1;q\rangle_n}q^{\alpha k}x^k$$

$$=\sum_{k,n=0}^{\infty}\frac{t^{n+k}q^{\frac{n^2+2nk+k^2-n-k}{2}}(1-q)^{2k+n}}{\langle 1,1+\alpha;q\rangle_k\langle 1;q\rangle_{n+k}}\langle -n-k;q\rangle_kq^{\frac{k^2+k}{2}+\alpha k}x^k$$

$$=\sum_n\frac{t^nq^{\binom{n}{2}}(1-q)^n}{\langle 1;q\rangle_n}\sum_k\frac{t^k(-1)q^{\binom{k}{2}+k^2+\alpha k}(1-q)^{2k}x^k}{\langle 1,1+\alpha;q\rangle_k}$$

$$=E_{\frac{1}{q}}(t)_0\phi_2(-;1+\alpha,\infty|q,xtq^{1+\alpha}(1-q)^2)$$

□

The generating functions for the q-Laguerre polynomials are special cases of

Theorem 5.5.

$$\sum_{n=0}^{\infty}\frac{\langle a;q\rangle_n}{\langle 1;q\rangle_n}t^n\ {}_{p+1}\phi_p(-n,(A);(C)|q,x)= \tag{200}$$
$$\frac{1}{(t;q)_a}\ {}_{p+2}\phi_{p+1}((A),a,\infty;(C)|q,x||-;\frac{q}{t}).$$

Proof.

$$\sum_{n=0}^{\infty}\frac{\langle a;q\rangle_n}{\langle 1;q\rangle_n}t^n\ {}_{p+1}\phi_p(-n,(A);(C)|q,x)=$$
$$\sum_{m=0}^{\infty}\frac{(-xt)^mq^{\binom{m}{2}-m^2}\langle (A),a;q\rangle_m}{\langle 1,(C);q\rangle_m}\sum_{n=0}^{\infty}\frac{\langle a+m;q\rangle_n}{\langle 1;q\rangle_n}t^nq^{-mn}= \tag{201}$$
$$\sum_{m=0}^{\infty}\frac{(-xt)^mq^{\binom{m}{2}-m^2}\langle (A),a;q\rangle_m}{\langle 1,(C);q\rangle_m(tq^{-m};q)_{a+m}}=$$
$$\sum_{m=0}^{\infty}\frac{x^m\langle (A),a;q\rangle_m}{\langle 1,(C);q\rangle_m(\frac{q}{t};q)_m(t;q)_a}=LHS.$$

□

Remark 14. This is a q-analogue of [58, 1.2, p. 328] and [9, (25), p. 947].

Corollary 5.6. *A confluent form.*

$$\sum_{n=0}^{\infty} \frac{t^n}{\{n\}_q!}\ {}_{p+1}\phi_p(-n,(A);(C)|q,x) = \tag{202}$$
$$E_q(t)\ {}_{p+2}\phi_{p+1}((A),\infty,\infty;(C)|q,x||-;\frac{q}{t(1-q)}).$$

Proof. Let $a \to \infty$ in (200) □

Remark 15. This is a q-analogue of [58, 1.7, p. 329] and [9, (27), p. 947].

Theorem 5.7. *Two q-extensions of* [19, 5.15, 5.29].

(203)

$$\sum_{n=0}^{\infty} \frac{\langle d;q\rangle_n}{\langle 1+\alpha;q\rangle_n} t^n P_{n,q}^{(\alpha,\beta)}(x) \equiv$$
$$\sum_{n=0}^{\infty} \frac{\langle d;q\rangle_n}{\langle 1;q\rangle_n} t^n\ {}_2\phi_1(-n,\beta+n;\alpha+1|q,xq^{-\beta+\alpha+1}) =$$
$$\sum_{m=0}^{\infty} \frac{(-xtq^{-\beta+\alpha+1})^m q^{\binom{m}{2}-m^2}\langle \beta+m,d;q\rangle_m}{\langle 1,1+\alpha;q\rangle_m} \sum_{n=0}^{\infty} \frac{\langle d+m,\beta+2m;q\rangle_n}{\langle 1,\beta+m;q\rangle_n} t^n q^{-mn}.$$

(204)

$$\sum_{n=0}^{\infty} t^n P_{n,q}^{(\alpha-n,\beta)}(x) q^{\binom{n}{2}-n\alpha} \equiv$$
$$\sum_{n=0}^{\infty} \frac{\langle 1+\alpha-n;q\rangle_n}{\langle 1;q\rangle_n} t^n q^{\binom{n}{2}-n\alpha}\ {}_2\phi_1(-n,\beta+n;\alpha+1-n|q,xq^{-\beta+\alpha+1-n}) \equiv$$
$$\sum_{n=0}^{\infty} \frac{\langle -\alpha;q\rangle_n}{\langle 1;q\rangle_n} (-t)^n\ {}_2\phi_1(-n,\beta+n;\alpha+1-n|q,xq^{-\beta+\alpha+1-n}) =$$
$$\sum_{m=0}^{\infty} \frac{(-xtq^{-\beta-m})^m \langle \beta+m;q\rangle_m}{\langle 1;q\rangle_m} \sum_{n=0}^{\infty} \frac{\langle -\alpha,\beta+2m;q\rangle_n}{\langle 1,\beta+m;q\rangle_n} (-t)^n q^{-mn}.$$

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Department of Mathematics, Uppsala University, P.O. Box 480, SE-751 06 Uppsala, Sweden