DEMONSTRATIO MATHEMATICA
Vol. XLIV No 2 2011

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q-ANALOGUES OF GENERAL REDUCTION FORMULAS BY BUSCHMAN AND SRIVASTAVA AND AN IMPORTANT q-OPERATOR REMINDING OF MACROBERT

Abstract. We find four q-analogues of general reduction formulas from Buschman and Srivastava together with some special cases, e.g. q-analogues of reduction formulas for Appell- and Kampé de Fériet functions. A proper q-analogue of the notation $\triangle(l;\lambda)$ by MacRobert, Meijer and Srivastava is given, and the definition of q-hypergeometric series is generalized accordingly.

1. Introduction

The umbral method for q-calculus [2] - [8], consisting of logarithmic q-shifted factorials, the tilde operator, a comfortable notation for q-powers, the symbol for real infinity, equivalent to the zero in Gasper-Rahman [10], the q–Kampé de Fériet function, compare with [3], are the main ingredients in this new method, which will increase our knowledge of q-calculus, advocated in the beginning of the last century by the late Cambridge student, reverend F. H. Jackson. All the topics above are not new; they have been presented in the book [9].

In this article, the important notation $\triangle(l;\lambda)$ of MacRobert [11], Meijer and Srivastava [15] for a certain array of l parameters is given its proper q-analogue with the aid of a generalized tilde operator; in this paper we only consider the cases $l = 2, 3$, but a general definition is given. A deep knowledge of the $\triangle(l;\lambda)$ operator is necessary to grasp the subtleties of multiple hypergeometric functions. This $\triangle$-operator has a very long history in the field of special functions, in particular in India, which we will come back to in later papers.

2000 *Mathematics Subject Classification*: Primary 33D70; Secondary 33D15.
Key words and phrases: Buschman and Srivastava reduction formulas, MacRobert $\triangle(l;\lambda)$ operator; reduction formulas for Appell functions.

Buschman and Srivastava [1] have proved a great number of double series identities with general terms. We will find q-analogues of most of these formulas like in [3]; the method of proof will be similar except that we now use the q-Dixon- and q-Watson summation formulas. Some of the obtained formulas are symmetric in two variables, just as in the undeformed case. We pick out a form of these formulas, which converges nicely for small values of x. A list of different formulations of the Buschman–Srivastava formulas and their q-analogues in various journals and books is given, for better orientation.

This paper is organized as follows: In this section we give a general introduction. In section 2, four q-analogues of Buschman–Srivastava formulas are given. In section 3, we apply the Buschman–Srivastava formulas to find q-analogues of reduction formulas for Appell and Kampé de Fériet functions; the $\triangle$ operator appears only in the Heine function. In other papers, the $\triangle$ operator can appear also in the q-Kampé de Fériet function.

Definition 1. The power function is defined by $q^a \equiv e^{alog(q)}$. Let $\delta > 0$ be an arbitrary small number. We will use the following branch of the logarithm: $-\pi + \delta < \mathrm{Im}(\log q) \leq \pi + \delta$. This defines a simply connected space in the complex plane.

The variables $a, b, c, \ldots \in \mathbb{C}$ denote certain parameters. The variables i, j, k, l, m, n, p, r will denote natural numbers except for certain cases where it will be clear from the context that i will denote the imaginary unit.

Let the q-shifted factorial be defined by

$$\langle a; q\rangle_n \equiv \begin{cases} 1, & n = 0; \\ \prod_{m=0}^{n-1}(1 - q^{a+m}) & n = 1, 2, \ldots. \end{cases} \tag{1}$$

Since products of q-shifted factorials occur so often, to simplify them we shall frequently use the more compact notation

$$\langle a_1, \ldots, a_m; q\rangle_n \equiv \prod_{j=1}^{m} \langle a_j; q\rangle_n. \tag{2}$$

Let the Γ_q-function be defined in the unit disk $0 < |q| < 1$ by

$$\Gamma_q(x) \equiv \frac{\langle 1; q\rangle_\infty}{\langle x; q\rangle_\infty}(1 - q)^{1-x}. \tag{3}$$

The following notation will prove convenient, since many of our formulas contain exponents with upper and lower indices, which become less legible in the Gasper–Rahman notation.

$$\mathrm{QE}(x) \equiv q^x. \tag{4}$$

The operator

$$\widetilde{} : \frac{\mathbb{C}}{\mathbb{Z}} \mapsto \frac{\mathbb{C}}{\mathbb{Z}}$$

is defined by

$$a \mapsto a + \frac{\pi i}{\log q}. \tag{5}$$

By (5) it follows that

$$\widetilde{\langle a; q \rangle_n} = \prod_{m=0}^{n-1} (1 + q^{a+m}). \tag{6}$$

Assume that $(m, l) = 1$, i.e. m and l relatively prime. The operator

$$\widetilde{\tfrac{m}{l}} : \frac{\mathbb{C}}{\mathbb{Z}} \mapsto \frac{\mathbb{C}}{\mathbb{Z}}$$

is defined by

$$a \mapsto a + \frac{2\pi i m}{l \log q}. \tag{7}$$

We will also need another generalization of the tilde operator.

$$_k\langle \widetilde{a}; q \rangle_n \equiv \prod_{m=0}^{n-1} \Big(\sum_{i=0}^{k-1} q^{i(a+m)} \Big). \tag{8}$$

This leads to the following q-analogue of [12, p.22, (2)].

Theorem 1.1 ([6]).

$$\langle a; q \rangle_{kn} = \prod_{m=0}^{k-1} \Big\langle \frac{a+m}{k}; q \rangle_n \times_k \langle \widetilde{\frac{a+m}{k}}; q \Big\rangle_n. \tag{9}$$

Definition 2. A q-analogue of a notation due to Thomas MacRobert (1884–1962) [11, p. 135] and Srivastava [15]. This notation was also often used for the Meijer G-function and the Fox H-function ($q = 1$).

$$\langle \triangle(q; l; \lambda); q \rangle_n \equiv \prod_{m=0}^{l-1} \Big\langle \frac{\lambda+m}{l}; q \Big\rangle_n \times_l \Big\langle \widetilde{\frac{\lambda+m}{l}}; q \Big\rangle_n. \tag{10}$$

When λ is a vector, we mean the corresponding product of vector elements. When λ is replaced by a sequence of numbers separated by commas, we mean the corresponding product as in the case of q-shifted factorials. The last factor in (10) corresponds to l^{nl}.

1.1. Definition of the q-Kampé de Fériet function We will give a definition reminding of [10], which allows easy confluence to diminish the dimension in (12), and has the advantage of beeing symmertic in the variables. Furthermore, q is allowed to be a vector and the full machinery of tilde operators and q-additions will be used.

In the following two definitions we put

$$\widehat{a} \equiv a \vee \tilde{a} \vee \widetilde{\tfrac{m}{n} a} \vee_k \tilde{a} \vee \triangle(q;l;\lambda). \tag{11}$$

The following definition is a q-analogue of [16, (24), p. 38], in the spirit of Srivastava.

Definition 3. Let

$$(a),(b),(g_i),(h_i),(a'),(b'),(g_i'),(h_i')$$

have dimensions

$$A,B,G_i,H_i,A',B',G_i',H_i'.$$

Let

$$1+B+B'+H_i+H_i'-A-A'-G_i-G_i' \geq 0, i=1,\ldots,n.$$

Then the generalized q-Kampé de Fériet function is defined by

$$\Phi^{A+A':G_1+G_1';\ldots;G_n+G_n'}_{B+B':H_1+H_1';\ldots;H_n+H_n'}\left[\begin{matrix}(\hat{a}):(\hat{g_1});\ldots;(\hat{g_n})\\(\hat{b}):(\hat{h_1});\ldots;(\hat{h_n})\end{matrix}|\vec{q};\vec{x}||\begin{matrix}(a'):(g_1');\ldots;(g_n')\\(b'):(h_1');\ldots;(h_n')\end{matrix}\right] \tag{12}$$

$$\equiv \sum_{\vec{m}} \frac{\langle(\hat{a});q_0\rangle_m (a')(q_0,m)\prod_{j=1}^n(\langle(\hat{g_j});q_j\rangle_{m_j}((g_j')(q_j,m_j)x_j^{m_j})}{\langle(\hat{b});q_0\rangle_m (b')(q_0,m)\prod_{j=1}^n(\langle(\hat{h_j});q_j\rangle_{m_j}(h_j')(q_j,m_j)\langle 1;q_j\rangle_{m_j})} \times(-1)^{\sum_{j=1}^n m_j(1+H_j+H_j'-G_j-G_j'+B+B'-A-A')}$$

$$\times \mathrm{QE}\left((B+B'-A-A')\binom{m}{2},q_0\right)\prod_{j=1}^n \mathrm{QE}\left((1+H_j+H_j'-G_j-G_j')\binom{m_j}{2},q_j\right).$$

We assume that no factors in the denominator are zero. We assume that $(a')(q_0,m),(g_j')(q_j,m_j),(b')(q_0,m),(h_j')(q_j,m_j)$ contain factors of the form $\langle a(\hat{k});q\rangle_k$, $(s;q)_k,(s(k);q)_k$ or $\mathrm{QE}(f(\vec{m}))$.

Definition 4. Generalizing Heine's series we shall define a q hypergeometric series by

$${}_{p+p'}\phi_{r+r'}\left[\begin{matrix}\hat{a}_1,\ldots,\hat{a}_p\\\hat{b}_1,\ldots,\hat{b}_r\end{matrix}|q;z||\begin{matrix}\prod_i f_i(k)\\\prod_j g_j(k)\end{matrix}\right] \tag{13}$$

$$\equiv \sum_{k=0}^{\infty}\frac{\langle\hat{a}_1;q\rangle_k\ldots\langle\hat{a}_p;q\rangle_k}{\langle 1,\hat{b}_1;q\rangle_k\ldots\langle\hat{b}_r;q\rangle_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+r+r'-p-p'} z^k\frac{\prod_i f_i(k)}{\prod_j g_j(k)}.$$

We assume that the $f_i(k)$ and $g_j(k)$ contain p' and r' factors of the form $\langle \widehat{a(k)};q\rangle_k$ or $(s(k);q)_k$ respectively. In case of $\triangle(q;l;\lambda)$, the index is adapted accordingly. When we have a sequence of elements a_i, we can denote them by (A).

1.2. Two lemmata In the following three proofs we will use the finite q-Dixon theorem.

Theorem 1.2. [3, p. 210 (39)]

$$(14)\qquad \begin{cases} {}_4\phi_3\left[\begin{matrix} -2k, b, c, \widetilde{1-k} \\ 1-2k-b, 1-2k-c, \widetilde{-k}\end{matrix}|q;q^{1-k-b-c}\right] \\ \equiv \sum_{j=0}^{2k}\binom{2k}{j}_q \frac{\langle b,c,\widetilde{1-k};q\rangle_j(-1)^j \mathrm{QE}(\binom{j}{2}+j(1-3k-b-c))}{\langle 1-2k-b,1-2k-c,\widetilde{-k};q\rangle_j} \\ = \frac{\langle 1-2k-b-c,\widetilde{1-k},\widetilde{b};q\rangle_k}{\langle 1-2k-c,\widetilde{1-k-b},b+k;q\rangle_k}\langle \frac{1}{2};q^2\rangle_k; \\ {}_4\phi_3\left[\begin{matrix} -k,b,c,\widetilde{1-\frac{k}{2}} \\ 1-k-b,1-k-c,\widetilde{-\frac{k}{2}}\end{matrix}|q;q^{1-\frac{k}{2}-b-c}\right] = 0, k \textit{ odd}.\end{cases}$$

In another proof we will use a q-analogue of the Watson formula [1].

Theorem 1.3. [6, p. 170 (43)]

$$(15)\qquad {}_4\phi_3\left[\begin{matrix} \frac{c}{2}, \frac{\widetilde{c}}{2}, a, -N \\ \frac{-N+1+a}{2}, \widetilde{\frac{-N+1+a}{2}}, c\end{matrix}|q;q\right] = \begin{cases} = \frac{\left\langle \frac{1}{2}, \frac{1+c-a}{2};q^2\right\rangle_{\frac{N}{2}}}{\left\langle \frac{1-a}{2}, \frac{1+c}{2};q^2\right\rangle_{\frac{N}{2}}}, & \textit{if } N \textit{ even;} \\ 0, & \textit{if } N \textit{ odd.}\end{cases}$$

2. q-analogues of Buschman–Srivastava double sums

The Buschman–Srivastava paper [1] was a landmark for the studies of multiple q-hypergeometric series. Some of these formulas had previously been published in other form by Shanker and Saran [13]. Srivastava and Jain [17] have found q-analogues of some of these formulas, some of which are included in the book [9]. The following table summarizes the connection between the various formulas and the methods of proof; the four references are in chronological order.

[13]	[14]	[1]	[18]	Proof	Equation no.
–	(4)	3.2	33	q-Vandermonde	[3](62), (66)
–	(5)	3.7	49	finite Bailey-Daum	[3](68)
–	(17)	3.3	44	finite Bailey-Daum	[3](81)
b p. 10	–	2.7, 3.4	46	finite q-Dixon	(16)

[13]	[14]	[1]	[18]	Proof	Equation no.
c p.10	–	2.8, 3.6	48	finite q-Dixon	(18)
–	–	2.9, 3.8	50	finite q-Dixon	(20)
a p.10	–	3.10, 3.5	47	q-Watson	(23)

We are now going to find a number of general double sums. Since the convergence problem is rather delicate, we try to choose the most proper form with respect to an arbitrary q-power. Sometimes we add this q-power afterwards, to save space in the proof. In the following, a statement like $a \neq k$ will mean $a \neq k, k \in \mathbb{N}$. Everywhere the symbol $\{C_n\}_{n=0}^{\infty}$ denotes a bounded sequence of complex numbers. It is assumed that both sides converge. Note that the formulas (18), (20) and (23) are symmetric in two variables.

Theorem 2.1. *A q-analogue of Buschman, Srivastava [1, p. 437 (2.7)].*

$$\sum_{m,n} \frac{C_{m+n}x^{m+n}(-1)^n\langle g;q\rangle_m\langle g,\widetilde{1-\frac{m+n}{2}};q\rangle_n \mathrm{QE}\left(-\frac{n}{2}-\frac{nm}{2}\right)}{\langle 1,h;q\rangle_m\langle 1,h,\widetilde{-\frac{m+n}{2}};q\rangle_n}$$
$$= \sum_{N=0}^{\infty} \frac{C_{2N}x^{2N}\langle g,h-g,\widetilde{1-N};q\rangle_N}{\langle 1,\widetilde{1},h,\frac{h}{2},\widetilde{\frac{h}{2}},\frac{h+1}{2},\widetilde{\frac{h+1}{2}};q\rangle_N} q^{\binom{N}{2}+Ng}, \quad -h \neq k. \tag{16}$$

Proof.

$$LHS = \sum_{N,n} \frac{C_N x^N(-1)^n\langle g;q\rangle_{N-n}\langle g,\widetilde{1-\frac{N}{2}};q\rangle_n \mathrm{QE}\left(\binom{n}{2}-\frac{nN}{2}\right)}{\langle 1,h;q\rangle_{N-n}\langle 1,h,\widetilde{-\frac{N}{2}};q\rangle_n} \tag{17}$$

$$= \sum_{N=0} \frac{C_N x^N q^{N(g-h)}\langle -g+1-N;q\rangle_N}{\langle 1,-h+1-N;q\rangle_N}$$
$$\sum_{n} \frac{(-1)^n\binom{N}{n}_q\langle g,-h+1-N,\widetilde{1-\frac{N}{2}};q\rangle_n \mathrm{QE}\left(\binom{n}{2}+n(h-g-\frac{N}{2})\right)}{\langle h,-g+1-N,\widetilde{-\frac{N}{2}};q\rangle_n}$$

$$\overset{\text{by}(14)}{=} \sum_{N=0} \frac{C_{2N}x^{2N}q^{2N(g-h)}\langle -g+1-2N;q\rangle_{2N}}{\langle 1,-h+1-2N;q\rangle_{2N}} \Gamma_q\begin{bmatrix} 1-2N-g,h,1-N,h-g+N \\ 1-2N,h-g,1-N-g,h+N \end{bmatrix}$$

$$= \sum_{N=0} \frac{C_{2N}x^{2N}\langle h-g,\widetilde{1-N},\widetilde{g};q\rangle_N\langle -g+1-2N;q\rangle_{2N}\langle \frac{1}{2};q^2\rangle_N \mathrm{QE}\left(\binom{2N}{2}+2Ng\right)}{\langle h,\frac{h}{2},\widetilde{\frac{h}{2}},\frac{h+1}{2},\widetilde{\frac{h+1}{2}},N+g,\widetilde{1-N-g};q\rangle_N\langle 1;q\rangle_{2N}}$$

$$= \sum_{N=0} \frac{C_{2N}x^{2N}\langle g,h-g,\widetilde{1-N},\widetilde{g};q\rangle_N}{\langle 1,\widetilde{1},h,\frac{h}{2},\widetilde{\frac{h}{2}},\frac{h+1}{2},\widetilde{\frac{h+1}{2}},\widetilde{1-N-g};q\rangle_N} = RHS. \blacksquare$$

Theorem 2.2. *A q-analogue of* [1, p. 438 (2.8)].

$$\sum_{m,n} \frac{C_{m+n}x^{m+n}(-1)^n\langle g,h;q\rangle_m\langle g,h,\widetilde{1-\frac{m+n}{2}};q\rangle_n}{\langle 1;q\rangle_m\langle 1,\widetilde{-\frac{m+n}{2}};q\rangle_n}\mathrm{QE}\left(\frac{-n+mn}{2}\right) \tag{18}$$

$$= \sum_{N=0}^{\infty} C_{2N}x^{2N}\frac{\langle g,h,\widetilde{1-N},\frac{g+h}{2},\frac{g+h+1}{2},\widetilde{\frac{g+h}{2}},\widetilde{\frac{g+h+1}{2}};q\rangle_N q^{\binom{N}{2}}}{\langle 1,\widetilde{1},g+h;q\rangle_N}, \quad -h-g\neq k.$$

Proof. We prove an equivalent formula.

$$\sum_{m,n} \frac{C_{m+n}x^{m+n}(-1)^n\langle g,h;q\rangle_m\langle g,h,\widetilde{1-\frac{m+n}{2}};q\rangle_n}{\langle 1;q\rangle_m\langle 1,\widetilde{-\frac{m+n}{2}};q\rangle_n}\mathrm{QE}\left(\frac{(n-3mn)}{2}+m\right) \times\mathrm{QE}\left(-m^2-n^2-(m+n)(g+h)\right) \tag{19}$$

$$= \sum_{N,n} C_N x^N(-1)^n\langle g,h;q\rangle_{N-n}\langle g,h,\widetilde{1-\frac{N}{2}};q\rangle_n \times\frac{\mathrm{QE}\left(\frac{(-n^2-n+Nn)}{2}+N-N^2-N(g+h)\right)}{\langle 1;q\rangle_{N-n}\langle 1,\widetilde{-\frac{N}{2}};q\rangle_n}$$

$$= \sum_{N,n} C_N x^N(-1)^n \frac{\binom{N}{n}_q\langle -g+1-N,-h+1-N;q\rangle_N\langle g,h,\widetilde{1-\frac{N}{2}};q\rangle_n\mathrm{QE}\left(\binom{n}{2}+n(1-h-g-\frac{3N}{2})\right)}{\langle 1;q\rangle_N\langle -h+1-N,-g+1-N,\widetilde{-\frac{N}{2}};q\rangle_n}$$

$$\overset{\text{by}(14)}{=} \sum_{N=0} \frac{C_{2N}x^{2N}\langle\widetilde{1-N},\widetilde{g},1-2N-g-h;q\rangle_N\langle 1-2N-g,1-2N-h;q\rangle_{2N}\langle\frac{1}{2};q^2\rangle_N}{\langle g+N,\widetilde{1-N}-g,1-2N-h;q\rangle_N\langle 1;q\rangle_{2N}}$$

$$= \sum_{N=0}^{\infty} \frac{C_{2N}x^{2N}\langle\widetilde{1-N},\widetilde{g},g+h+N;q\rangle_N\langle g,h;q\rangle_{2N}\mathrm{QE}\big(-4N^2+2N-N(3g+2h)\big)}{\langle N+h,\widetilde{1-N}-g,g+N,1,\widetilde{1};q\rangle_N}$$

$$= \sum_{N=0}^{\infty} \frac{C_{2N}x^{2N}\langle g,h,\widetilde{1-N},\triangle(q;2;g+h);q\rangle_N\mathrm{QE}\big(\binom{N}{2}-4N^2+2N-N(2g+2h)\big)}{\langle 1,\widetilde{1},g+h;q\rangle_N}.$$

Finally, multiply C_n by $\mathrm{QE}\left(2\binom{n}{2}+n(g+h)\right)$. ∎

Theorem 2.3. *A q-analogue of* [1, p. 438 (2.9)].

$$\sum_{m,n}\frac{C_{m+n}x^{m+n}(-1)^m\langle \widetilde{1-\tfrac{m+n}{2}};q\rangle_n}{\langle 1,\nu,\sigma,\widetilde{-\tfrac{m+n}{2}};q\rangle_n\langle 1,\nu,\sigma;q\rangle_m}\mathrm{QE}\left(-\frac{n}{2}-\frac{3mn}{2}+\frac{(m+n)^2}{4}\right) \tag{20}$$
$$=\sum_{N=0}^{\infty}\frac{C_{2N}x^{2N}\langle -1+\nu+\sigma;q\rangle_{3N}\langle\widetilde{1-N};q\rangle_N(-1)^N}{\langle 1,\widetilde{1},\nu,\sigma;q\rangle_N\langle \nu,\sigma,-1+\nu+\sigma;q\rangle_{2N}},\ \nu,\sigma,\nu+\sigma-1\neq -k.$$

Proof.

$$LHS \tag{21}$$
$$=\sum_{N=0}^{\infty}\sum_{n=0}^{N}\frac{C_Nx^N(-1)^{N-n}\langle\widetilde{1-\tfrac{N}{2}};q\rangle_n}{\langle 1,\nu,\sigma,\widetilde{-\tfrac{N}{2}};q\rangle_n\langle 1,\nu,\sigma;q\rangle_{N-n}}\mathrm{QE}\left(3\binom{n}{2}-\frac{3nN}{2}+n+\frac{N^2}{4}\right)$$
$$=\sum_{N=0}^{\infty}\frac{C_Nx^Nq^{\frac{N^2}{4}}(-1)^N}{\langle 1,\nu,\sigma;q\rangle_N}\sum_{n=0}^{N}\frac{\langle -N,-\nu+1-N,-\sigma+1-N,\widetilde{1-\tfrac{N}{2}};q\rangle_n}{\langle 1,\nu,\sigma,\widetilde{-\tfrac{N}{2}};q\rangle_n}$$
$$\times q^{n(\frac{3N}{2}-1+\nu+\sigma)}$$
$$=\sum_{N=0}^{\infty}\frac{C_{2N}x^{2N}\langle\widetilde{1-N},1-\widetilde{2N}-\nu,2N+\nu+\sigma-1;q\rangle_N\langle\tfrac{1}{2};q^2\rangle_N}{\langle 1,\nu,\sigma;q\rangle_{2N}\langle\sigma,\widetilde{\nu+N},1-\nu-N;q\rangle_N},$$

where we have used (14) for the q-Dixon theorem. ∎

Before we prove the next formula, we remind the reader that the following q-analogue of [1, p. 440 (3.10)] has been found by Srivastava and Jain [17, p.217, 2.2]:

$$\sum_{m,n=0}^{\infty}\frac{C_{m+n}x^{m+n}(-1)^n\langle a,\widetilde{a};q\rangle_m\langle b,\widetilde{b};q\rangle_n}{\langle 1,2a;q\rangle_m\langle 1,2b;q\rangle_n} \tag{22}$$
$$=\sum_{N=0}^{\infty}\frac{C_{2N}x^{2N}\langle a+b,\widetilde{a+b};q\rangle_{2N}}{\langle 1,a+\tfrac{1}{2},b+\tfrac{1}{2},a+b;q^2\rangle_N}.$$

Theorem 2.4. *Another q-analogue of* [1, p. 440 (3.10)].

$$\sum_{m,n=0}^{\infty}\frac{C_{m+n}x^{m+n}(-1)^n\mathrm{QE}\left(\binom{m}{2}-ng\right)\langle g;q\rangle_m\langle h,\widetilde{h};q\rangle_n}{\langle 1,2g;q\rangle_m\langle 1,2h,-m-\widetilde{g}+1-n;q\rangle_n} \tag{23}$$
$$=\sum_{N=0}^{\infty}\frac{C_{2N}x^{2N}\langle h+\widetilde{g+N},\tfrac{g+h}{2},\widetilde{\tfrac{g+h}{2}},\tfrac{g+h+1}{2},\widetilde{\tfrac{g+h+1}{2}};q\rangle_N\mathrm{QE}\left(\binom{2N}{2}\right)}{\langle g+h,\widetilde{g},g+\tfrac{1}{2},\widetilde{g+\tfrac{1}{2}},h+\tfrac{1}{2},\widetilde{g+N},\widetilde{h+\tfrac{1}{2}},\widetilde{1},1;q\rangle_N}.$$

Proof. We prove the equivalent formula

$$\text{(24)}\qquad \sum_{m,n=0}^{\infty} \frac{C_{m+n}x^{m+n}(-1)^n \mathrm{QE}\left(-\binom{n}{2}-mn+mg\right)\langle g;q\rangle_m\langle h,\widetilde{h};q\rangle_n}{\langle 1,2g;q\rangle_m\langle 1,2h,\widetilde{-m-g+1-n};q\rangle_n}$$

$$= \sum_{N=0}^{\infty} \frac{C_{2N}x^{2N}\langle \widetilde{h+g+N},\frac{g+h}{2},\widetilde{\frac{g+h}{2}},\frac{g+h+1}{2},\widetilde{\frac{g+h+1}{2}};q\rangle_N q^{2gN}}{\langle g+h,\widetilde{g},g+\frac{1}{2},\widetilde{g+\frac{1}{2}},h+\frac{1}{2},\widetilde{g+N},\widetilde{h+\frac{1}{2}},\widetilde{1},1;q\rangle_N}.$$

(25) *LHS*

$$= \sum_{N=0}^{\infty}\sum_{n=0}^{N} \frac{C_N\langle g;q\rangle_{N-n}\langle h,\widetilde{h};q\rangle_n x^N(-1)^n \mathrm{QE}\left(-\binom{n}{2}-(N-n)n+(N-n)g\right)}{\langle 1,2g;q\rangle_{N-n}\langle 1,2h,\widetilde{-g+1-N};q\rangle_n}$$

$$= \sum_{N=0}^{\infty}\sum_{n=0}^{N} \frac{C_N\langle -g+1-N;q\rangle_N\langle h,\widetilde{h},-2g+1-N;q\rangle_n x^N(-1)^n \mathrm{QE}\left(\binom{n}{2}+n(1-N)\right)}{\langle 1;q\rangle_{N-n}\langle -2g+1-N;q\rangle_N\langle -g+1-N,1,2h,\widetilde{-g+1-N};q\rangle_n}$$

$$= \sum_{N=0} \frac{C_N\langle -g+1-N;q\rangle_N x^N}{\langle 1,-2g+1-N;q\rangle_N}$$

$$\times\sum_{n=0}^{N} \frac{\binom{N}{n}_q\langle h,\widetilde{h},-2g+1-N;q\rangle_n(-1)^n \mathrm{QE}\left(\binom{n}{2}+n(1-N)\right)}{\langle -g+1-N,2h,\widetilde{-g+1-N};q\rangle_n}$$

$$\overset{\text{by (15)}}{=} \sum_{N=0}^{\infty} \frac{C_{2N}\langle -g+1-2N;q\rangle_{2N}x^{2N}\langle \frac{1}{2},h+g+N;q^2\rangle_N}{\langle 1,-2g+1-2N;q\rangle_{2N}\langle \frac{1}{2}+h,g+N;q^2\rangle_N}$$

$$= \sum_{N=0}^{\infty} \frac{C_{2N}\langle g;q\rangle_{2N}x^{2N}\langle \frac{1}{2},h+g+N;q^2\rangle_N q^{2gN}}{\langle 1,2g;q\rangle_{2N}\langle \frac{1}{2}+h,g+N;q^2\rangle_N}$$

$$= \sum_{N=0}^{\infty} \frac{C_{2N}x^{2N}\langle \widetilde{h+g+N},\frac{g+h}{2},\widetilde{\frac{g+h}{2}},\frac{g+h+1}{2},\widetilde{\frac{g+h+1}{2}};q\rangle_N q^{2gN}}{\langle g+h,\widetilde{g},g+\frac{1}{2},\widetilde{g+\frac{1}{2}},h+\frac{1}{2},\widetilde{g+N},\widetilde{h+\frac{1}{2}},\widetilde{1},1;q\rangle_N}.$$

Finally, multiply C_n by $\mathrm{QE}\left(\binom{n}{2}-ng\right)$. ∎

3. Reduction formulas

We now specialize the very general formulas to reduction formulas for Appell- and Kampé de Fériet functions. We will need the $\triangle$ notation, since otherwise there will not be enough space to write out the formulas.

Theorem 3.1. *A q-analogue of a reduction formula for the second Appell function.*

$$\sum_{m,n=0}^{\infty} \frac{\langle \lambda; q\rangle_{m+n} x^{m+n} (-1)^n \langle g; q\rangle_m \langle g, \widetilde{1-\tfrac{m+n}{2}}; q\rangle_n \mathrm{QE}\left(\frac{-mn-n}{2}\right)}{\langle 1, h; q\rangle_m \langle 1, h, \widetilde{-\tfrac{m+n}{2}}; q\rangle_n} \tag{26}$$
$$= {}_7\phi_7 \left[\begin{matrix} \triangle(q;2;\lambda), g, h-g \\ \triangle(q;2;h), h, \widetilde{1}, \infty \end{matrix} |q; -x^2 q^g|| \begin{matrix} \langle \widetilde{1-k}; q\rangle_k \\ - \end{matrix} \right].$$

Proof. Put $C_k = \langle \lambda; q\rangle_k$ in (16). ∎

Remark 1. The righthand sides of formulas (26) and (28) converge quicker than the LHS because of the q-power with negative exponent, the double sum and the minus sign on the left. The other formulas in this section have similar properties.

Theorem 3.2. *A q-analogue of a reduction formula for the third Appell function. By using vectors, this formula can easily be extended to a q-analogue of* [16, p. 31 (48)].

$$\sum_{m,n=0}^{\infty} \frac{x^{m+n} (-1)^n \langle g, h; q\rangle_m \langle g, h, \widetilde{1-\tfrac{m+n}{2}}; q\rangle_n}{\langle \mu; q\rangle_{m+n} \langle 1; q\rangle_m \langle 1, \widetilde{-\tfrac{m+n}{2}}; q\rangle_n} \mathrm{QE}\left(\frac{-n+mn}{2}\right) \tag{27}$$
$$= {}_7\phi_7 \left[\begin{matrix} g, h, \triangle(q;2;g+h) \\ \triangle(q;2;\mu), g+h, \widetilde{1}, \infty \end{matrix} |q; -x^2|| \begin{matrix} \langle \widetilde{1-k}; q\rangle_k \\ - \end{matrix} \right].$$

Proof. Put $C_k = \frac{1}{\langle \mu; q\rangle_k}$ in (18). ∎

Corollary 3.3. *A q-analogue of* [1, p. 439 (3.4)] *and* [16, p. 31 (46)]

$$\Phi^{p:2;3}_{p:1;2} \left[\begin{matrix} \vec{\lambda} : g, \infty; g, \infty \\ \vec{\mu} : h; h \end{matrix} |q; x, -xq^{-\frac{1}{2}}|| \begin{matrix} \langle \widetilde{1-\tfrac{m+n}{2}}; q\rangle_n q^{-\frac{mn}{2}} \\ \langle \widetilde{-\tfrac{m+n}{2}}; q\rangle_n \end{matrix} \right] \tag{28}$$
$$= {}_{6+4p}\phi_{6+4p} \left[\begin{matrix} \triangle(q;2;\vec{\lambda}), g, h-g, 3\infty \\ \triangle(q;2;\vec{\mu}, h), h, \widetilde{1} \end{matrix} |q; -x^2 q^g|| \begin{matrix} \langle \widetilde{1-k}; q\rangle_k \\ - \end{matrix} \right].$$

Proof. Put $C_n = \frac{\langle \vec{\lambda}; q\rangle_n}{\langle \vec{\mu}; q\rangle_n}$ in (16). ∎

Corollary 3.4. *A q-analogue of* [1, p. 439 (3.5)]

$$\Phi^{p:1;2}_{p:1;2}\left[\begin{matrix}\vec{\lambda}:g;h,\widetilde{h}\\ \vec{\mu}:2g;2h\end{matrix}|q;-x,x||\begin{matrix}q^{mn}\\ \langle\widetilde{m+g};q\rangle_n\end{matrix}\right] \tag{29}$$

$$={}_{5+4p}\phi_{8+4p}\left[\begin{matrix}\triangle(q;2;\vec{\lambda},g+h)\\ \triangle(q;2;\vec{\mu}),g+h,g+\frac{1}{2},h+\frac{1}{2},\widetilde{g+\frac{1}{2}},\widetilde{h+\frac{1}{2}},\widetilde{1},\widetilde{g}\end{matrix}\right.$$
$$\left.|q;x^2q||\begin{matrix}\langle h+\widetilde{g+k};q\rangle_k\\ \langle\widetilde{k+g};q\rangle_k\end{matrix}\right].$$

Proof. Put $C_n=\frac{\langle\vec{\lambda};q\rangle_n}{\langle\vec{\mu};q\rangle_n}$ in (23). ∎

Theorem 3.5. *A q-analogue of* [1, p. 439 (3.8)] *and* [16, p. 32 (50)]

(30)

$$\sum_{m,n}\frac{\langle\vec{\lambda};q\rangle_{m+n}x^{m+n}(-1)^m\langle 1\widetilde{-\frac{m+n}{2}};q\rangle_n}{\langle\vec{\mu};q\rangle_{m+n}\langle 1,\nu,\sigma,\widetilde{-\frac{m+n}{2}};q\rangle_n\langle 1,\nu,\sigma;q\rangle_m}\mathrm{QE}\left(-\frac{n}{2}-\frac{3mn}{2}+\frac{(m+n)^2}{4}\right)$$
$$={}_{16+4p}\phi_{15+4p}\left[\begin{matrix}\triangle(q;2;\vec{\lambda}),\triangle(q;3;\nu+\sigma-1),9\infty\\ \triangle(q;2;\vec{\mu},\nu,\sigma,\nu+\sigma-1),\nu,\sigma,\widetilde{1}\end{matrix}|q;-x^2||\begin{matrix}\langle\widetilde{1-k};q\rangle_k\\ -\end{matrix}\right].$$

Proof. Put $C_n=\frac{\langle\vec{\lambda};q\rangle_n}{\langle\vec{\mu};q\rangle_n}$ in (20). The $\triangle(q;3;\nu+\sigma-1)$ corresponds to six q-shifted factorials, this explains the 9∞. ∎

This last formula is the crown of our efforts in this section, and beautifully unites the notation used so far. The formula [16, p. 32 (50)] is also the last one in the corresponding chapter. We will come back to more q-analogues from [16] in later papers.

4. Discussion

We would like to remind that the umbral notation is equivalent to Gasper and Rahman [10]; however, the $\triangle(q;l;\lambda)$ operator cannot be readily expressed in their notation. The same goes for the factor $\langle\widetilde{1-k};q\rangle_k$, which elucidates the integration property in q-calculus. There are more comments at the end of the article [3].

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Received Jujy 17, 2010.