Advances in Dynamical Systems and Applications
ISSN 0973-5321, Volume 3, Number 2, pp. 251–282 (2008)
http://campus.mst.edu/adsa

q-Stirling Numbers, an Umbral Approach

Thomas Ernst
Uppsala University
Department of Mathematics
P.O. Box 480, SE-751 06 Uppsala, Sweden
thomas@math.uu.se

Abstract

Three different approaches to q-difference operators are given, the first one applies to $\mathbb{C}(q)[x]$ and the last two to $\mathbb{C}(q)[q^x]$. For the first one (Hahn–Cigler), definitions and basic formulas for the two q-Stirling numbers are given. For the second (Carlitz–Gould), and third approach (Jackson), the respective q-Taylor formulas are used to find a q-binomial coefficient identity. Three different formulas for Carlitz' q-analogue of sums of powers are found. The first one uses a double sum for q-Stirling numbers. The last two are multiple sums with q-binomial coefficients.

AMS Subject Classifications: Primary 05A40, 11B73; Secondary 05A10, 39A13.
Keywords: q-Stirling numbers, Hahn–Cigler approach, Carlitz–Gould approach, Jackson approach, Carlitz sum, q-Schwatt formula, quartet of formulas.

1 Introduction

The aim of this paper is to describe how different q-difference operators combine with q-Stirling numbers to form various q-formulas. Contrary to [28] where formal power series were considered, the aim is here concentrated to functions of q^x, or equivalently functions of the q-binomial coefficients. We will find many q-analogues of Stirling number identities from Jordan [56] and the elementary textbooks by Cigler [16] and Schwatt [75]. A historical survey of the early use of Stirling numbers in connection with series expansions and umbral calculus in Germany will also be given. This is a continuation of the survey of umbral calculus which was given in [28].

James Stirling (1692–1770) was born in Scotland and studied in Glasgow and Oxford. In 1717 Stirling went to Venice; probably he had been promised a chair of mathematics there, but for some reason the appointment was never realized. In spite of this, he

Received May 21, 2007; Accepted November 15, 2008
Communicated by Lance Littlejohn

continued his mathematical research. He also attended the University of Padua, where he got to know Nicolaus Bernoulli (I), who occupied the chair there.

Stirling later expressed Maclaurin's formula in a different form using what is now called Stirling's numbers of the second kind [35, p. 102]. Because of his long sojourn in Italy, the Stirling numbers are well known there, as can be seen from the reference list.

A.T. Vandermonde (1735–1796) is best known for his determinant and for the Vandermonde theorem for hypergeometric series [84]. Vandermonde also introduced the following notation in 1772 [84].

Let $(x)_n = x(x-1)(x-2)\cdots(x-n+1)$ be the falling factorial. Stirling numbers of the first kind are the coefficients in the expansion $(x)_n = \sum_{k=0}^{n} s(n,k)(x)^k$. The Stirling numbers of the second kind are given by $x^n = \sum_{k=0}^{n} S(n,k)(x)_k$.

In combinatorics, unsigned Stirling numbers of the first kind $|s(n,k)|$ count the number of permutations of n elements with k disjoint cycles.

Tables of these numbers were published by Grünert [44, p. 279], De Morgan [21, p. 253] and Cayley [11].

Stirling numbers have wideranging applications in computer technology [41] and in numerical analysis [30]. One reason is that computers use difference operators rather than derivatives and these numbers are used in the transformation process.

Stirling numbers also have applications in statistics as was shown in the monograph by Jordan [55, p, 14].

We will now briefly describe the q-umbral method invented by the author [22]– [27]. This method is a mixture of Heine 1846 [48] and Gasper–Rahman [31]. The advantages of this method have been summarized in [25, p. 495].

Definition 1.1. The power function is defined by $q^a \equiv e^{a\log(q)}$. We always use the principal branch of the logarithm.

The variables

$$a, b, c, a_1, a_2, \ldots, b_1, b_2, \ldots \in \mathbb{C}$$

denote certain parameters. The variables i, j, k, l, m, n, p, r will denote natural numbers except for certain cases where it will be clear from the context that i will denote the imaginary unit. In the whole paper, the symbol $\equiv$ denotes definitions, except where it is clear from the context that it denotes congruences. The q-analogues of a complex number a and of the factorial function are defined by:

$$\{a\}_q \equiv \frac{1-q^a}{1-q},\ q \in \mathbb{C}\backslash\{1\}, \tag{1.1}$$

$$\{n\}_q! \equiv \prod_{k=1}^{n}\{k\}_q,\ \{0\}_q! \equiv 1,\ q \in \mathbb{C}, \tag{1.2}$$

Let the q-shifted factorial (compare [33, p.38]) be defined by

$$\langle a;q\rangle_n \equiv \begin{cases} 1, & n=0; \\ \prod_{m=0}^{n-1}(1-q^{a+m}) & n=1,2,\ldots, \end{cases} \tag{1.3}$$

The Watson notation [31] will also be used

$$(a;q)_n \equiv \begin{cases} 1, & n=0; \\ \prod_{m=0}^{n-1}(1-aq^m), & n=1,2,\ldots. \end{cases} \tag{1.4}$$

Let the Gauss q-binomial coefficient be defined by

$$\binom{n}{k}_q \equiv \frac{\langle 1;q\rangle_n}{\langle 1;q\rangle_k \langle 1;q\rangle_{n-k}}, \tag{1.5}$$

for $k=0,1,\ldots,n$.

Euler found the following q-analogue of the exponential function

$$\mathrm{e}_q(z) \equiv \sum_{n=0}^{\infty} \frac{z^n}{\langle 1;q\rangle_n}. \tag{1.6}$$

Nowadays another q-exponential function is more often used: If $|q|>1$, or $0<|q|<1$ and $|z|<|1-q|^{-1}$,

$$\mathrm{E}_q(z) = \sum_{k=0}^{\infty} \frac{1}{\{k\}_q!} z^k. \tag{1.7}$$

By the Euler equation (1.6), we can replace $\mathrm{E}_q(z)$ by

$$\frac{1}{(z(1-q);q)_\infty},\ |z(1-q)|<1,\ 0<|q|<1.$$

So by meromorphic continuation, the meromorphic function $\frac{1}{(z(1-q);q)_\infty}$, with simple poles at $\frac{q^{-k}}{1-q}$, $k\in\mathbb{N}$, is a good substitute for $\mathrm{E}_q(z)$ in the whole complex plane. We shall however continue to call this function $\mathrm{E}_q(z)$, since it plays an important role in the operator theory. For convenience, we can say that we work in $(\mathbb{C}(q))[[x]]$.

Let the q-Pochhammer symbol $\{a\}_{n,q}$ be defined by

$$\{a\}_{n,q} \equiv \prod_{m=0}^{n-1}\{a+m\}_q. \tag{1.8}$$

The following notation will be convenient.

$$\mathrm{QE}(x) \equiv q^x. \tag{1.9}$$

The Nalli–Ward–AlSalam q-addition (NWA) is given by

$$(a \oplus_q b)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q a^k b^{n-k},\ n = 0, 1, 2, \ldots. \tag{1.10}$$

Furthermore, we put

$$(a \ominus_q b)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q a^k (-b)^{n-k},\ n = 0, 1, 2, \ldots. \tag{1.11}$$

There is a q-addition dual to the NWA, which will be presented here for reasons to be given shortly. The following polynomial in 3 variables x, y, q originates from Gauss. The Jackson–Hahn–Cigler q-addition (JHC) is the function

$$(x \boxplus_q y)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q q^{\binom{k}{2}} y^k x^{n-k},\ n = 0, 1, 2, \ldots. \tag{1.12}$$

The following general inversion formula will prove useful in the sequel.

Theorem 1.2. *Gauss inversion [3, p. 96], a corrected version of [34, p. 244]. A q-analogue of [70, p. 4]. The following two equations for arbitrary sequences a_n, b_n are equivalent.*

$$a_n = q^{-f(n)} \sum_{l=0}^{n} (-1)^l q^{\binom{l}{2}} \binom{n}{l}_q b_{n-l}, \tag{1.13}$$

$$b_n = \sum_{i=0}^{n} q^{f(i)} \binom{n}{i}_q a_i. \tag{1.14}$$

Proof. It will suffice to prove that

$$a_n = q^{-f(n)} \sum_{l=0}^{n} (-1)^l \binom{n}{l}_q q^{\binom{l}{2}} \sum_{i=0}^{n-l} q^{f(i)} \binom{n-l}{i}_q a_i. \tag{1.15}$$

The first sum is zero except for $i = n$ and $l = 0$. □

In the history of mathematics, Stirling numbers appeared in many different disguises. Before Nielsen coined this name, the most frequent appearance of the so-called second Stirling number was as submultiple of the Euler formula for k-th differences of powers [39]. This formula is

$$S(n,k) = \frac{1}{k!} \triangle^k x^n |_{x=0} \equiv \frac{1}{k!} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} i^n. \tag{1.16}$$

In the eighteenth and nineteenth centuries many articles about Bernoulli numbers contained Stirling numbers in disguise. There appeared a pseudo-Stirling number $S(n,k)k!$ in a work by Euler 1755. Tables for these pseudo-Stirling numbers were published in [49, p. 9] and [42, p. 71].

Formulas for Bernoulli polynomials were written as sums with Stirling numbers as coefficients, like in [89, p. 211] and [72, p. 96]. The following table illustrates some different notations for $S(n,k)$.

Grünert [43]	Grünert [44]	Björling [4]	Saalschütz [72]	Worpitzky [89]
A_n^k	C_k^n	C_k^n	α_k^p	α_k^n

A formula related to (1.16), which forms the basis for q-analysis is

$$\sum_{n=0}^{m}(-1)^n \binom{m}{n}_q q^{\binom{n}{2}} u^n = (u;q)_m. \tag{1.17}$$

According to Ward [87, p. 255] and Kuperschmidt [62, p. 244], this identity was first obtained by Euler. Gauss 1876 [32] also found this formula.

As will be seen, the corresponding expression for the second q-Stirling number has a slightly different character than the left hand side of (1.17).

We will find many new formulas for q-Stirling numbers. This paper first appeared as preprint in October–December 2005. In February 2006 Johann Cigler reminded the author about some corrections and that he also had written a tutorial on a similar subject [17]. The q-Stirling numbers of Cigler and the author are equal. Whenever an equation appears, which also appeared in [17], it will be mentioned and the page number (November 2006) will be given.

q-Stirling numbers are of the greatest benefit in q-calculus. This however has not been fully acknowledged until now. In a book by Don Knuth [61], it is shown that the q-Stirling number of the second kind gives the running time of the algorithm for a computer program. A related result for Markov processes was obtained by Crippa, D.; Simon, K.; Trunz, P. [18].

As Sharma and Chak [77, p. 326] remarked, the operator D_q, defined by

$$(D_q\varphi)(x) \equiv \begin{cases} \dfrac{\varphi(x)-\varphi(qx)}{(1-q)x}, & \text{if } q \in \mathbb{C}\backslash\{1\},\ x \neq 0; \\ \dfrac{d\varphi}{dx}(x) & \text{if } q = 1; \\ \dfrac{d\varphi}{dx}(0) & \text{if } x = 0 \end{cases} \tag{1.18}$$

plays the same role for polynomials in x as the difference operator in Chapter 4

$$\triangle_{\mathrm{CG},q}^{1} f(x) \equiv f(x+1)-f(x),\ \triangle_{\mathrm{CG},q}^{n+1} f(x) \equiv \triangle_{\mathrm{CG},q}^{n} f(x+1) - q^n \triangle_{\mathrm{CG},q}^{n} f(x) \tag{1.19}$$

does for polynomials in q^x. If we want to indicate the variable which the q-difference operator is applied to, we write $(D_{q,x}\varphi)(x,y)$ for the operator. The same notation will also be used for a general operator.

All the next 5 equations were used by Euler. They have the following form, where E is the forward shift operator and $\triangle = E - I$.

Theorem 1.3. *Newton–Gregory–Taylor series*

$$f(x) = \sum_{k=0}^{\infty} \binom{x}{k} (\triangle^k f)(0). \tag{1.20}$$

A product expansion

$$\triangle^n f(x) = (E - I)^n f(x). \tag{1.21}$$

An equivalent formula

$$\triangle^n f(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} E^{n-k} f(x). \tag{1.22}$$

An inverted formula.

$$E^n f(x) = \sum_{i=0}^{n} \binom{n}{i} \triangle^i f(x). \tag{1.23}$$

Leibniz' formula (1710)

$$\triangle^n (fg) = \sum_{i=0}^{n} \binom{n}{i} \triangle^i f \; (\triangle^{n-i} E^i)) g. \tag{1.24}$$

In the footsteps of Euler, a strong combinatorial school grew up in Germany. This so-called Grünert–Gudermann school will be treated briefly from a historical standpoint in Chapter 2. The reason is that Grünert and other members of this school used the Stirling numbers at different occasions. The corresponding English and French combinatorial schools have been treated in [28].

We will now briefly explain the different contents of Chapters 3–5 from a conceptual point of view. In Chapter 3 we only consider functions in $\mathbb{C}(q)[x]$. In Chapters 4 and 5, we mainly consider functions $f, g \in \mathbb{C}(q)[q^x]$. The main purpose of Chapter 3 is to introduce the q-Stirling numbers and to begin to study their properties.

In each of Chapters 3, 4, 5 for the respective $\triangle_q$ operator, we will, when possible, find q-analogues of formulas (1.20)–(1.24). For clarity, we will keep the same order of these equations in each chapter. As minimum 4 of these formulas occur, we will refer to them as the quartet of formulas.

In 1951 D.B. Sears wrote an important paper [76], where transformations for basic hypergeometric functions were derived from a slightly different difference operator. In Sears' paper, the variable x was instead the index for a q-shifted factorial. By inversion of the basis two different sets of equations were obtained.

In a previous paper [28] the author used a formal power series approach to find many formulas for q-Bernoulli polynomials etc. The shift operator was two q-additions (Ward, Jackson) and the quartet of formulas was given with ordinary binomial coefficients.

The $\triangle$ operators in Chapter 4 (Carlitz–Gould) and in Chapter 5 (Jackson) are very similar; in fact

$$\triangle_{\mathrm{J},q}^{n} f(q^x) = q^{-nx-\binom{n}{2}} \triangle_{\mathrm{CG},q}^{n} f(q^x). \tag{1.25}$$

The two symbols, sometimes called the difference and sum calculus, correspond respectively to differentiation and integration in the continuous calculus. We will find q-analogues of the inverse operators $\triangle$ and $\sum$ in Chapter 4.

In the footsteps of Faulhaber, Fermat, Jacob Bernoulli and De Moivre, we will find an expression for the Carlitz function

$$S_{\mathrm{C},m,q}(n) \equiv \sum_{i=0}^{n-1} \{i\}_q^m q^i \tag{1.26}$$

in Chapters 3 and 4.

In Chapter 6 we will prove a q-binomial coefficient identity by using the operator $\triangle_{\mathrm{CG},q}$ with corresponding q-Taylor formula. After simplification, it is shown that the operator $\triangle_{\mathrm{J},q}$ with corresponding q-Taylor formula gives the same formula.

2 The Grünert–Gudermann Combinatorial School

The goal of the Gudermann combinatorial school was to develop functions in power series according to the Taylor formula. The Taylor formula was originally formulated with finite difference ratios, so-called fluxions. There were previously two designations for finite differences after Taylor and Cousin. Then Hindenburg introduced the notation $\frac{k}{y}$ in *Archiv der reinen und angewandten Mathematik* [50, p. 94] 1795.

In 1800 Arbogast has suggested to write a D (derivative) instead of the $\dfrac{dy}{dx}$ of Leibniz, in order to simplify the notation. This had a main influence on the development in England and in Germany. This can be seen from the different designations in two publications of Hindenburg. In 1795 the journal *Archiv der reinen und angewandten Mathematik* contained some work on the Taylor theorem, which was expressed by a difference operator.

However in 1803 Hindenburg [51, p. 180] used the symbol D and obviously noticed the difference between the two. The above-mentioned magazine also contained some military reports, among other things from Lambert; Hindenburg was also a physicist. We will see that this had a strong influence on Gudermann.

Christof Gudermann (1798–1852), promoted by his close friendship with Crelle, was first *Gymnasiallehrer* in Cleve, later professor in Münster. He first wrote only in German, and later alternating in Latin, which was the common scientific language at that time, and could therefore reach international range. Gudermann wrote an excellent Latin in a time, when the Latin was already declining in Europe.

Crelle was very concerned about the mathematical questions of his time, and could find publishers for Gudermann's textbooks.

Gudermann had a decisive influence as the teacher of Karl Weierstrass. It is reported that thirteen listeners came to the first lecture of Gudermann on elliptical functions. At the end of the term only one had remained, i.e. Weierstrass. Gudermann was the first one who discovered Weierstrass' extraordinary mathematical gift. Weierstrass was inspired by Gudermann's theories about series expansions and often expressed his large gratitude for his old teacher, and Weierstrass further developed and modified the combinatorial school of Gudermann.

Affected by Lambert, who introduced the hyperbolic functions, Gudermann developed among other things the function $\frac{1}{\cosh(x)}$ in powers of x, and thereby availed himself of the work by Scherk on the so-called Euler numbers.

The Gudermann names for the trigonometric functions have had many successors up to the year 1908 [7, p. 173].

Gudermann also used the sign for sums of Rothe and expressed $\sin x$ und $\cos x$ as infinite products [46, p. 68]. In this connection it is interesting that Rothe und Schweins formulated the q-binomial theorem, but without proof.

Christian Kramp (1760–1826) took over the designation D by Arbogast and developed it further in 1808. The combinatorial school of Vandermonde and Kramp enjoyed a popularity in the years 1772–1856. The goal was to divide the so-called *Fakultäten* in four classes: positive, negative, whole and broken exponents. Each class had its own laws, similarly as for the q-factorial (1.3).

Influenced by Kramp, Bessel improved this idea in his detailed paper, and finally Weierstrass brought the *Fakultäten* onto complex level in 1856.

Gudermann very often developed his functions by using Taylors formula; he used a forerunner of Pochhammer's symbol — disguised in Kramp's notation. One could say that this circle formed its own school around Gudermann. This school consisted among other people of Johann August Grünert (1797–1872), editor of *Archiv der Mathematik und Physik*, which had started in 1841, and Oscar Schlömilch (1823–1901), editor of *Zeitschrift für Mathematik und Physik*, which had started in 1856. These two magazines differed from Crelle's journal, which had a more purely mathematical content. Grünert, a pupil of Pfaff and Gauss, wrote early on *Fakultätenreihen*, and made tables of the Stirling numbers [42, p. 71], [44, p. 279].

The Stirling numbers were later used in series expansions for Bernoulli functions [89, p. 210]. This Gudermann school also had advocates in Sweden, e.g. Malmsten and Björling, who both contributed to the Grünert Archiv. This magazine also contained publications on hyperbolic functions and spherical trigonometry. The last subject is a modern name for *analytische Sphärik*, which was treated in [45].

In 1825 Grünert started a mathematical seminar in Greifswald and later let his students use his private mathematical library.

The Gudermann–Grünert–Schlömilch school also found advocates elsewhere. Some of them, of the first and second generation, were Sonine, Schläfli, Ettingshausen, J. Petzval (1807–1891), Gegenbauer, F. Neumann, Beltrami und F. Rogel.

One could say that this was in former times a beginning of the AMS 33 (special functions with applications) in Europe.

Grünert had a conflict with Grassmann 1862, therefore his name is not mentioned in Klein's eminent book [59]; Klein also treats Gudermann unfairly. Approximately in 1853, when Grünert was 56, the *Archiv der Mathematik und Physik* began its decline. After the death of Grünert 1872, Reinhold Hoppe (1816–1900) took over the editorship. Like Schlömilch, Hoppe was an advocate for, among other things, the umbral calculus.

3 The Hahn–Cigler–Carlitz–Johnson Approach

The main purpose of this chapter is to introduce and study the q-Stirling numbers. This chapter is a partial continuation of papers by Hahn [47, p. 6 2.2], Cigler [13, p. 102–104], Carlitz [8] and Johnson [54, p. 217]. The last three papers use the same q-Stirling numbers. We start with 3 definitions followed by 3 examples. First a most important polynomial.

Definition 3.1. A q-analogue of the polynomial from [16, p. 20]. Cigler [13, p. 102], [17, p. 38] calls this polynomial *Hauptfolge*.

$$(x)_{k,q} \equiv \prod_{m=0}^{k-1} (x - \{m\}_q). \tag{3.1}$$

The following notation of Cigler [14, p. 107], [17, p. 39, 3.25] will be used.

Definition 3.2.

$$E^l_{\mathrm{C},q} f(x) \equiv f(xq^l + \{l\}_q). \tag{3.2}$$

Definition 3.3. [13, p. 102], [14, p. 107] This is a special case of the Hahn operator [47, p. 6 2.2].

$$\triangle_{\mathrm{H},q} f(x) \equiv \frac{f(qx+1) - f(x)}{1 + (q-1)x}. \tag{3.3}$$

Example 3.4. [17, p. 39], [13, p. 102], a q-analogue of [16, p. 20, 2.5]

$$\triangle_{\mathrm{H},q} (x)_{k,q} = \{k\}_q (x)_{k-1,q}. \tag{3.4}$$

Example 3.5. [14, p. 107]

$$\triangle_{\mathrm{H},q} E_{\mathrm{C},q} = q E_{\mathrm{C},q} \triangle_{\mathrm{H},q}. \tag{3.5}$$

Example 3.6. A q-analogue of [2, p. 237, (27)].

$$(\{k\}_q)_{l,q} = \{l\}_q! \binom{k}{l}_q q^{\binom{l}{2}}. \tag{3.6}$$

The first quartet of formulas supplemented by some equivalent ones turns out to be less useful than the last two quartets, as far as q-Taylor formulas are concerned.

Theorem 3.7. *A q-Taylor formula from [13, p. 103]. A q-analogue of [6, p. 11], [16, p. 25].*

$$f(x) = \sum_{k=0}^{\infty} \frac{(\triangle_{\mathrm{H},q}^{k} f)(0)}{\{k\}_q!}(x)_{k,q}. \tag{3.7}$$

Theorem 3.8. *The operator $\triangle_{\mathrm{H},q}^{n}$ can be expressed by the following two operator equations. Cigler [17, p. 41 (3.34)], [13, p. 103 (40)].*

$$q^{\binom{n}{2}} \triangle_{\mathrm{H},q}^{n} = \frac{1}{(1+(q-1)x)^n} \prod_{k=0}^{n-1} (E_{\mathrm{C},q} - q^k). \tag{3.8}$$

Theorem 3.9. *[17, p. 41 (3.33)]*

$$\triangle_{\mathrm{H},q}^{n}(x)_{k,q} = q^{-\binom{n}{2}}(1+(q-1)x)^{-n} \sum_{l=0}^{n} (-1)^{n-l} q^{\binom{n-l}{2}} \binom{n}{l}_q (xq^l + \{l\}_q)_{k,q}. \tag{3.9}$$

Proof. Induction on n. □

We will rewrite the above theorem in a slightly different way

Theorem 3.10.

$$\triangle_{\mathrm{H},q}^{n} f(x) = q^{-\binom{n}{2}}(1+(q-1)x)^{-n} \sum_{l=0}^{n} (-1)^{n-l} q^{\binom{n-l}{2}} \binom{n}{l}_q E_{\mathrm{C},q}^{l} f(x). \tag{3.10}$$

Proof. Operate on $(x)_{k,q}$. □

This formula can be inverted.

Theorem 3.11.

$$E_{\mathrm{C},q}^{n} f(x) = \sum_{i=0}^{n} q^{\binom{i}{2}} \binom{n}{i}_q (1+(q-1)x)^i \triangle_{\mathrm{H},q}^{i} f(x). \tag{3.11}$$

Proof. Use the above inversion theorem with $f(n) = \binom{n}{2}$. □

Corollary 3.12. *A Leibniz theorem.*

$$\triangle^n_{\mathrm{H},q}(fg) = q^{-\binom{n}{2}} \sum_{i=0}^{n} q^{\binom{i}{2}+\binom{n-i}{2}} \binom{n}{i}_q \triangle^i_{\mathrm{H},q} f(\triangle^{n-i}_{H,q} E^i_{\mathrm{C},q})g. \tag{3.12}$$

Proof. Same as [56, p. 96 f]. □

We are now ready for the definition of q-Stirling numbers, which will occupy us for the next two chapters. The second Stirling numbers, as below but for $q = 1$ occurred in Stirling's book [79, p. 8]. However Stirling didn't use any symbol for these numbers.

Definition 3.13. The q-Stirling number of the first kind $s(n,k)_q$ and the q-Stirling number of the second kind $S(n,k)_q$ are defined by [43, p.358], [17, p. 38 (3.13–14)], [13, p. 103], [17, p. 38] and [54, p. 217, 4.11].

$$(x)_{n,q} \equiv \sum_{k=0}^{n} s(n,k)_q x^k, \tag{3.13}$$

$$x^n \equiv \sum_{k=0}^{n} S(n,k)_q (x)_{k,q}. \tag{3.14}$$

Remark 3.14. We use the same conventions for Stirling numbers as Cigler [16, p. 34], Jordan [56, p. 142], Gould [39], Vein & Dale [86, p. 306] and Milne [64, p. 90]. Other definitions usually differ in sign, as in [6, p. 114], [41], [9], [37] where all (q-)Stirling numbers are positive.

Remark 3.15. Schwatt [75, ch. 5] denotes the $S(n,k)$ by $a_{n,k}$ without knowing that they are Stirling numbers. The book by Schwatt contains some very interesting series calculations, which we will come back to shortly.

The following recursions follow at once [13, p. 103]. The second one, a q-analogue of [44, p. 248, p. 256], [4, p. 287] and [75, p. 81 (4)], also occurred in [43, p.360], [83, p. 85 13.1], [54, p. 213, 3.6], [61, 7215, exc 29], [15, p. 146, (9)].

$$s(n+1,k)_q = s(n,k-1)_q - \{n\}_q \, s(n,k)_q. \tag{3.15}$$

$$S(n+1,k)_q = S(n,k-1)_q + \{k\}_q \, S(n,k)_q. \tag{3.16}$$

The orthogonality relation is the following q-analogue of [56, p. 182], [16, p. 35]

Theorem 3.16. *The two q-Stirling numbers viewed as matrices are inverses of each other.*

$$\sum_k S(m,k)_q s(k,n)_q = \delta_{m,n}. \tag{3.17}$$

The following table lists some of the first $s(n,k)_q$. Compare [16, p. 34], [56, p. 144] and [86, p. 306].

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
$n=0$	1	0	0	0	0
$n=1$	0	1	0	0	0
$n=2$	0	-1	1	0	0
$n=3$	0	$1+q$	$-(2+q)$	1	0
$n=4$	0	$-\{3\}_q!$	$3+4q+3q^2+q^3$	$-3-2q-q^2$	1

The following table lists some of the first $S(n,k)_q$. Compare [16, p. 35].

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
$n=0$	1	0	0	0	0
$n=1$	0	1	0	0	0
$n=2$	0	1	1	0	0
$n=3$	0	1	$2+q$	1	0
$n=4$	0	1	$3+3q+q^2$	$3+2q+q^2$	1

There are a number of simple rules to check the computation of the q-Stirling numbers of the first kind, as the following q-analogues of [56, p. 145 ff] show. Put $x=1$ in (3.13) to obtain

$$\sum_{k=1}^{n} s(n,k)_q = 0,\ n>1. \tag{3.18}$$

Put $x=-1$ in (3.13) to obtain

$$\sum_{k=1}^{n} |s(n,k)_q| = (-1)^{n-1}\prod_{m=0}^{n-1}(1+\{m\}_q),\ q>0. \tag{3.19}$$

The q-Stirling numbers of the first kind have a particularly simple expression.

Theorem 3.17.

$$s(n,k)_q = (-1)^{k-n} e_{n-k}(1,\{2\}_q,\ldots,\{n-1\}_q), \tag{3.20}$$

where e_k denotes the elementary symmetric polynomial.

Proof. Use (3.13). □

Corollary 3.18. *[17, p. 38].*

$$s(n,1)_q = (-1)^{n-1}\{n-1\}_q!. \tag{3.21}$$

Proof. Put $k=1$ in (3.20). □

There is an exact formula by Carlitz for $S(n,k)_q$, which will be quite useful.

Theorem 3.19. *[17, p. 41 (3.35)], [13, p. 104], [8, p. 990, 3.3], [83, p. 86 13.2] ($c = 1$). A q-analogue of [56, p. 169, (3)], [16, p. 37, 2.31], [82, p. 495], [75, p. 83 (19)]. Compare [44, p. 257].*

$$S(n,k)_q = (\{k\}_q! q^{\binom{k}{2}})^{-1} \sum_{i=0}^{k} \binom{k}{i}_q (-1)^i q^{\binom{i}{2}} \{k-i\}_q^n. \tag{3.22}$$

This formula can also be written as the following q-analogue of [82].

$$S(n,k)_q = \frac{1}{\{k\}_q!} \triangle_{\mathrm{H},q}^n (x)_{k,q}|_{x=0}. \tag{3.23}$$

There are a number of simple rules to check the computation of the q-Stirling numbers of the second kind. We start with some q-analogues of Jordan's book [56, p. 170. f] about finite differences intended both for mathematicians and statisticians.

Theorem 3.20.

$$(-1)^n = \sum_{k=0}^{n} S(n,k)_q (-1)^k \prod_{m=0}^{k-1} (1 + \{k\}_q). \tag{3.24}$$

Proof. Put $x = -1$ in (3.14). □

Theorem 3.21.

$$S(n+1,n)_q - S(n,n-1)_q = \{n\}_q. \tag{3.25}$$

Proof. Put $k = n$ in (3.16). □

There are two kinds of generating functions for $S(n,k)_q$. The first one is as follows.

Theorem 3.22. *[17, p. 42 (3.36)], [61, 7.2.1.5, answer 29], a q-analogue of [56, p. 175 (2)], [41, p. 337, 7.47], [83, p. 64, 2.6] and [16, p. 37, 2.30].*

$$\sum_{n=m}^{\infty} S(n,m)_q t^n = \frac{t^m}{\prod_{l=1}^{m} (1 - t\{l\}_q)}, \; |t| < \frac{1}{m}. \tag{3.26}$$

This can be expressed in two other ways. A q-analogue of [56, p. 193. (1)], which serves as definition of q-reciprocal factorial.

$$\sum_{n=m}^{\infty} S(n,m)_q z^{-n-1} = \frac{1}{(z)_{m+1,q}} \equiv (z)_{-(m+1),q}, \; z > m. \tag{3.27}$$

A q-analogue of [56, p. 193. (2)] and [16, p. 36, 2.29].

$$\sum_{n=m}^{\infty} S(n,m)_q(-x)^{-n} = \frac{(-1)^m}{\prod_{l=1}^{m}(x+\{l\}_q)},\ x>m. \tag{3.28}$$

By the orthogonality relation we obtain a q-analogue of [56, p. 193. (3)].

$$x^{-k} = \sum_{m=k}^{\infty} \frac{|s(m,k)_q|}{\prod_{l=1}^{m}(x+\{l\}_q)}. \tag{3.29}$$

The q-Stirling numbers can be used to obtain several exact formulas for q-derivatives and q-integrals as follows.

Theorem 3.23.

$$\int_0^1 (t)_{n,q}\, d_q(t) = \sum_{k=1}^{n} \frac{s(n,k)_q}{\{k+1\}_q}. \tag{3.30}$$

Proof. q-integrate (3.13). □

Theorem 3.24. *A q-analogue of [56, p. 194 (5)].*

$$D_q^s \frac{1}{(z)_{m+1,q}} = \sum_{n=m}^{\infty} S(n,m)_q\{-n-s\}_{s,q} z^{-n-1-s}. \tag{3.31}$$

Proof. Use (3.27). □

Theorem 3.25. *A q-analogue of [56, p. 194 (6)].*

$$\int^z \frac{1}{(t)_{m+1,q}}\, d_q(t) = \sum_{n=m}^{\infty} \frac{S(n,m)_q}{\{-n\}_q z^n} + k. \tag{3.32}$$

Proof. Use (3.27). □

The second generating function for $S(n,m)_q$ is as follows.

Theorem 3.26. *[17, p. 42 (3.38)]. A q-analogue of [19, p. 206 (2a)], [83, p. 64, 2.7].*

$$\sum_{n=k}^{\infty} \frac{S(n,k)_q t^n}{\{n\}_q!} = (\{k\}_q! q^{\binom{k}{2}})^{-1} \sum_{i=0}^{k} \binom{k}{i}_q (-1)^i q^{\binom{i}{2}} \mathrm{E}_q(t\{k-i\}_q). \tag{3.33}$$

Proof.

$$LHS = \sum_{n=k}^{\infty} \frac{t^n}{\{n\}_q!}(\{k\}_q! q^{\binom{k}{2}})^{-1} \sum_{i=0}^{k} \binom{k}{i}_q (-1)^i q^{\binom{i}{2}} \{k-i\}_q^n =$$
$$(\{k\}_q! q^{\binom{k}{2}})^{-1} \sum_{i=0}^{k} \binom{k}{i}_q (-1)^i q^{\binom{i}{2}} \sum_{n=0}^{\infty} \frac{(t\{k-i\}_q)^n}{\{n\}_q!} = RHS. \tag{3.34}$$

□

There is a generating function for q-Stirling numbers of the first kind.

Theorem 3.27. *A q-analogue of [56, p. 185 (1)].*

$$\sum_{k=m}^{n} s(n,k)_q \binom{k}{m} q^m = q^{n-1} s(n-1,m)_q + q^n s(n-1,m-1)_q. \tag{3.35}$$

This formula can be inverted.

Theorem 3.28. *A q-analogue of [56, p. 185].*

$$s(n,k)_q = \sum_{m=k}^{n} q^{n-m-1}(-1)^{m+k} \binom{m}{k} [s(n-1,m)_q + q s(n-1,m-1)_q]. \tag{3.36}$$

Corollary 3.29. *A q-analogue of [56, p. 186 (4)].*

$$\sum_{k=1}^{n} s(n,k)_q k = \begin{cases} 1, & n=1; \\ q^{n-2}(-1)^n \{n-2\}_q! & n>1. \end{cases} \tag{3.37}$$

Proof. Put $m = 1$ in (3.35). □

Corollary 3.30. *A q-analogue of [56, p. 186 (5)].*

$$\sum_{n=2}^{m} S(m,n)_q \, q^{n-2}(-1)^n \{n-2\}_q! = m-1. \tag{3.38}$$

Proof. Apply $\sum_{n=1}^{m} S(m,n)_q$ to both sides of (3.37). Then use the orthogonality relation.

□

The following 3 theorems are proved in exactly the same way as in Jordan [56].

Theorem 3.31. *A q-analogue of [56, p. 187 (10)].*

$$\sum_{k=1}^{n} s(n,k)_q S(k+1,i)_q = \{n\}_q \binom{0}{n-i} + \binom{0}{n+1-i}. \tag{3.39}$$

Theorem 3.32. *A q-analogue of [56, p. 188 (11)].*

$$\sum_{k=0}^{n} S(n,k)_q [s(k+1,l)_q + \{k\}_q s(k,l)_q] = \delta_{l,n+1}. \tag{3.40}$$

Theorem 3.33. *A q-analogue of [56, p. 188 (15)].*

$$\sum_{k=1}^{n+1} S(n+1,k)_q = \sum_{k=1}^{n} (1+\{k\}_q) S(n,k)_q, n > 0. \tag{3.41}$$

The following operator [53] will be useful. In its earliest form with $q = 1$ it dates back to Euler and Abel [1, B. 2, p. 41], who used it in differential equations.

Definition 3.34.

$$\theta_q \equiv x D_q. \tag{3.42}$$

Theorem 3.35. *Cigler [17, p. 37 (3.8)]. A q-analogue of the Grünert operator formula [44, 247], [39, p. 455, 4.8], [78, p. 181], [83, p. 64, 2.1], [75, p. 81, (2)], [56, p. 196 (2)], [63, p. 95].*

$$\theta_q^n = \sum_{k=0}^{n} S(n,k)_q \, q^{\binom{k}{2}} x^k D_q^k. \tag{3.43}$$

Proof. Induction. □

This leads to the following inverse formula.

Theorem 3.36. *Cigler [17, p. 37 (3.9)]. A q-analogue of [80, p 548], [78, p 183].*

$$q^{\binom{n}{2}} x^n D_q^n = \sum_{k=1}^{n} s(n,k)_q \theta_q^k. \tag{3.44}$$

Proof. Use the orthogonality relation for q-Stirling numbers. □

The previous formula can be expressed in another way.

Theorem 3.37. *Jackson [52, p. 305]. A q-analogue of the 1844 Boole formula [5], [78, p 183], [12, p 24, (2.1)].*

$$q^{\binom{n}{2}} x^n D_q^n = \prod_{k=0}^{n-1} (\theta_q - \{k\}_q). \tag{3.45}$$

Proof. Use (3.13). □

Example 3.38. A q-analogue of [56, p. 196]. Let $f(x) = (x \oplus_q 1)^n$ and apply (3.43) to get

$$\sum_{k=0}^{n} \binom{n}{k}_q x^k \{k\}_q^m = \sum_{k=0}^{\min(m,n)} S(m,k)_q\, q^{\binom{k}{2}} x^k \{n-k+1\}_{k,q} (x \oplus_q 1)^{n-k}. \tag{3.46}$$

Put $x = 1$ to get

$$\sum_{k=0}^{n} \binom{n}{k}_q \{k\}_q^m = \sum_{k=0}^{\min(m,n)} S(m,k)_q\, q^{\binom{k}{2}} \{n-k+1\}_{k,q} (1 \oplus_q 1)^{n-k}. \tag{3.47}$$

If we put $m = 1$ or $m = 2$ in (3.47), we get q-analogues of the mean and variance of the binomial distribution from Melzak [63, p. 96].

Put $x = -1$ in (3.46) to get

$$\sum_{k=0}^{n} \binom{n}{k}_q (-1)^k \{k\}_q^m = \sum_{k=0}^{\min(m,n)} S(m,k)_q\, q^{\binom{k}{2}} (-1)^k \{n-k+1\}_{k,q} (1 \ominus_q 1)^{n-k}. \tag{3.48}$$

Example 3.39. Let $f(x) = (x \boxplus_q 1)^n$ and apply (3.43) to get

$$\sum_{k=0}^{n} \binom{n}{k}_q q^{\binom{k}{2}} x^{n-k} \{n-k\}_q^m = \sum_{k=0}^{\min(m,n)} S(m,k)_q\, q^{\binom{k}{2}} x^k \{n-k+1\}_{k,q} (x \boxplus_q 1)^{n-k}. \tag{3.49}$$

Put $x = 1$ to get

$$\sum_{k=0}^{n} \binom{n}{k}_q q^{\binom{k}{2}} \{n-k\}_q^m = \sum_{k=0}^{\min(m,n)} S(m,k)_q\, q^{\binom{k}{2}} \{n-k+1\}_{k,q} (1 \boxplus_q 1)^{n-k}. \tag{3.50}$$

Put $x = -1$ to get (3.22).

Theorem 3.40. *A q-analogue of Cauchy [10, p. 161].*

$$D_q^m(\mathrm{E}_q(\alpha x) f(x)) = \mathrm{E}_q(\alpha x)(D_q \oplus_q \alpha\epsilon)^m f(x). \tag{3.51}$$

We continue with a few equations with operator proofs in the spirit of Gould and Schwatt.

Theorem 3.41. *Almost a q-analogue of [39, p 455, 4.9].*

$$\sum_{k=0}^{n} \{k\}_q{}^p x^k = \sum_{k=0}^{p} S(p,k)_q\, q^{\binom{k}{2}} x^k D_q^k \left(\frac{x^{n+1} - 1}{x - 1} \right). \tag{3.52}$$

We obtain the following limit.

Theorem 3.42.

$$\sum_{k=0}^{n} \{k\}_q{}^p = \sum_{k=0}^{p} S(p,k)_q \, q^{\binom{k}{2}} \lim_{x\to 1} D_q^k \left(\frac{x^{n+1}-1}{x-1} \right). \tag{3.53}$$

Theorem 3.43. *A q-analogue of [39, p. 456, 4.10], [75, p. 85 (38)], [40, p. 490].*

$$\sum_{k=0}^{\infty} \{k\}_q{}^p x^k = \sum_{k=0}^{p} S(p,k)_q q^{\binom{k}{2}} \frac{x^k \{k\}_q!}{(x;q)_{k+1}}, \ |x| < 1. \tag{3.54}$$

Proof. Let $n \to \infty$ in (3.52). □

Theorem 3.44. *A q-analogue of [39, p. 456, 4.11].*

$$\theta_q^n \mathrm{E}_q(x) = \mathrm{E}_q(x) \sum_{k=0}^{n} S(n,k)_q \, q^{\binom{k}{2}} x^k. \tag{3.55}$$

The following q-analogue of Bell numbers is the same as Milne [64, p. 99].

Definition 3.45. Compare [13, p. 104] $(x = 1)$. The q-Bell number is given by

$$B_q(n) \equiv \sum_{k=0}^{n} S(n,k)_q \, q^{\binom{k}{2}}. \tag{3.56}$$

Theorem 3.46. *The q-Dobinsky theorem [64, p. 108, 4.5] is q-analogue of [75, p. 84].*

$$B_q(n) = \mathrm{E}_{\frac{1}{q}}(-1) \sum_{k=0}^{\infty} \frac{\{k\}_q^n}{\{k\}_q!}. \tag{3.57}$$

This can be generalized as follows.

Theorem 3.47. *A q-analogue of [75, p 84 (26)].*

$$\sum_{k=0}^{\infty} \frac{\{k\}_q{}^p x^k}{\{k\}_q!} = \sum_{k=0}^{p} S(p,k)_q q^{\binom{k}{2}} x^k \mathrm{E}_q(x). \tag{3.58}$$

We will now return to the Carlitz q-analogue of [16, p. 13], [85, p. 575], [75, p. 86] for sums of powers. This function was also treated by Kim [58] from a different point of view.

Definition 3.48. Carlitz [8, p. 994–995].

$$S_{\mathrm{C},m,q}(n) \equiv \sum_{i=0}^{n-1} \{i\}_q^m q^i, \ S_{\mathrm{C},0,q}(1) \equiv 1. \tag{3.59}$$

We will now follow Cigler's computations ($q = 1$) and finally arrive at a formula which expresses $S_{\mathrm{C},m,q}(n)$ as a general double sum.

As a corollary, we obtain a result corresponding to an equation by Järvheden for sums of squares.

Remark 3.49. In [88] and [74], two completely different approaches to the problem of finding q-analogues of sums of consecutive powers of integers were presented.

Lemma 3.50. *A q-analogue of [16, p. 20, 2.6].*

$$\sum_{i=0}^{n-1}(\{i\}_q)_{j,q}q^i = \frac{(\{n\}_q)_{j+1,q}}{\{j+1\}_q},\ j < n. \tag{3.60}$$

Proof. Use induction on n. □

The Carlitz sum can be expressed as a double sum of q-Stirling numbers.

Corollary 3.51. *A q-analogue of [16, p. 35]. Compare [8, p. 994, 6.1].*

$$S_{\mathrm{C},m,q}(n) = \sum_j \frac{S(m,j)_q}{\{j+1\}_q}\sum_{l=0}^{j+1} s(j+1,l)_q\{n\}_q^l. \tag{3.61}$$

Proof.

$$\begin{aligned} LHS &= \sum_{i=1}^{n-1}\sum_{j=0}^{m} S(m,j)_q(\{i\}_q)_{j,q}q^i \\ &= \sum_{j=0}^{m} S(m,j)_q\frac{(\{n\}_q)_{j+1,q}}{\{j+1\}_q} = RHS. \end{aligned} \tag{3.62}$$

□

Tables of sums of powers have occupied mathematicians for centuries. Just as one example, De Moivre [20, 5] tabulated sums of powers up to $m = 10$. De Moivre's l corresponds to our $n-1$. As an example, we compute a q-analogue for $m = 2$.

Corollary 3.52. *A q-analogue of [57, p. 98].*

$$S_{\mathrm{C},2,q}(n) = \frac{(\{n\}_q)_{3,q}}{\{3\}_q} + \frac{(\{n\}_q)_{2,q}}{\{2\}_q},\ n \geq 2. \tag{3.63}$$

Proof. By (3.60),

$$\sum_{i=0}^{n-1}\{i\}_q(\{i\}_q - 1)q^i = \frac{(\{n\}_q)_{3,q}}{\{3\}_q}. \tag{3.64}$$

□

In general, $S_{\mathrm{C},m,q}(n)$ contains the factor $(\{n\}_q)_{2,q}$.

4 The Carlitz–Gould Approach

The following operators were introduced by Carlitz [8, p. 988] 1948. Schendel [73], Gould [36], Milne [64], Zeng [90] and Phillips [69] used the same technique. Applications from approximation theory can be found in Phillips [69]. Observe that the q-Stirling number of the second kind used by Milne [64, p. 93] is $q^{\binom{k}{2}}S(n,k)_q$.

Definition 4.1. The Carlitz–Gould q-difference is defined by

$$\triangle_{\mathrm{CG},q}^{1}f(x)\equiv f(x+1)-f(x),\ \triangle_{\mathrm{CG},q}^{n+1}f(x)\equiv \triangle_{\mathrm{CG},q}^{n}f(x+1)-q^{n}\triangle_{\mathrm{CG},q}^{n}f(x). \tag{4.1}$$

Remark 4.2. We get the above definition by putting $y=-1$ in Schendel [73].

Now follow the Carlitz–Gould quartet and two examples.

Theorem 4.3. *The following q-Taylor formula applies [8, 2.5 p. 988], [38, 7.2, p. 856], [36, 2.11, p. 91].*

$$f(x)=\sum_{k=0}^{\infty}\frac{\triangle_{\mathrm{CG},q}^{k}f(0)}{\{k\}_q!}\{x-k+1\}_{k,q}. \tag{4.2}$$

Proof. Apply $\triangle_{\mathrm{CG},q}^{s}$ to both members and finally put $x=0$. □

Theorem 4.4. *[37, p. 283, 2.13], [90], [36, 2.10, p. 91], [69, p. 46, 1.118], [64, p. 91] and a q-analogue of [16, p. 26]. Compare [73, p. 82].*

$$\triangle_{\mathrm{CG},q}^{n}f(x)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}_q q^{\binom{k}{2}}E^{n-k}f(x), \tag{4.3}$$

where the shift operator E is given by

$$E^n f(x)\equiv f(x+n). \tag{4.4}$$

Proof. Use induction. □

This formula can be inverted.

Theorem 4.5.

$$E^n f(x)=\sum_{i=0}^{n}\binom{n}{i}_q\triangle_{\mathrm{CG},q}^{i}f(x). \tag{4.5}$$

Proof. This is the general inversion formula again, compare the corrected version of [34, p. 244]. □

Corollary 4.6. *[69, p. 47 1.122], a q-analogue of [56, p. 97, 10], [16, p. 27, 2.13], [65, p. 35, 2]. Assume that the functions $f(x)$ and $g(x)$ depend on q^x. Then*

$$\triangle_{\mathrm{CG},q}^{n}(fg)=\sum_{i=0}^{n}\binom{n}{i}_q\triangle_{\mathrm{CG},q}^{i}f\triangle_{\mathrm{CG},q}^{n-i}E^i g. \tag{4.6}$$

Proof. Compare [56, p. 96 f].

$$
\begin{aligned}
&LHS = \sum_{k=0}^{n}(-1)^k \binom{n}{k}_q q^{\binom{k}{2}} E^{n-k} f\ E^{n-k} g = \\
&\sum_{k=0}^{n}(-1)^k \binom{n}{k}_q q^{\binom{k}{2}} \sum_{i=0}^{n-k} \binom{n-k}{i}_q \triangle_{\mathrm{CG},q}^{i} f\ E^{n-k} g = \\
&\sum_{i=0}^{n} \binom{n}{i}_q \triangle_{\mathrm{CG},q}^{i} f\ \sum_{k=0}^{n-i}(-1)^k \binom{n-i}{k}_q q^{\binom{k}{2}} E^{n-k} g = RHS.
\end{aligned}
\tag{4.7}
$$

□

Example 4.7.

$$
\triangle_{\mathrm{CG},q}^{n}(q^{mx}) = \begin{cases} q^{mx}(-1)^n q^{\binom{n}{2}} \langle 1+m-n;q\rangle_n,\ m \geq n. \\ 0,\ m < n. \end{cases} \tag{4.8}
$$

Proof. Use (4.3). □

Example 4.8. Compare [64, p. 92, 1.11]. For $m \leq n$.

$$
\triangle_{\mathrm{CG},q,x}^{m} \langle x+\gamma;q\rangle_n = \langle n-m+1;q\rangle_m \langle x+m+\gamma;q\rangle_{n-m} q^{m(x+\gamma+m-1)}. \tag{4.9}
$$

This is equivalent to the following formula from [36, 2.12, p. 91], a q-analogue of [65, p. 26, 6].

$$
\triangle_{\mathrm{CG},q}^{m} \binom{x}{n}_q = \binom{x}{n-m}_q q^{m(x+m-n)},\ m \leq n. \tag{4.10}
$$

Proof. Use induction. □

Theorem 4.9. *Milne [64, p. 93]. An exact formula for the second q-Stirling number with the CG-operator.*

$$
S(n,k)_q = (\{k\}_q! q^{\binom{k}{2}})^{-1} \triangle_{\mathrm{CG},q}^{n} \{x\}_q^n | x = 0. \tag{4.11}
$$

We will now follow Schwatt [75, ch. 5] and develop a calculus for the Carlitz function $S_{\mathrm{C},m,q}(n)$ from the previous chapter. First a lemma.

Lemma 4.10. *A q-analogue of [75, p. 86, (50)].*

$$
\sum_{s=k}^{n-1} \binom{s}{k}_q q^s = \binom{n}{k+1}_q q^k. \tag{4.12}
$$

Theorem 4.11. *A q-analogue of [75, p. 86, (51)].*

$$S_{\mathrm{C},m,q}(n) = \sum_{k=0}^{m} (-1)^k q^k \binom{n}{k+1}_q \sum_{a=0}^{k} (-1)^a \binom{k}{a}_q q^{\binom{k-a}{2}} \{a\}_q^m. \tag{4.13}$$

Proof. Write the LHS as

$$\theta_q^m \sum_{k=1}^{n-1} (xq)^k |_{x=1}, \tag{4.14}$$

and use (3.43), (4.12) and (3.22). □

Theorem 4.12. *A q-analogue of [75, p. 87 (63)].*

$$S_{\mathrm{C},m,q}(n) = \sum_{k=0}^{n} \binom{n}{k}_q \sum_{a=0}^{k} (-1)^a \binom{k}{a}_q q^{\binom{a}{2}} \sum_{i=1}^{k-a-1} \{i\}_q^m q^i. \tag{4.15}$$

Proof. Use (4.2) and (4.3). □

Now let $b_{m,k,q}$ denote the coefficient of $q^k \binom{n}{k+1}_q$ in $S_{\mathrm{C},m,q}(n)$. Then by (4.13) we obtain the following recurrence, which is almost a q-analogue of [75, p. 88 (69)].

$$b_{m,k,q} - \{k\}_q b_{m-1,k,q} = q^{k-1}\{k\}_q b_{m-1,k-1,q}. \tag{4.16}$$

We obtain the following expressions for $S_{\mathrm{C},m,q}(n)$ expressed as linear combinations of q-binomial coefficients.

Theorem 4.13. *Almost a q-analogue of Schwatt [75, p. 88 (69)].*

$$S_{\mathrm{C},1,q}(n) = q\binom{n}{2}_q. \tag{4.17}$$

$$S_{\mathrm{C},2,q}(n) = q\binom{n}{2}_q + q^3(1+q)\binom{n}{3}_q. \tag{4.18}$$

$$\begin{aligned} S_{\mathrm{C},3,q}(n) = {} & q\binom{n}{2}_q + q^3(1+q+(1+q)^2)\binom{n}{3}_q + \\ & q^6(1+q)(1+q+q^2)\binom{n}{4}_q. \end{aligned} \tag{4.19}$$

$$\begin{aligned} S_{\mathrm{C},4,q}(n) = {} & q\binom{n}{2}_q + q^3(1+q)(2+q+(1+q)^2)\binom{n}{3}_q + q^6(1+q+q^2) \\ & \times (1+q+(1+q)^2+\{3\}_q!)\binom{n}{4}_q + q^{10}\{4\}_q!\binom{n}{5}_q. \end{aligned} \tag{4.20}$$

By the q-Pascal identity we obtain the following q-analogue of Munch [66, p. 14].

Theorem 4.14.

$$S_{\mathrm{C},2,q}(n) = q^3 \binom{n}{3}_q + q \binom{n+1}{3}_q. \tag{4.21}$$

$$S_{\mathrm{C},3,q}(n) = q^6 \binom{n}{4}_q + 2q^3(1+q) \binom{n+1}{4}_q + q \binom{n+2}{4}_q. \tag{4.22}$$

$$\begin{aligned} S_{\mathrm{C},4,q}(n) = q^{10} \binom{n}{5}_q + q^6(3+5q+3q^2) \binom{n+1}{5}_q + \\ + q^3(3+5q+3q^2) \binom{n+2}{5}_q + q \binom{n+3}{5}_q. \end{aligned} \tag{4.23}$$

We can now introduce the sum operator mentioned in the introduction.

Definition 4.15. The inverse CG difference is defined by

$$\triangle_{\mathrm{CG},q}^{-1} f(k)|_0^n \equiv \sum_0^n f(x) \delta_q(x) \equiv \sum_{k=0}^{n-1} f(k). \tag{4.24}$$

Example 4.16.

$$\triangle_{\mathrm{CG},q}^{-1} (1-q^l) \langle n+1;q \rangle_{l-1}\, q^n \equiv \sum_{k=0}^{n-1} (1-q^l) \langle k+1;q \rangle_{l-1} q^k = \langle n;q \rangle_l. \tag{4.25}$$

Corollary 4.17.

$$\{n\}_q \{n+1\}_q = \{2\}_q \sum_{i=1}^n \{i\}_q q^{i-1}. \tag{4.26}$$

$$\sum_{i=1}^n \{i\}_q q^{2i} = \frac{\{n\}_{2,q}}{\{2\}_q} - \frac{\{n\}_{3,q}(1-q)}{\{3\}_q}. \tag{4.27}$$

It is possible to develop a calculus similar to $S_{\mathrm{C},m,q}(n)$ for the sum (4.27), but we have not pursued this path.

By (4.3) and (4.9) we obtain the following.

Theorem 4.18. *[67, p. 110].*

$$\begin{aligned} &\sum_{n=0}^m (-1)^n \binom{m}{n}_q q^{\binom{n}{2}} \langle x+1;q \rangle_{m-n} \langle x-y+1-n+m;q \rangle_n = \\ &\langle y-m+1;q \rangle_m q^{m(x-y+m)},\ x,y \in \mathbb{C}. \end{aligned} \tag{4.28}$$

We will return to this equation in the next chapter. Now we say goodbye to the Carlitz–Gould approach and continue with a similar operator which has the advantage of being both a q-derivative and difference operator at the same time.

5 The Jackson q-derivative as Difference Operator

This chapter will be about how the Jackson q-derivative can be used as difference operator operating on the space of all q-shifted factorials. We illustrate the technique with some examples. The similarity with the operator from the previous chapter is striking and will apparently lead to many multiple q-equations. However it turns out that most of these are doublets, as is shown in the example from the last chapter.

For functions of q^x, the Cigler operator ϵ [13] will be replaced by E in q-Leibniz theorems as below.

Theorem 5.1.

$$D^n_{q,q^x}\langle\gamma+x;q\rangle_k=(-1)^n\{k-n+1\}_{n,q}\langle\gamma+x+n;q\rangle_{k-n}q^{\binom{n}{2}+n\gamma},n\leq k. \tag{5.1}$$

Example 5.2. We apply the operator D^m_{q,q^x} to (4.28). Then

$$\begin{aligned}&LHS=\sum_{n=0}^{m}(-1)^n\binom{m}{n}_q q^{\binom{n}{2}}\sum_{i=0}^{m}\binom{m}{i}_q D^i_{q,q^x}\langle x+1;q\rangle_{m-n}\\&E^iD^{m-i}_{q,q^x}\langle x-y+1-n+m;q\rangle_n=\sum_{n=0}^{m}(-1)^n\binom{m}{n}_q q^{\binom{n}{2}}\times\\&\binom{m}{m-n}_q(-1)^m\{m-n\}_q!q^{\binom{m-n}{2}+m-n}\{n\}_q!\,q^{\binom{n}{2}+n(-y+1-n+m)}\\&=(-1)^mq^{\binom{m}{2}+m}\{m\}_q!\sum_{n=0}^{m}(-1)^n\binom{m}{n}_q q^{\binom{n}{2}-ny}=\\&q^{m^2-my}\langle y+1-m;q\rangle_m\{m\}_q!=RHS.\end{aligned} \tag{5.2}$$

Now instead rewrite (4.28) in the form

$$\begin{aligned}&\sum_{n=0}^{m}\binom{m}{n}_q\langle x+1;q\rangle_{m-n}\langle y-x-m;q\rangle_nq^{n(x+m)+y(m-n)}=\\&\langle y-m+1;q\rangle_mq^{m(x+m)},\ x,y\in\mathbb{C},\end{aligned} \tag{5.3}$$

and operate with D^m_{q,q^y} on both sides to obtain

$$\begin{aligned}&LHS=\sum_{n=0}^{m}\binom{m}{n}_q\langle x+1;q\rangle_{m-n}q^{n(x+m)}\sum_{i=0}^{m}\binom{m}{i}_q D^i_{q,q^y}q^{y(m-n)}\\&E^iD^{m-i}_{q,q^y}\langle y-x-m;q\rangle_n=\sum_{n=0}^{m}\binom{m}{n}_q\langle x+1;q\rangle_{m-n}q^{n(x+m)}\times\\&\binom{m}{m-n}_q\{m-n\}_q!(-1)^n\{n\}_q!q^{\binom{n}{2}-n(x+m)}\\&=\{m\}_q!\sum_{n=0}^{m}\binom{m}{n}_q\langle x+1;q\rangle_{m-n}(-1)^nq^{\binom{n}{2}}.\end{aligned} \tag{5.4}$$

The RHS is

$$(-1)^m \{m\}_q! q^{\binom{m}{2}+m(x+1)}. \tag{5.5}$$

After simplification this last equality is equivalent to a confluent form of the second q-Vandermonde identity.

Inspired by the previous calculation we make the following definition.

Definition 5.3. The Jackson q-difference is defined by

$$\begin{aligned} &\triangle_{\mathrm{J},x,q} f(q^x) \equiv \triangle_{\mathrm{J},q} f(q^x) \\ &\equiv (f(q^{x+1}) - f(q^x))q^{-x} \equiv -(1-q) D_{q,q^x} f(q^x), \end{aligned} \tag{5.6}$$

$$\triangle_{\mathrm{J},q}^{n+1} = \triangle_{\mathrm{J},q} \triangle_{\mathrm{J},q}^{n}. \tag{5.7}$$

The following equation is obtained.

Theorem 5.4.

$$\triangle_{\mathrm{J},q} \left(q^{\binom{k}{2}} \binom{x}{k}_q \right) = q^{\binom{k-1}{2}} \binom{x}{k-1}_q. \tag{5.8}$$

Proof. Use the q-Pascal identity. □

Corollary 5.5.

$$\triangle_{\mathrm{J},q}^{m} \binom{x}{n}_q = \binom{x}{n-m}_q q^{-mn+\binom{m+1}{2}},\ m \le n. \tag{5.9}$$

Example 5.6.

$$\sum_{n=0}^{\infty} q^{\binom{n}{2}} \binom{x}{n}_q t^n = \mathrm{e}_q(-tq^x \boxplus_q t). \tag{5.10}$$

Proof. Use the q-binomial theorem. □

We now present the Jackson quartet.

Theorem 5.7. *The following q-Taylor formula applies.*

$$f(x) = \sum_{k=0}^{\infty} \binom{x}{k}_q q^{\binom{k}{2}} \triangle_{\mathrm{J},q}^{k} f(0). \tag{5.11}$$

Theorem 5.8. *A q-analogue of [16, p. 26]. Compare [73, p. 82].*

$$\triangle_{\mathrm{J},q}^{n} f(q^x) = q^{-nx-\binom{n}{2}} \sum_{k=0}^{n} (-1)^k \binom{n}{k}_q q^{\binom{k}{2}} E^{n-k} f(q^x). \tag{5.12}$$

Proof. Use the corresponding equation for the q-derivative. □

This formula can be inverted.

Theorem 5.9.

$$E^n f(q^x) = \sum_{k=0}^{n} q^{xk+\binom{k}{2}} \binom{n}{k}_q \triangle_{\mathrm{J},q}^k f(q^x). \tag{5.13}$$

Corollary 5.10. *A q-analogue of [56, p. 97, 10], [16, p. 27, 2.13], [65, p. 35, 2].*

$$\triangle_{\mathrm{J},q}^n (f(q^x)g(q^x)) = \sum_{k=0}^{n} \binom{n}{k}_q \triangle_{\mathrm{J},q}^k f(q^x)(\triangle_{\mathrm{J},q}^{n-k} E^k) g(q^x). \tag{5.14}$$

Proof. Use the Leibniz theorem for the q-derivative. □

Example 5.11.

$$\triangle_{\mathrm{J},q}^n (q^{mx}) = \begin{cases} q^{x(m-n)}(-1)^n \langle 1+m-n; q\rangle_n, \; m \geq n. \\ 0, \; m < n. \end{cases} \tag{5.15}$$

6 Applications

The developed technique leads to easy proofs of q-binomial coefficient identities. The following example can also be proved from the q-Vandermonde identity.

Example 6.1. A q-analogue of the important formula [16, p. 27], [70, p. 15, (9)], [71, p. 65].

$$\begin{aligned} &\binom{x}{m}_q \binom{x}{n}_q = \sum_{k=0}^{m+n} \mathrm{QE}\left((k-n)(k-m)\right) \times \\ &\binom{k}{n}_q \binom{n}{m+n-k}_q \binom{x}{k}_q . \end{aligned} \tag{6.1}$$

Proof. We have

$$\begin{aligned} &\triangle_{\mathrm{CG},q}^k \binom{x}{m}_q \binom{x}{n}_q = \sum_{l=0}^{k} \binom{k}{l}_q \mathrm{QE}\left(l(x+l-n)+(k-l)(x+k-m)\right) \times \\ &\binom{x+l}{m-k+l}_q \binom{x}{n-l}_q . \end{aligned} \tag{6.2}$$

Now use the q-Taylor formula (4.2) with $y=0$, $f(x) = \binom{x}{m}_q \binom{x}{n}_q$. □

Remark 6.2. If we use $\triangle_{\mathrm{J},q}$ instead, we get

$$\begin{aligned}\binom{x}{m}_q\binom{x}{n}_q = \sum_{k=0}^{m+n} \mathrm{QE}\left(-\binom{m}{2}-\binom{n}{2}+\binom{m+n-k}{2}+\binom{k}{2}\right)\times \\ \binom{k}{n}_q\binom{n}{m+n-k}_q\binom{x}{k}_q.\end{aligned} \tag{6.3}$$

This equation is however equivalent to (6.1).

Acknowledgments. I would like to thank Don Knuth and Johann Cigler for their kind advice. Karl Dilcher's Bernoulli bibliography has also been an invaluable help. Several computations have been checked by *Mathematica*.

References

[1] N.H. Abel, *OEuvres completes de Niels Henrik Abel* Christiana: Imprimerie de Grondahl and Son; New York and London: Johnson Reprint Corporation. VIII, 621 p. (1965).

[2] L. Aceto, and D. Trigiante, The matrices of Pascal and other greats. *Amer. Math. Monthly* 108 (2001), no. 3, 232–245.

[3] M. Aigner, *Combinatorial theory*. Grundlehren der Mathematischen Wissenschaften 234. Springer-Verlag, Berlin-New York, 1979.

[4] E.G. Björling, In determinationem coefficientium …*Crelle J*. **28**. (1844), 284–288.

[5] G. Boole, On a general method in analysis. *Philosophical transactions*, 134, 225–282 (1844).

[6] G. Boole, *A treatise on the calculus of finite differences*. 2nd ed. 1872.

[7] F. Cajori, *A History of Mathematical Notations*. V2. Chicago 1929.

[8] L. Carlitz, q-Bernoulli numbers and polynomials. *Duke math. J.* 15 (1948), 987–1000.

[9] L. Carlitz, Stirling pairs. *Rend. Sem. Mat. Univ. Padova* 59 (1978), 19–44.

[10] A. Cauchy, Exercices de mathématique, seconde année. Paris 1827.

[11] A. Cayley, Table of $\Delta^m 0^n \div \Pi(m)$ up to $m = n = 20$. *Cambr. Trans*. XIII. 1–4. (1881)

[12] S.K. Chatterjea, Operational formulae for certain classical polynomials. I. *Quart. J. Math. Oxford* Ser. (2) 14 (1963), 241–246.

[13] J. Cigler, Operatormethoden für q-Identitäten. *Monatshefte für Math.* **88**,(1979), 87–105.

[14] J. Cigler, Operatormethoden für q-Identitäten II: q-Laguerre-Polynome. *Monatshefte für Math.* **91**,(1981) 105–117.

[15] J. Cigler, A characterisation of the q-exponential polynomials. (Eine Charakterisierung der q-Exponentialpolynome.) *Sitzungsber., Abt. II, sterr. Akad. Wiss., Math.-Naturwiss. Kl.* **208**, 143–157 (1999).

[16] J. Cigler, *Differenzenrechnung*. Wien 2001.

[17] J. Cigler, *Elementare q-Identitäten* Skriptum Wien 2006

[18] D. Crippa, and K. Simon, and P. Trunz, Markov processes involving q-Stirling numbers. *Comb. Probab. Comput.***6**, No.2, 165–178 (1997)

[19] L. Comtet, *Advanced combinatorics*. Reidel 1974.

[20] A. De Moivre, *Miscellanea Analytica*. London 1730.

[21] A. De Morgan, *The differential and integral Calculus*Baldwin and Cradock, London 1842. Elibron Classics, 2002.

[22] T. Ernst, *The history of q-calculus and a new method,* Uppsala 2000.

[23] T. Ernst, q-Generating functions for one and two variables. *Simon Stevin*, **12** no. 4, 2005, 589–605.

[24] T. Ernst, *A new method for q-calculus,* Uppsala dissertations 2002.

[25] T. Ernst, A method for q-calculus. *J. nonlinear Math. Physics* **10** No.4 (2003), 487–525.

[26] T. Ernst, Some results for q-functions of many variables. *Rendiconti di Padova*,**112** (2004), 199–235.

[27] T. Ernst, *q-Analogues of some operational formulas*. Preprint 2003.

[28] T. Ernst, q-Bernoulli and q-Euler Polynomials, An Umbral Approach. *International journal of difference equations* **1** no. 1 2006, 13–62.

[29] L. Euler, *Institutiones calculi differentialis* 1755, new printing Birkhuser 1913.

[30] C-E. Fröberg, *Lärobok i numerisk analys*. Stockholm 1962.

[31] G. Gasper, and M. Rahman, *Basic hypergeometric series,* Cambridge, 1990.

[32] C.F. Gauss, *Werke 2*. 1876, 9–45.

[33] I.M. Gelfand, and M.I. Graev, and V.S. Retakh, General hypergeometric systems of equations and series of hypergeometric type. (Russian)*Uspekhi Mat. Nauk* **47** (*1992*), no. 4, (286),3–82, 235, translation in *Russian Math. Surveys* **47** (1992), no. 4 1–88.

[34] J. Goldman, and G.C. Rota, On the foundations of combinatorial theory. IV. Finite vector spaces and Eulerian generating functions. *Stud. Appl. Math.* **49** (*1970*), 239–258.

[35] H.H. Goldstine, *A history of numerical analysis from the 16th through the 19th century. Studies in the History of Mathematics and Physical Sciences*, Vol. 2. Springer 1977.

[36] H.W. Gould, The q-series generalization of a formula of Sparre Andersen. *Math. Scand.* **9** (1961), 90–94.

[37] H.W. Gould, The q-Stirling numbers of first and second kinds. *Duke Math. J.* **28** (1961), 281–289.

[38] H.W. Gould, The operator $(a^x\Delta)^n$ and Stirling numbers of the first kind. *Amer. Math. Monthly* **71** (1964), 850–858.

[39] H.W. Gould, Euler's formula for n-th differences of powers. *Am. Math. Mon.* **85**, 450–467 (1978).

[40] H.W. Gould, Evaluation of sums of convolved powers using Stirling and Eulerian numbers. *The Fibonacci quarterly* **16**, 6 (1978), 488–497.

[41] R.L. Graham, and D.E. Knuth, and O. Patashnik, *Concrete mathematics. A foundation for computer science*. Second edition. Addison-Wesley Publishing Company, Reading, MA, 1994.

[42] J.A. Grünert, Math. Abhandlungen, Erste Sammlung, Altona: Hammerich 1822.

[43] J.A. Grünert, Summierung der Reihe... *Borchardt J.* **2**. (1827), 358–363.

[44] J.A. Grünert, Über die Summierung der Reihen von der Form . . . *Borchardt J*. **25**. (1843), 240–279.

[45] C. Gudermann, *Grundriss der analytischen Sphärik*. Köln 1830.

[46] C. Gudermann, *Potenzial oder cyklisch-hyperbolische Functionen*. Berlin 1833.

[47] W. Hahn, Über Orthogonalpolynome, die q-Differenzengleichungen genügen. *Mathematische Nachrichten* **2** (1949), 4–34.

[48] E. Heine, Über die Reihe... *J. reine angew. Math*. **32**, (1846) 210–212.

[49] J. Herschel, *Collection of examples of the applications of calculus of finite differences.* Cambridge 1820.

[50] C.F. Hindenburg, *Archiv der reinen und angewandten Mathematik*. Leipzig, 1795.

[51] C.F. Hindenburg, *Über combinatorische Analysis und Derivations-calcul*. Leipzig 1803.

[52] F.H. Jackson, On q-difference equations. *American J. Math.* **32** (*1910*), 305–314.

[53] F.H. Jackson, On basic double hypergeometric functions. *Quart. J. Math.* **13** (1942) 69–82.

[54] W.P. Johnson, Some applications of the q-exponential formula. *Proceedings of the 6th Conference on Formal Power Series and Algebraic Combinatorics* (New Brunswick, NJ, 1994). *Discrete Math*. **157** (1996), no. 1-3, 207–225.

[55] Ch. Jordan, *Statistique mathématique*. Paris, Gauthier–Villars. (1927)

[56] Ch. Jordan, *Calculus of finite differences*. Third Edition. Chelsea Publishing Co., New York 1950.

[57] B. Järvheden, *Elementär kombinatorik* Studentlitteratur 1976.

[58] T. Kim, Sums of powers of consecutive q-integers. *Adv. Stud. Contemp. Math. (Kyungshang)* **9** (2004), no. 1, 15–18.

[59] F. Klein, *Vorlesungen ber die Entwickelung der Mathematik im 19. Jahrhundert*. Bd. I+II. Bearbeitet von R. Courant und St. Cohn–Vossen. Springer 1979.

[60] D. Knuth, Two notes on notation. *Am. Math. Mon*. **99**, No.5, 403–422 (1992).

[61] D. Knuth, *The Art of Computer Programming,* Volume 4, to be published 2005.

[62] B.A. Kupershmidt, q-Newton binomial: from Euler to Gauss. *J. Nonlinear Math. Phys.* **7** (*2000*), no. 2, 244–262. 123–140.

[63] Z.A. Melzak, *Companion to concrete mathematics*, vol. 1. Mathematical techniques and various applications. Pure and Applied Mathematics. John Wiley & Sons, New York-London-Sydney, 1973.

[64] S.C. Milne, q-analog of restricted growth functions, Dobinski's equality, and Charlier polynomials. *Trans. Amer. Math. Soc*. **245** (1978), 89–118.

[65] L.M. Milne-Thomson, *The Calculus of Finite Differences*. Macmillan and Co., Ltd., London, 1951.

[66] O.J. Munch, On power product sums. *Nordisk Mat. Tidskr*. **7** (1959), 5–19.

[67] M. Nishizawa, Evaluation of a certain q-determinant. *Linear Algebra Appl.* **342** (*2002*), 107–115.

[68] N.E. Nörlund, Mémoire sur les polynomes de Bernoulli. *Acta Math*. **43** (1920), 121–196.

[69] G.M. Phillips, Interpolation and approximation by polynomials. *CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC,* 14. Springer-Verlag, New York, 2003.

[70] J. Riordan, *Combinatorial identities*. Reprint of the 1968 original. Robert E. Krieger Publishing Co., Huntington, N.Y., 1979.

[71] *Finite operator calculus*. Edited by Gian-Carlo Rota. With the collaboration of P. Doubilet, C. Greene, D. Kahaner, A. Odlyzko and R. Stanley. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1975.

[72] L. Saalschütz, Vorlesungen über die Bernoulli'schen Zahlen, ihren Zusammenhang mit den Secanten-Coefficienten und ihre wichtigeren Anwendungen. Berlin. J. Springer. VIII + 208 S. 8°. (1893)

[73] L. Schendel, Zur Theorie der Functionen. (x) *Borchardt J*. **LXXXIV**. 80–84. (1877)

[74] M. Schlosser, q-analogues of the sums of consecutive integers, squares, cubes, quarts and quints. *Electron. J. Combin*. **11** (2004), no. 1

[75] I.J. Schwatt, *An introduction to the operations with series*. Philadelphia: The Press of the University ol Pennsylvannia, 1924.

[76] D.B. Sears, On the transformation theory of basic hypergeometric functions. (English)Proc. London Math. Soc (2) **53**, (1951) 158–180.

[77] A. Sharma, and A.M. Chak, The basic analogue of a class of polynomials. *Riv. Mat. Univ. Parma* **5** (1954), 325–337.

[78] E. Stephens, *The elementary theory of operational mathematics*. McGraw-Hill (1937).

[79] J. Stirling, *Methodus differentialis*. London, 1730.

[80] L. Toscano, Sull'iterazione degli operatori xD e Dx. (On integration of the operators xD and Dx). *[J] Ist. Lombardo, Rend.*, II. Ser. **67**, 543–551 (1934).

[81] L. Toscano, Differenze finite e derivate dei polinomi di Iacobi. (Italian) *Matematiche*, Catania **10** (1955), 44–56.

[82] L. Toscano, Numeri di Stirling generalizzati e operatori permutabili di secondo ordine. (Italian) *Matematiche* (Catania) **24** (1969), 492–518.

[83] L. Toscano, L'operatore xD e i numeri di Bernoulli e di Eulero. (Italian) *Matematiche* **31**(1976), 63–89 (1977).

[84] A.T. Vandermonde, *Histoire de l'Académie Royale des Sciences* 1772, p. 489–498.

[85] H.S. Vandiver, Simple explicit expressions for generalized Bernoulli numbers of the first order. *Duke Math. J*. **8** (1941), 575–584.

[86] R. Vein and P. Dale, *Determinants and their applications in mathematical physics*. Applied Mathematical Sciences, 134. Springer-Verlag, New York, 1999.

[87] M. Ward, A calculus of sequences. *Amer. J. Math.* **58** (*1936*), 255–266.

[88] O. Warnaar, *Electron. J. Combin.* **11** (2004).

[89] J. Worpitzky, Studien über die Bernoullischen und Eulerschen Zahlen. *J. f. Math.* **94** (1883), 203–232.

[90] J. Zeng, The q-Stirling numbers, continued fractions and the q-Charlier and q-Laguerre polynomials. *J. Comput. Appl. Math*. **57** (1995), no. 3, 413–424.