Linear Algebra and its Applications 439 (2013) 1167–1182

Contents lists available at SciVerse ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

An umbral approach to find q-analogues of matrix formulas

Thomas Ernst *

Department of Mathematics, Uppsala University, P.O. Box 480, SE-751 06 Uppsala, Sweden

ARTICLE INFO

Article history:
Received 10 December 2011
Accepted 18 March 2013
Available online 12 April 2013

Submitted by Peter Šemrl

AMS classification:
Primary: 15A24, 15A15, 33D15
Secondary: 33D99, 15A18

Keywords:
NWA q-addition
Matrix power series
q-Cauchy–Vandermonde determinant
q-Stirling numbers
q-Exponential function
Ring
LU factorization

ABSTRACT

A general introduction is given to the logarithmic q-analogue formulation of mathematical expressions with a special focus on its use for matrix calculations. The fundamental definitions relevant to q-analogues of mathematical objects are given and form the basis for matrix formulations in the paper. The umbral approach is used to find q-analogues of significant matrices. Finally, as an explicit example, a new formula for q-Cauchy–Vandermonde determinant containing matrix elements equal to q-numbers introduced by Ward is proved by using a new type of q-Stirling numbers together with Lagrange interpolation in $\mathbb{Z}(q)$.

1. Introduction

The q-analogue-formulation of mathematical expressions has been the subject of many recent studies, e.g. [5], and many areas have been covered. This paper is part in a series of papers about the connection between q-calculus and linear algebra, each of which can be read independently of one another. Other papers in this series, apart from my recent book [10], are about q-unit and q-Pascal matrices [9] and q-Cauchy, q-Bernoulli and q-Euler matrices.

The present investigation has been carried out with a special emphasis on q-analogues of matrices, an area where still much work remains to be done. Many different approaches are probably possible, but in the present investigation the q-umbral method introduced by the author [7] has been used for

* Tel.: +46 18261924.
E-mail address: thomas@math.uu.se

http://dx.doi.org/10.1016/j.laa.2013.03.018

relevant matrix formulations and applications. The fundamental definitions of q-calculus are presented and form the basis for the matrix formulations.

This paper is organized as follows: In Section 2 the first definitions and theorems are given, among others the q-binomial coefficients, the q-multinomial coefficients and the related Ward numbers. The q-binomial coefficients make the way for the NWA and JHC q-additions, which form the basis for a q-umbral calculus in the spirit of Rota and Taylor [20] with an infinite alphabet, consisting of certain letters or umbra, presented in Section 3. This calculus has been enriched with the powerful commutative monoid concept. In Section 4 we introduce the matrix nomenclature based on the τ operator, which enables two matrix multiplications and two scalar products. Under certain constraints of this q-scalar product, we are able to prove a q-deformed version of the Cauchy–Schwarz inequality. In Section 4.2 we consider q-derivatives and q-exponential functions of matrices, and finally a formula for $\mathrm{E}_q(tA)$ expressed as function of the eigenvalues of the matrix A. In Section 5 we consider a specific example, the q-Cauchy matrix, compare with [9], with matrix elements made up from Ward numbers. The fundamental q-difference equation for q-Appell polynomials, with the Polya–Vein matrix on the right hand side, formally column-wise has the q-Cauchy matrix with different initial values as solution. By a wise choice of q-Stirling numbers $\in \mathbb{Z}(q)$, we are able to perform the standard block Gaussian elimination of the defining relation for these numbers in order to compute the determinant of the q-Cauchy matrix. We compute the LU factorization of the q-Cauchy matrix for $n \leq 5$, which has the same L-matrix (Pascal matrix) as for $q = 1$. Finally, a short proof of the LU factorization of the Cauchy–Vandermonde matrix in [17, p. 115] is given.

2. Fundamental definitions and theorems

In this section we introduce the preliminary notation for the method used in the paper.

Definition 1. The power function is defined by $q^a \equiv e^{a\log(q)}$. Let $\delta > 0$ be an arbitrary small number. We will use the following branch of the logarithm: $-\pi + \delta < \mathrm{Im}\,(\log q) \leq \pi + \delta$. This defines a simply connected space in the complex plane. The variables

$$a, b, c, a_1, a_2, \ldots, b_1, b_2, \ldots \in \mathbb{C}$$

denote parameters in hypergeometric series or q-hypergeometric series.

The variables i, j, k, l, m, n, p, r will denote natural numbers except for certain cases where it will be clear from the context that i will denote the imaginary unit. Let the q-shifted factorial be defined by

$$\langle a; q\rangle_n \equiv \begin{cases} 1, & n = 0; \\ \prod_{m=0}^{n-1} (1 - q^{a+m}) & n = 1, 2, \ldots. \end{cases} \tag{1}$$

Gauss' q-binomial coefficients [12] are an important ingredient and will in q-form be written

$$\binom{n}{k}_q \equiv \frac{\langle 1; q\rangle_n}{\langle 1; q\rangle_k \langle 1; q\rangle_{n-k}}, \tag{2}$$

for $k = 0, 1, \ldots, n$. The q-binomial coefficient $\binom{n}{k}_q$ is a polynomial of degree $k(n-k)$ in q with integer coefficients, whose sum equals $\binom{n}{k}$. We will use the following boundary values for these coefficients.

$$\binom{n}{-m}_q \equiv 0,\ n = 0, \pm 1, \pm 2 \ldots,\ m = 1, 2, \ldots, \tag{3}$$

$$\binom{n}{n+m}_q \equiv 0,\ n = 0, 1, 2 \ldots,\ m = 1, 2, \ldots. \tag{4}$$

The q-multinomial coefficient is defined by

$$\binom{n}{k_1, \ldots, k_l}_q \equiv \frac{\langle 1; q\rangle_n}{\prod_{i=1}^{l}\langle 1; q\rangle_{k_i}}, \tag{5}$$

for the sequence $\{k_i\}_{i=1}^{l} = 0, 1, \ldots, n$ and $\sum_{i=1}^{l} k_i = n$. The special case $l = 2$ in (5) corresponds to the q-binomial coefficient $\binom{n}{k}_q$ above.

The q-multinomial coefficient will be used in Theorem 2.2 and then, further on, on p. 15 in the q-Cauchy matrix.

The q-analogues of a complex number a, the factorial function, the Pochhammer symbol and of the derivate D [5] are defined by

$$\{a\}_q \equiv \frac{1-q^a}{1-q},\ q \in \mathbb{C}\backslash\{1\}, \tag{6}$$

$$\{n\}_q! \equiv \prod_{k=1}^{n}\{k\}_q,\ \{0\}_q! \equiv 1,\ q \in \mathbb{C}, \tag{7}$$

$$\{a\}_{n,q} \equiv \prod_{m=0}^{n-1}\{a+m\}_q. \tag{8}$$

$$(\mathrm{D}_q\varphi)(x) \equiv \frac{\varphi(x)-\varphi(qx)}{(1-q)x},\ q \in \mathbb{C}\backslash\{1\}. \tag{9}$$

The notation (8) will appear much later, in formula (79). If we want to indicate the variable on which the q-difference operator is applied, we denote the operator $(\mathrm{D}_{q,x}\varphi)(x, y)$. If $|q| > 1$, or $0 < |q| < 1$ and $|z| < |1-q|^{-1}$, the q-exponential function $\mathrm{E}_q(z)$ is defined by

$$\mathrm{E}_q(z) \equiv \sum_{k=0}^{\infty}\frac{1}{\{k\}_q!}z^k. \tag{10}$$

The corresponding q-trigonometric functions are

$$\mathrm{Sin}_q(x) \equiv \frac{1}{2i}(\mathrm{E}_q(ix) - \mathrm{E}_q(-ix)), \tag{11}$$

and

$$\mathrm{Cos}_q(x) \equiv \frac{1}{2}(\mathrm{E}_q(ix) + \mathrm{E}_q(-ix)). \tag{12}$$

If $f(x) \in \mathbb{C}[x]$, the polynomials with complex coefficients, then the function $\epsilon : \mathbb{C}[x] \mapsto \mathbb{C}[x]$ is defined by

$$\epsilon f(x) \equiv f(qx). \tag{13}$$

The following theorem is a special case of the definition for the NWA q-addition.

Theorem 2.1 [4]. *Let A and B be linear operators on $\mathbb{C}[x]$ with*

$$BA = qAB. \tag{14}$$

Then

$$(A+B)^n = \sum_{k=0}^{n} \binom{n}{k}_q A^k B^{n-k}, \qquad n = 0, 1, 2, \ldots. \tag{15}$$

We now come to the most important operator in this paper, which makes the way for q-analogues of the Vandermonde determinant and addition theorems for important special functions.

Definition 2 [10, p. 24]**.** Let a and b be elements of a ring. Then the NWA q-addition is given by

$$(a \oplus_q b)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q a^k b^{n-k}, \qquad n = 0, 1, 2, \ldots. \tag{16}$$

Furthermore,we put

$$(a \ominus_q b)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q a^k (-b)^{n-k}, \qquad n = 0, 1, 2, \ldots. \tag{17}$$

We now put $B = \mathrm{D}_q$ and $A = \epsilon$ in (15). Since the requirements are fulfilled, we conclude that we can also define the NWA q-addition for noncommuting A and B when (14) is valid. The NWA q-addition is a generalization of (15).

The situation where the commutative requirement is weakened (like in Theorem 2.1) happens rarely in the paper and is mainly used for the case of the q-difference and ϵ, which are almost commutative.

The following q-analogue of integer powers will be used in the definition of the important q-Cauchy matrix (75), and therefore they need to be introduced here. There is a Ward number $\overline{n}_q$

$$\overline{n}_q \equiv 1 \oplus_q 1 \oplus_q \ldots \oplus_q 1, \tag{18}$$

where the number of 1 in the RHS (right hand side) is n.

The following theorem reminding of [22, p. 258] shows how Ward numbers usually appear in applications.

Theorem 2.2.

$$(\overline{n}_q)^k = \sum_{m_1+\ldots+m_n=k} \binom{k}{m_1, \ldots, m_n}_q. \tag{19}$$

We have the following special cases:

$$(\overline{0}_q)^k = \delta_{k,0}; \;\; (\overline{n}_q)^0 = 1; \;\; (\overline{n}_q)^1 = n. \tag{20}$$

The following table lists some of the first $(\overline{n}_q)^k$.

	$k=2$	$k=3$	$k=4$
$n=1$	1	1	1
$n=2$	$3+q$	$4+2q+2q^2$	$5+3q+4q^2+3q^3+q^4$
$n=3$	$6+3q$	$10+8q+8q^2+q^3$	$3\left(5+5q+7q^2+6q^3+3q^4+q^5\right)$

and for $n=4$

$k=2$	$k=3$	$k=4$
$10+6q$	$4\left(5+5q+5q^2+q^3\right)$	$5\left(7+9q+13q^2+12q^3+7q^4+3q^5\right)+q^6$

The following polynomial (21) in 3 variables $a,\ b,\ q$ originates from Gauss. This JHC q-addition is neither commutative nor associative and complements the NWA q-addition. The JHC has more than one zero element.

Definition 3. Let a and b be elements of a ring. The Jackson–Hahn–Cigler q-addition (JHC) [4, p. 91], [14, p. 362], [15, p. 78] is the function

$$(a \boxplus_q b)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q q^{\binom{k}{2}}\, b^k a^{n-k} = a^n \left(-\frac{b}{a}; q\right)_n, \qquad n = 0, 1, 2, \ldots. \tag{21}$$

The JHC q-subtraction is defined analogously:

$$(a \boxminus_q b)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q q^{\binom{k}{2}}\, (-b)^k a^{n-k}, \qquad n = 0, 1, 2, \ldots. \tag{22}$$

3. The q-umbral calculus

We first give an improved and extended version of the q-umbral calculus from [7], heavily influenced by [3,20]. This umbral calculus has a close connection to the work of Nörlund [18] on Bernoulli polynomials. This was discussed elsewhere [7].

Definition 4. Let $\{\theta_i\}_0^\infty$ and $\{\phi_i\}_0^\infty$ denote two complex-valued sequences. We denote the identity and the forward operators by I and E. The identity operator and the forward operator are defined by

$$I(\theta_i) \equiv \theta_i,\ E(\theta_i) \equiv \theta_{i+1}, \qquad i \geq 0 \tag{23}$$

The following operator gives an example of the applications of the second q-addition JHC. The Carlitz–Gould [13] q-difference operator is defined by

$$\triangle^1_{\mathrm{CG},q}\theta_0 \equiv (E-I)\theta_0,\ \triangle^{l+1}_{\mathrm{CG},q}\theta_0 \equiv \triangle^{l}_{\mathrm{CG},q}E\theta_0 - q^l \triangle^{l}_{\mathrm{CG},q}\theta_0. \tag{24}$$

Or equivalently,

$$\triangle_{\mathrm{CG},q} \sim E \boxminus_q I. \tag{25}$$

The last formula requires knowledge of the umbral calculus to be defined shortly.

We are now going to present a q-umbral calculus, which is somewhat influenced by [20, p. 696]. We have however made certain changes from the original Rota-Taylor paper; compare with the recent book [10].

Definition 5. We introduce a ring (monoid) $(A, +, \star)$, consisting of a set A, where the monoid refers to the Ward q-addition, and an additional multiplication $\star$ is added. A q-umbral calculus contains such a set A called the alphabet, with elements called letters or umbrae.

These letters are denoted $\alpha,\ \beta$ etc. The most common alphabets are

(1) $\mathbb{R}$
(2) $\mathbb{C}$.

In each case, we keep to one alphabet, and all letters must belong here.

Assume that α, β are distinct umbrae, then a new umbra is obtained by : $\alpha * \beta$, where $* \in \{\oplus_q, \boxplus_q, \ominus_q, \boxminus_q\}$. For convenience of writing, the operators $\boxplus_q$ and $\boxminus_q$ should be as far right as possible.

Let M be a subset of A. Then $\langle M \rangle$ denotes the set generated by M together with the four operations above.

The set $\mathbb{R}_q$ is defined as follows.

$$\mathbb{R}_q \equiv \langle \mathbb{R} \rangle. \tag{26}$$

There is a certain linear q-functional $eval$, $\mathbb{R}[x] \times \mathbb{R}_q \to \mathbb{R}$, with $eval(0) = a_0$, called the evaluation. In the following, an arbitrary $eval \in \mathbb{R}[x]$ will be used.

Let $\vee$ denote a logical disjunction. Two umbrae $\alpha \in \mathbb{R}_q \vee \mathbb{R}$ and $\beta \in \mathbb{R}_q \vee \mathbb{R}$ are said to be equivalent, denoted $\alpha \sim \beta$ if $eval(f, \alpha) = eval(f, \beta)$. The set of equivalent umbrae form an equivalence class and $\sim$ is an equivalence relation i.e.,

$$\alpha \sim \alpha, \tag{27}$$

$$\alpha \sim \beta, \beta \sim \gamma \Leftrightarrow \alpha \sim \gamma. \tag{28}$$

If α and β are not equivalent, we write

$$\alpha \nsim \beta. \tag{29}$$

The equivalence class which contains α is denoted by $[\alpha]$. Equality between equivalence classes is denoted by $=$. Thus $[\alpha] = [\beta] \Leftrightarrow \alpha \sim \beta$.

Theorem 3.1 [16, p. 345]**.** *The q-addition (16) has the following properties, for $\alpha, \beta, \gamma \in A$:*

$$(\alpha \oplus_q \beta) \oplus_q \gamma \sim \alpha \oplus_q (\beta \oplus_q \gamma). \tag{30}$$

$$\alpha \oplus_q \beta \sim \beta \oplus_q \alpha. \tag{31}$$

Proof. The first property (associativity) is proved as follows: It would suffice to prove that

$$[(\alpha \oplus_q \beta) \oplus_q \gamma]^n = [\alpha \oplus_q (\beta \oplus_q \gamma)]^n. \tag{32}$$

But this is equivalent to

$$\sum_{k=0}^{n} \binom{n}{k}_q \sum_{l=0}^{k} \binom{k}{l}_q a^l b^{k-l} c^{n-k} = \sum_{k'=0}^{n} \binom{n}{k'}_q a^{k'} \sum_{l'=0}^{n-k'} \binom{n-k'}{l'}_q b^{l'} c^{n-k'-l'}. \tag{33}$$

Now put $l = k'$ and $l' = k - l$ to conclude the proof. The proof of the commutative law is obvious. $\square$

There is a distinguished element θ of the alphabet called the zero, with the property

$$x \boxminus_q x \sim \theta, \;\; eval(f, \theta) = a_0. \tag{34}$$

In case of matrices, the RHS shall be interpreted as a constant multiplied with the unit matrix.

The elements α and $\beta \in \mathbb{R}$ are called inverse to each other, if $\alpha \boxplus_q \beta \sim \theta$.

The NWA and the JHC are dual operators, meaning that

$$f(x \oplus_q \alpha \boxminus_q \alpha) = f(x). \tag{35}$$

There is also the concept of equation within the alphabet. If

$$\alpha \oplus_q \beta \sim \gamma, \tag{36}$$

or

$$\alpha \boxplus_q \beta \sim \gamma, \tag{37}$$

we can solve for α. The solutions will be

$$\alpha \sim \gamma \boxminus_q \beta, \tag{38}$$

or

$$\alpha \sim \gamma \ominus_q \beta. \tag{39}$$

We can only calculate $-\beta$ explicitly from (37) as

$$-\beta \sim \alpha \boxminus_q \gamma \tag{40}$$

This phenomenon is related to missing commutativity.

4. First matrix calculations

Matrix elements will always be denoted (i, j). Here i denotes the row and j denotes the column. The matrix elements range from 0 to $n - 1$. Juxtaposition of matrices will always be interpreted as matrix multiplication. Definitions 2 and 3 can obviously be generalized to commuting matrices A and B of the same dimension.

4.1. The involution τ

The inversion operator τ, the theme of this section, is used when we are looking for q-analogues of Lie groups. These pseudo matrix groups will be defined as the q-exponential functions (62) and (63) of the Lie algebra from the undeformed case [8]. The involution τ will also be used for the definition of certain q-determinants and in scalar products.

To understand the following definitions properly, one has to emphasize, that the entries are umbrae rather than normal real numbers. This explains how τ is acting and how these definitions really differ from their standard linear algebra counterparts. Contrary to the conventional case $q = 1$, we will define at least two matrix multiplications.

Definition 6. The inversion of the basis $q \to \frac{1}{q}$ is denoted τ. Sometimes we only want to invert factors which depend on a certain variable. In order to do this, we can specify the function argument. The inversion of the basis $q \to \frac{1}{q}$ of functions depending on x is denoted τ_x.

Definition 7. There are two matrix multiplications: $\cdot$ and $\cdot_q$. The notation $\cdot$ is used for ordinary matrix multiplication whereas $\cdot_q$ is defined as follows: Let $\alpha(q)$ and $\beta(q)$ be two $n \times n$ matrices with matrix elements α_{ij} and β_{ij}, respectively. Then we define

$$(\alpha \cdot_q \beta)_{ij} \equiv \sum_{m=0}^{n-1} \alpha_{im} \tau(\beta_{mj}). \tag{41}$$

We can modify this product slightly, writing

$$(\alpha \cdot_{x,q} \beta)_{ij} \equiv \sum_{m=0}^{n-1} \alpha_{im} \tau_x(\beta_{mj}). \tag{42}$$

Definition 8. Let $\alpha = [\alpha_{ij}]$ be an $m \times n$ matrix with matrix elements α_{ij}. The conjugate of α is the $m \times n$ matrix $\overline{\alpha} = [\overline{\alpha_{ij}}]$.

Let $0 < q < 1$. The conjugate transpose of α is the $n \times m$ matrix $\alpha^{\star} \equiv [\overline{(\alpha_{ij})}]^T$.

Theorem 4.1. *Properties of the conjugate transpose. Let $\alpha(q)$ and $\beta(q)$ be $n \times n$ matrices and q real. Then*

$$(\alpha^{\star})^{\star} \sim \alpha, \tag{43}$$

$$(\alpha \oplus_q \beta)^{\star} \sim \alpha^{\star} \oplus_q \beta^{\star}, \tag{44}$$

$$(z\alpha)^{\star} \sim (\overline{z})\alpha^{\star}, \text{ for any scalar } z \in \mathbb{C}, \tag{45}$$

$$(\alpha \cdot \beta)^{\star} = \beta^{\star} \cdot \alpha^{\star} \tag{46}$$

Proof. Formula (45) is proved as follows.

$$\left((z\alpha)^{\star}\right)_{ij} \sim \left((\overline{z\alpha})^T\right)_{ij} \sim \overline{z} \cdot \left((\overline{\alpha})^T\right)_{ij} \sim \overline{z}\left(\alpha^{\star}\right)_{ij}. \tag{47}$$

Formula (46) is proved as follows.

$$\begin{aligned} \left((\alpha \cdot \beta)^{\star}\right)_{ij} &= \left(\sum_{m=0}^{n-1} \alpha_{im}\beta_{mj}\right)^{\star} = \left(\sum_{m=0}^{n-1} \overline{\beta}_{jm}\overline{\alpha}_{mi}\right) \\ &= \left((\overline{\beta})^T \cdot (\overline{\alpha})^T\right)_{ij} = \left((\beta^{\star} \cdot \alpha^{\star})\right)_{ij}. \quad \square \end{aligned} \tag{48}$$

We can define several q-scalar products.

Definition 9. The Euclidean q-scalar product of two vectors e_1 and e_2, each of dimension n, with matrix elements e_{1k} and e_{2k} is defined by the following mapping $(\mathbb{R}_q)^{2n} \mapsto \mathbb{R}$:

$$e_1 \cdot_q e_2 \equiv \sum_{k=1}^{n} (e_{1;k})^1 (\tau(e_{2;k}))^1. \tag{49}$$

In the rest of this subsection we only consider vectors e_i with the property

$$e_i \cdot_q e_i \geq 0. \tag{50}$$

The corresponding norm of a vector e_1 is defined by

$$\|e_1\| \equiv \sqrt{e_1 \cdot_q e_1}. \tag{51}$$

We note the following linearity relation:

$$(a \oplus_q b) \cdot_q (c \oplus_q d) = a \cdot_q c + a \cdot_q d + b \cdot_q c + b \cdot_q d. \tag{52}$$

For vectors we have the q-Cauchy–Schwarz inequality:

Theorem 4.2. *For all vectors e_1, and e_2, which are functions of a certain number of variables (including q) such that inequality (50) is valid in a certain region, the following inequality holds in this region:*

$$(e_1 \cdot_q e_2)\tau(e_1 \cdot_q e_2) \le \|e_1\|^2\|e_2\|^2. \tag{53}$$

Proof.

$$\begin{aligned} 0 &\overset{\text{by}(50)}{\le} (re_1 \oplus_q te_2) \cdot_q (re_1 \oplus_q te_2) \overset{\text{by}(52)}{=} r^2(e_1 \cdot_q e_1) + tr(e_2 \cdot_q e_1) \\ &+ rt(e_1 \cdot_q e_2) + t^2(e_2 \cdot_q e_2) = r^2\|e_1\|^2 + rt(e_1 \cdot_q e_2 + \tau(e_1 \cdot_q e_2)) + t^2\|e_2\|^2. \end{aligned} \tag{54}$$

Now put $r = \|e_2\|$ and $t = -\|e_2\|^{-1}(e_1 \cdot_q e_2)$ to obtain

$$\|e_1\|^2\|e_2\|^2 - [(e_1 \cdot_q e_2)^2 + (e_1 \cdot_q e_2)\tau(e_1 \cdot_q e_2)] + (e_1 \cdot_q e_2)^2 \ge 0. \tag{55}$$

Finally we obtain

$$\|e_1\|^2\|e_2\|^2 \ge (e_1 \cdot_q e_2)\tau(e_1 \cdot_q e_2) \quad \square. \tag{56}$$

4.2. *Functions of matrices*

In this subsection we treat q-matrices in a more general way. The concept of formal power series is introduced. The matrix norm is defined as for the case $q = 1$. The two q-exponential functions also play an important role, and some of the concepts from the previous subsection can be used in this context. The addition formula (64) is the general form for the first matrix multiplication $\cdot$ in $SO_q(2)$ and $SU_q(2)$ [8]. The addition formula (65) is the general form for the second matrix multiplication $\cdot_q$ in $SO_q(2)$ and $SU_q(2)$ [8].

We will now follow the reasoning in Zurmühl [23, p. 267ff], but in q-deformed form.

Definition 10. Let A be a square matrix and put

$$f(A) \equiv \sum_{k=0}^{\infty} a_k A^k. \tag{57}$$

A matrix power series (and generally an arbitrary matrix series) is exactly then called convergent, when every element of the partial sums converges with increasing number of terms. We don't have to consider the convergence of the n^2 matrix elements; on the contrary, it suffices to check the convergence of the usual power series $\{f(\lambda_i)\}_{i=1}^k$ for the eigenvalues λ_i of the matrix A. Therefore, we instead consider the following partial sum of (57):

$$f(A)_m \equiv \sum_{k=0}^{m} a_k A^k, \tag{58}$$

which in the limit goes to

$$f(A) = \lim_{m\to\infty} f(A)_m. \tag{59}$$

The q-derivate of a matrix is defined as the matrix with matrix elements equal to the q-derivate of the original matrix elements.

Theorem 4.3.

$$\mathrm{D}_q AB = \mathrm{D}_q(A)\epsilon B + A\mathrm{D}_q B. \tag{60}$$

Proof. We show that the matrix elements are equal and use formula (15) in the process.

$$\begin{aligned} \mathrm{D}_q(AB)_{ij} &= \sum_{m=0}^{n-1} \mathrm{D}_q(A_{im}B_{mj}) \\ &= \sum_{m=0}^{n-1} \mathrm{D}_q(A_{im})\epsilon B_{mj} + (A_{im}\mathrm{D}_q B_{mj} = \big(\mathrm{D}_q(A)\epsilon B + A\mathrm{D}_q B\big)_{ij} \quad \square. \end{aligned} \tag{61}$$

Definition 11. Let A be an $n \times n$ matrix, $0 < |q| < 1$. Then

$$\mathrm{E}_q(A) \equiv \sum_{k=0}^{\infty} \frac{1}{\{k\}_q!} A^k, \tag{62}$$

$$\mathrm{E}_{\frac{1}{q}}(A) \equiv \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}}}{\{k\}_q!} A^k, \tag{63}$$

where $\{k\}_q!$ is defined by (7). The convergence regions for these two series can be computed by the matrix eigenvalues and the well-known fact that the convergence radius of a matrix power series is less or equal to the corresponding power series for the matrix eigenvalues.

Theorem 4.4. *If A and B are commuting matrices, we have the following two formulas, which can be interpreted as pseudogroup morphisms in the spirit of [8, p. 159].*

$$\mathrm{E}_q(A \oplus_q B) = \mathrm{E}_q(A)\mathrm{E}_q(B). \tag{64}$$

$$\mathrm{E}_q(A \boxplus_q B) = \mathrm{E}_q(A) \cdot_q \mathrm{E}_q(B). \tag{65}$$

Theorem 4.5. *Let the eigenvalues of the quadratic $n \times n$ matrix A be $\{\lambda_i\}_{i=1}^k$ with multiplicities $\{n_i\}_{i=1}^k$, $\sum_{i=1}^k n_i = n$. According to the Cayley–Hamilton theorem, we can compute $\mathrm{E}_q(tA)$ as*

$$\mathrm{E}_q(tA) = \sum_{i=1}^{k} P_i(A)\mathrm{E}_q(t\lambda_i)V(\lambda_i), \tag{66}$$

where $V(\lambda_i)$ is the eigenvector of the eigenvalue λ_i and $P_i(\lambda_j)$ can be found by putting $(A-I)^j v = 0$ for all $j <$ the multiplicity of the eigenvalue λ_i. Each element in $\mathrm{E}_q(tA)$ is a linear combination of terms of the form $t^j\mathrm{E}_q(t\lambda)$, where λ is an eigenvalue of A.

Proof. Use the reasoning in [23, p. 268 f]. $\square$

Example 1. If

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{67}$$

then $\lambda = \pm i$ and

$$\mathrm{E}_q(tA) = \begin{pmatrix} \mathrm{Cos}_q(t) & \mathrm{Sin}_q(t) \\ -\mathrm{Sin}_q(t) & \mathrm{Cos}_q(t) \end{pmatrix}. \tag{68}$$

If a given matrix has been transformed to Jordan form $J = \Lambda + N$, with commuting terms Λ (diagonal) and N (nilpotent), and then computing $\mathrm{E}_q(\Lambda)$ and $\mathrm{E}_q(N)$, where the second series terminates, formula (64) gives a way of computing $\mathrm{E}_q(\Lambda \oplus_q N)$.

We will need the following additional relation before we embark on the q-Cauchy matrix.

Definition 12. The NWA q-shift operator [p. 242, 3.1] is given by

$$\mathrm{E}(\oplus_q)(t^n) \equiv (t \oplus_q 1)^n \tag{69}$$

This can be generalized to

$$\mathrm{E}(\oplus_q)^a(x^n) \equiv (x \oplus_q a)^n \tag{70}$$

We will use this q-shift operator in the definition of an important matrix in the next section.

Definition 13. We define a set $\mathcal{U}$ with respect to composition of the operator $\mathrm{E}\oplus_q$, in which for every ordered pair of letters $\alpha, \beta \in \mathcal{U}$

$$\mathrm{E}(\oplus_q)^\alpha \mathrm{E}(\oplus_q)^\beta(x^n) \equiv (x \oplus_q \alpha \oplus_q \beta)^n. \tag{71}$$

Theorem 4.6. *The set $\mathcal{U}$ is a commutative semigroup (or monoid) with respect to composition of the operator* $\mathrm{E}\oplus_q$.

Proof. Use the associativity and commutativity of $\oplus_q$ and $\ominus_q$. There is no additive zero element. □

Finally, we note that the q-difference operator is an infinitesimal generator of the semigroup. This q-derivative is the infinitesimal representation of the q-shift operator, just like the Lie algebra representations are infinitesimal representations associated to group representation in Lie group theory.

$$\mathrm{E}(\oplus_q) = \mathrm{E}_q(\mathrm{D}_q). \tag{72}$$

Formula (72) is a q-analogue of

$$\mathrm{E} = e^{\mathrm{D}}. \tag{73}$$

5. An explicit example, the *q*-Cauchy matrix

In this final section we treat the important q-Cauchy matrix, which is closely connected to the Stirling numbers. It is also known under the name Vandermonde matrix. The Cauchy matrix has applications in interpolation theory. Oruc and Phillips have given an explicit factorization of the Cauchy–Vandermonde matrix in [17]. We show that the q-Cauchy matrix can be factorized in a few cases, and that the corresponding determinant has an expected value. We also give a short proof of the main theorem in [17, p. 115].

Definition 14. The following abbreviation will be used.

$$\xi(t) \equiv (1, t, t^2, \ldots, t^{n-1})^T. \tag{74}$$

Definition 15. A q-analogue of the matrix formulation by Aceto and Trigiante [1, p. 234, (15)]. The q-Cauchy matrix is given by

$$W_{n,q}(t) \equiv (\xi(t)\ \ \mathrm{E}(\oplus_q)\xi(t)\ \ \mathrm{E}(\oplus_q)^{\overline{2}_q}\xi(t)\ \cdots\ \mathrm{E}(\oplus_q)^{\overline{n-1}_q}\xi(t))$$
$$\equiv (\xi(t)\ \ \xi(t\oplus_q 1)\ \ \xi(t\oplus_q \overline{2_q})\ \cdots\ \xi(t\oplus_q \overline{(n-1)_q})). \tag{75}$$

The columns are powers of the variable t and the rows are equally spaced with increment $\oplus_q 1$.

Theorem 5.1. *The matrix elements of the q-Cauchy matrix are given by*

$$W_{n,q}(t)(i,j) = (t \oplus_q \overline{j_q})^i, \qquad i,j = 0,\ldots,n-1. \tag{76}$$

The following matrix is the expression for the q-difference operator in $ke - ke$-bases, a q-analogue of Arponen's paper [2]. This is the motivation for the similar notation.

Definition 16. A q-analogue of the Polya–Vein matrix [1, (1), p. 232], [21, p. 278 (5)], [19, p. 257]. The $n \times n$ matrix $\mathbf{D}_{n,q}$ is given by

$$\mathbf{D}_{n,q}(i,i-1) \equiv \{i\}_q, \qquad i = 1,\ldots,n-1,$$
$$\mathbf{D}_{n,q}(i,j) \equiv 0, \qquad j \neq i-1. \tag{77}$$

We make the convention that all $n \times n$ matrices are denoted by the first index n.

The matrix $\mathrm{D}_{n,q}$ has the property

$$\mathbf{D}_{n,q} e_i = \{i+1\}_q e_{i+1}, \qquad i = 0,\ldots,n-2, \tag{78}$$

where $e_i,\ i = 0,\ldots,n-1$ denote the standard unit basis vectors in $\mathbb{R}^n$.
We immediately obtain

$$\mathbf{D}_{n,q}^k e_i = \{i+1\}_{k,q} e_{i+k}, \qquad k = 0,\ldots,n-1, \tag{79}$$

or equivalently a q-analogue of [21, p. 279].

$$\mathbf{D}_{n,q}^k (i+k,i) = \{i+1\}_{k,q}. \tag{80}$$

According to our notation, $e_i = 0,\ i \geq n$ because $\mathbf{D}_{n,q}^k = 0,\ k \geq n$.

Example 2. The 4×4 dimensional Polya–Vein matrix acquires the following structure in q-formulation.

$$\mathbf{D}_{4,q} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \{2\}_q & 0 & 0 \\ 0 & 0 & \{3\}_q & 0 \end{pmatrix}. \tag{81}$$

The square of the previous matrix has the value

$$\mathbf{D}_{4,q}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \{1\}_{2,q} & 0 & 0 & 0 \\ 0 & \{2\}_{2,q} & 0 & 0 \end{pmatrix}. \tag{82}$$

If $y(t)$ is a vector of length n, the following q-difference equation in $\mathbb{R}^n$ of fundamental importance can be formulated:

$$\mathrm{D}_{q,t}y(t) = \mathbf{D}_{n,q}y(t),\, y(0) = y_0, \qquad -\infty < t < \infty. \tag{83}$$

The general solution of (83) is the q-Appell polynomial of degree ν and order m [7]. The initial values are then the vector of q-Bernoulli-, q-Euler- or q-Hermite numbers of order m etc. [7]. The initial value can also be the vector function e_0. The solution is then the vector function $\xi(t)$.

Theorem 5.2. *A q-analogue of the matrix theorem by Aceto and Trigiante [1, p. 234]. The vector function $\xi(t)$ satisfies (83) with $y_0 = e_0$. The q-Cauchy matrix satisfies (83) formally column-wise with different initial guesses.*

We are now going to find a new formula for a q-Cauchy–Vandermonde determinant expressed as a product of q-Ward numbers. To this end we first state a lemma and a definition.

Lemma 5.3 [11, p. 60]. *If A and B are quadratic matrices, not necessarily of the same dimension, then*

$$\begin{vmatrix} A & C \\ 0 & B \end{vmatrix} = detA\ detB. \tag{84}$$

The proof by induction uses the following q-Stirling numbers.

Definition 17. We define certain q-Stirling numbers $\{s_{n,k}(q)\}_{n,k=0}^{\infty} \in \mathbb{Z}(q)$ by the following system of equations:

$$\sum_{k=0}^{n} s_{n,k}(q)(\bar{i}_q)^k = \delta_{i,n}\{n\}_q!, \qquad i = 0, 1, \ldots, n. \tag{85}$$

Let A be equal to $W_{n+1,q}(0)^T$ and let X denote the vector of $s_{n,k}(q)$ of length $n+1$. For each fixed n in (85), we have a system of $n+1$ equations $AX = B$, with nonsingular A and nonzero B (because of $\delta_{i,n}$), which uniquely determines the $s_{n,k}(q)$. The nonsingularity of A is nothing but the nonsingularity of $W_{n+1,q}(0)$, and could easily be explained referring to its determinant, which is discussed in detail in the following proof. The $s_{n,k}(q)$ are defined as zero for $k > n$. These q-Stirling numbers form a Lagrange interpolation in $\mathbb{Z}(q)$.

The following special values are obtained for these q-Stirling numbers:

$$s_{n,n}(q) = 1,\ s_{n,0}(q) = \delta_{0,n}, \tag{86}$$

$$s_{2,1}(q) = -1,\ s_{3,1}(q) = \frac{1+q+2q^2}{1+q}, \tag{87}$$

$$s_{3,2}(q) = \frac{-2-2q-2q^2}{1+q}. \tag{88}$$

Theorem 5.4. *A q-Cauchy determinant for q-Ward numbers can be set up as follows.*

$$det(W_{n,q}(0)) = \prod_{j=1}^{n-2}\{n-j\}_q^j. \tag{89}$$

The following proof uses one step of block Gaussian elimination from (85).

Proof. We use induction. The theorem is obviously true for $n = 1$. Assume that it is true for $n - 1$. Then we have

$$det(W_{n,q}(0)) = \begin{vmatrix} W_{n-1,q}(0) & A \\ B & (\overline{n-1}_q)^{n-1} \end{vmatrix}, \tag{90}$$

where

$$A \equiv ((\overline{n-1}_q)^0, (\overline{n-1}_q)^1, \ldots, (\overline{n-1}_q)^{n-2})^T, \tag{91}$$

$$B \equiv ((\bar{0}_q)^{n-1}, (\bar{1}_q)^{n-1}, \ldots, (\overline{n-2}_q)^{n-1}). \tag{92}$$

By (84) it suffices to prove that if we add $s_{n-1,0}(q)$ times row 0, $s_{n-1,1}(q)$ times row $1, \ldots,$ $s_{n-1,n-2}(q)$ times row $n-2$ to the last row, with the following constraint:

$$(\bar{i}_q)^{n-1} + \sum_{k=0}^{n-2} s_{n-1,k}(q)(\bar{i}_q)^k = 0, \qquad i = 0, 1, \ldots, n-2, \tag{93}$$

then we get for the matrix element $(n-1, n-1)$:

$$(\overline{n-1}_q)^{n-1} + \sum_{k=0}^{n-2} s_{n-1,k}(q)(\overline{n-1}_q)^k = \{n-1\}_q!. \tag{94}$$

The result now follows from formula (85). □

After having found this formula, the natural question is: Can we do LU factorizations of the corresponding q-Cauchy matrices? Mathematica calculations show that this is possible for many values of n. The L matrix is the same Pascal matrix as in [17, p. 115], with $x_i = i$, independent of q, and the rows used for pivoting are $0, 1, \ldots, n-1$. Below we give the LU factorization for the matrix $W_{5,q}(0)$ in (76).

Example 3.

$$W_{5,q}(0) = L_5 U_{5,q}, \tag{95}$$

with

$$L_5 \equiv \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix}, \tag{96}$$

$$U_{5,q} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & \{2\}_q! & 2\{3\}_q & (1+q^2)(3+3q+q^2) \\ 0 & 0 & 0 & \{3\}_q! & 3(1+q)(1+q^2)\{3\}_q \\ 0 & 0 & 0 & 0 & \{4\}_q! \end{pmatrix}. \tag{97}$$

We will now give a short proof of LU factorization of the Cauchy–Vandermonde matrix in [17, p. 115].

Definition 18. Let the Cauchy–Vandermonde matrix $\mathbf{V}$ be given by

$$\mathbf{V} \equiv \begin{vmatrix} 1 & x_0 & \vdots & x_0^n \\ 1 & x_1 & \vdots & x_1^n \\ \vdots & \ddots & \vdots & \\ 1 & x_n & \vdots & x_n^n \end{vmatrix}. \tag{98}$$

Theorem 5.5 [17, p. 115]. *Let h_k denote the complete symmetric polynomial (the sum of all distinct monomials of degree k in the variables $x_0, x_1, \ldots, x_n$). The matrix $\boldsymbol{V}$ has a LU factorization, where the matrix elements of $\boldsymbol{L}$ and $\boldsymbol{U}$ are given by*

$$l_{i,j} = \prod_{t=0}^{j-1} \frac{x_i - x_{j-t-1}}{x_j - x_{j-t-1}}, \qquad 0 \le j \le i \le n, \tag{99}$$

$$u_{i,j} = \mathrm{h}_{j-i}(x_0, \ldots, x_i) \prod_{t=0}^{i-1} (x_i - x_t), \qquad 0 \le i \le j \le n. \tag{100}$$

Proof. We use induction. When substituting the matrix elements of $\mathbf{V}$, $\mathbf{L}$ and $\mathbf{U}$, we find that

$$x_i^j = x_0^j + \sum_{k=0}^{i-1} \prod_{m=0}^{k} (x_i - x_m) \mathrm{h}_{j-k-1}(x_0, \ldots, x_{k+1}). \tag{101}$$

Formula (101) holds for $j = 0$. Assume that (101) holds for $j > 0$ and try to prove that it also holds for $j + 1$. It will suffice to prove that

$$x_i^{j+1} = x_0^{j+1} + \sum_{k=0}^{i-1} \prod_{m=0}^{k} (x_i - x_m) \mathrm{h}_{j-k}(x_0, \ldots, x_{k+1}). \tag{102}$$

The following recursion formula for h is very useful in this proof:

$$\mathrm{h}_r(x_0, \ldots, x_{n-1}) = \mathrm{h}_r(x_0, \ldots, x_{n-2}) + x_{n-1} \mathrm{h}_{r-1}(x_0, \ldots, x_{n-1}), \qquad n, r \ge 1. \tag{103}$$

Put

$$K(\vec{x}, i, j) \equiv x_i \times (101) - (102), \tag{104}$$

where in the second term substitution for (103) has been made. We find that

$$\begin{aligned} K(\vec{x}, i, j) = {} & x_i \left(x_0^j + \sum_{k=0}^{i-1} \mathrm{h}_{j-k-1}(x_0, \ldots, x_{k+1}) \prod_{m=0}^{k} (x_i - x_m) \right) - x_0^{j+1} \\ & - \sum_{k=0}^{i-1} \prod_{m=0}^{k} (x_i - x_m) \left(\mathrm{h}_{j-k}(x_0, \ldots, x_k) + x_{k+1} \mathrm{h}_{j-k-1}(x_0, \ldots, x_{k+1}) \right). \end{aligned} \tag{105}$$

It now suffices to prove that $K(\vec{x}, i, j) = 0$. The four terms

$$x_i x_0^j - x_0^{j+1} - (x_i - x_0)x_0^j$$

cancel out. The remaining terms all contain the general term

$$\mathrm{h}_{j-l}(x_0, \ldots, x_l), \qquad 1 \le l \le i,$$

with coefficient equal to

$$\begin{aligned} & x_i \prod_{m=0}^{l-1} (x_i - x_m) - \prod_{m=0}^{l} (x_i - x_m) - x_l \prod_{m=0}^{l-1} (x_i - x_m) \\ & \quad = (x_i - x_l - (x_i - x_l)) \prod_{m=0}^{l-1} (x_i - x_m) = 0. \quad \square \end{aligned} \tag{106}$$

6. Conclusion

q-Calculus contains both pure mathematics and its applications. This could be the beginning of a new branch, which connects q-special functions with matrix theory. We have constructed a solid basis for the future development of this branch. Several papers on Pascal matrices and LU-factorization have recently been published, but this paper considerably broadens the subject.

Acknowledgements

I thank Karl-Heinz Fieseler, Leif Karlsson and the referee for their assistance.

References

[1] L. Aceto, D. Trigiante, The matrices of Pascal and other greats, Amer. Math. Monthly 108 (3) (2001) 232–245.
[2] T. Arponen, A matrix approach to polynomials, Linear Algebra Appl. 359 (2003) 181–196.
[3] E.T. Bell, Postulational bases for the umbral calculus, Amer. J. Math. 62 (1940) 717–724.
[4] J. Cigler, Operatormethoden für q-Identitäten, Monatsh. Math. 88 (1979) 87–105.
[5] T. Ernst, A method for q-calculus, J. Nonlinear Math. Phys. 10 (2003) 487–525.
[6] T. Ernst, Les déterminants généralisés de Cauchy, Adv. Dynam. Syst. Appl. 1 (2006) 59–77.
[7] T. Ernst, q-Bernoulli and q-Euler polynomials, an umbral approach, Int. J. Differ. Equations 1 (2006) 13–62.
[8] T. Ernst, q-deformed matrix pseudo-groups, Royal Flemish Academy of Belgium, 2010.
[9] T. Ernst, q-Leibniz functional matrices with applications to q-Pascal and q-Stirling matrices, Adv. Stud. Contemp. Math. 22 (4) (2012) 151–162.
[10] T. Ernst, A Comprehensive Treatment of q-Calculus, Birkhäuser, 2012.
[11] P.A. Fuhrmann, A polynomial approach to linear algebra. Universitext, Springer-Verlag, New York, 1996.
[12] C.F. Gauss, Summatio quarundam serierum singularium, Opera 2 (1876) 16–17.
[13] H.W. Gould, The q-series generalization of a formula of Sparre Andersen, Math. Scand. 9 (1961) 90–94.
[14] W. Hahn, Beiträge zur Theorie der Heineschen Reihen, Math. Nachr. 2 (1949) 340–379.
[15] F.H. Jackson, On basic double hypergeometric functions, Quart. J. Math. Oxford Ser. 13 (1942) 69–82.
[16] P. Nalli, Sopra un procedimento di calcolo analogo alla integrazione, (Italian) Palermo Rend. 47 (1923) 337–374.
[17] H. Oruc, G. Phillips, Explicit factorization of the Vandermonde matrix, Linear Algebra Appl. 315 (2000) 113–123.
[18] N.E. Nörlund, Mémoire sur les polynomes de Bernoulli, Acta Math. 43 (1920) 121–196.
[19] R. Redheffer, Induced transformations of the derivative-vector, Amer. Math. Monthly 83 (1976) 255–259.
[20] G.-C. Rota, B.D. Taylor, The classical umbral calculus, SIAM J. Math. Anal. 25 (1994) 694–711.
[21] P.R. Vein, Matrices which generate families of polynomials and associated infinite series, J. Math. Anal. Appl. 59 (1977) 278–287.
[22] M. Ward, A calculus of sequences, Amer. J. Math. 58 (1936) 255–266.
[23] R. Zurmühl, Matrizen und ihre technischen Anwendungen. 4. neubearb. Aufl., Springer, 1964.