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Hessenberg matrix for sums of Hermitian positive definite matrices and weighted shifts

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ABSTRACT

In this work, we introduce an algebraic operation between bounded Hessenberg matrices and we analyze some of its properties. We call this operation m-sum and we obtain an expression for it that involves the Cholesky factorization of the corresponding Hermitian positive definite matrices associated with the Hessenberg components.

This work extends a method to obtain the Hessenberg matrix of the sum of measures from the Hessenberg matrices of the individual measures, introduced recently by the authors for subnormal matrices, to matrices which are not necessarily subnormal.

Moreover, we give some examples and we obtain the explicit formula for the m-sum of a weighted shift. In particular, we construct an interesting example: a subnormal Hessenberg matrix obtained as the m-sum of two not subnormal Hessenberg matrices.

0. Introduction

We consider a positive measure μ supported on a compact set $\text{Supp}(\mu)$ of the complex plane. In the space $\mathcal{P}$ of polynomials, we consider the inner product of two polynomials $Q, R \in \mathcal{P}$ given by

$$\langle Q, R\rangle_\mu = \int_{\text{Supp}(\mu)} Q(z)\overline{R}(z)d\mu(z).$$

It is well known that there exists a unique sequence of orthonormal polynomials (ONPS) $\{P_n\}_{n=0}^{\infty}$ with positive leading coefficients (see [1,2] or [3]).

The moment matrix of the measure μ, given by $M = (c_{jk})_{j,k=0}^{\infty}$, where $c_{jk} = \int_\Omega z^j\overline{z}^k d\mu, j, k \in \mathbb{Z}_+$, is the matrix of the inner product with respect to the canonical basis.

In the space $\mathcal{P}^2(\mu)$, the closure of the polynomial space $\mathcal{P}$, with the metric induced by $\langle\ .\ ,\ .\rangle_\mu$, we consider the multiplication by z operator

$$zP_n(z) = \sum_{k=0}^{n+1} d_{k,n}P_k(z), \quad n \geq 0, \tag{1}$$

with $P_0 = 1$ when $c_{00} = 1$.

These coefficients give rise to an infinite matrix $D = (d_{i,j})_{i,j=0}^{\infty}$, where $d_{i,j} = 0$ if $i > j+1$, hence this is an upper Hessenberg matrix. The numbers $d_{n+1,n}$ are the quotients of the leading coefficients of P_n and P_{n+1} and hence they are positive. This matrix D defines a bounded subnormal operator [4] in ℓ^2.

When the support of the measure is a compact set of the real line, the Hessenberg matrix is symmetric, hence tridiagonal, and therefore it is the Jacobi matrix for the orthogonal polynomials. For a measure on the unit circle we obtain the GGT

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representation (see [5]) which is an unitary Hessenberg matrix if μ does not belong to the Szegö class. In the Szegö class one still has $D^*D = I$, and hence the operator is quasinormal (and thus subnormal). In both cases (real line and unit circle) the measure μ corresponds to the spectral measure of the normal extension of the subnormal operator D (see [4]).

In [6], Mantica calculates the Jacobi matrix associated with a sum of measures from the Jacobi matrices of each of the measures (see also [7,8]). Recently [9], the authors have introduced a method, extending Mantica's spectral techniques, to obtain the Hessenberg matrix of a sum of measures from the Hessenberg matrices of the component measures. The problem studied was the following: let $\mu_1, \mu_2, \ldots, \mu_m$ be positive measures on compact sets of the complex plane with Hessenberg matrices $D^{(1)}, D^{(2)}, \ldots, D^{(m)}$, and let $p_1, p_2, \ldots, p_m$ be positive numbers such that $\sum_{i=1}^{m} p_i = 1$. Then, how can we compute the upper Hessenberg matrix for the measure $\mu = \sum_{i=1}^{m} p_i\mu_i$ in terms of the matrices $D^{(1)}, D^{(2)}, \ldots, D^{(m)}$? To solve this problem we used the subnormality property of such Hessenberg operators.

In this work we deal with the same problem when the bounded Hessenberg matrices may represent operators which are not subnormal, i.e., when there is not a measure as solution of the associated moment problem.

Every infinite Hermitian positive definite matrix M defines an inner product in the space $\mathcal{P}$ of polynomials and then, there still exists a unique orthonormal polynomials sequence with positive leading coefficients (ONPS) $\{P_n\}_{n=0}^{\infty}$ associated with the matrix M. We can consider again the multiplication by z operator in the space $\mathcal{P}^2(M)$, the closure of the polynomial space $\mathcal{P}$. Then, we have the infinite upper Hessenberg matrix $D = (d_{jk})_{j,k=0}^{\infty}$ of this operator with respect to the basis of ONPS $\{P_n\}_{n=0}^{\infty}$.

In order to solve the problem, we introduce an algebraic operation between bounded Hessenberg matrices. This algebraic operation consists in obtaining the Hessenberg matrix associated with the Hermitian positive definite matrix given by the convex sum of the corresponding Hermitian positive definite matrices associated with the component Hessenberg matrices. Since this operation is related with the sum of measures when the component Hessenberg matrices are subnormal, or with the sum of the Hermitian positive definite matrices (usually denoted by M) when the Hessenberg matrices are not subnormal, we will call this operation as m-sum.

In the first section we include some background and previous results that we will need later.

In the second section, we define the m-sum and we prove some of its properties. We obtain an expression that involves the Cholesky factorization of the corresponding Hermitian positive definite matrices associated with the Hessenberg components. Moreover, we see that the algorithm given in [9] to compute finite sections of the sum of Hessenberg matrices can be applied to the m-sum of not subnormal Hessenberg matrices. We give some examples to compute the m-sum, obtaining the exact value of the finite sections of the Hessenberg matrix.

Finally, in the last section, we obtain the explicit formula for the m-sum of a weighted shift. Using Stampfli's results [10], we construct different examples, in order to study which properties (subnormality, hyponormality) are preserved under m-sum. In particular, we construct an interesting example of a subnormal Hessenberg matrix which is the m-sum of two not subnormal Hessenberg matrices.

For general information on orthogonal polynomials and subnormal operators, we recommend the books [1,4,2,11,3] by Chihara, Conway, Freud, Halmos and Szegö, respectively.

1. Previous results

Recall that a bounded operator S on a Hilbert space, S is normal if $S^*S = SS^*$, where S^* is the adjoint operator, and it is subnormal if it is the restriction of a normal operator to an invariant subspace (see [4,11]).

In all this work, we will consider bounded upper Hessenberg matrix operators. In this case, the Hermitian positive definite matrix M is a moment matrix if and only if the associated Hessenberg matrix D defines a subnormal operator [12,4,13].

1.1. Hessenberg matrix associated with a sum of measures

Consider a family of measures $\{\mu_i\}_{i=1}^{m}$ with compact support $\Omega_i \subset \mathbb{C}$ and $\mu_i(\Omega_i) = 1$. Let μ be the sum measure, i.e.,

$$d\mu = \sum_{i=1}^{m} p_i d\mu_i,$$

where $\sum_{i=1}^{m} p_i = 1$ and $p_i \geq 0$ for all $i = 1, 2, \ldots, m$. Let $\{P_n\}_{n=0}^{\infty}$ be the ONPS for this sum measure μ.

Let $\{D^{(i)}\}_{i=1}^{m}$ be the associated Hessenberg matrices, and let $D = (d_{ij})_{i,j=0}^{\infty}$ be the Hessenberg matrix associated with μ.

Note that the matrices $D^{(i)}$ are bounded as operators on $\ell^2(\{0, 1, 2, \ldots\})$, shortly ℓ^2, because the support of every μ_i is compact. Also, remark that every matrix defines a subnormal operator in ℓ^2 [12,14,13]. These two properties were used in [9] to prove the following result which extends Mantica's spectral techniques [6] to the complex plane, and provides a technique to calculate D in terms of $\{D^{(i)}\}_{i=1}^{m}$. This technique will be applied in the next section to the case of Hessenberg matrices which are not subnormal.

Theorem 1. *Let the sum measure μ, the* ONPS *$\{P_n\}_{n=0}^{\infty}$ and the Hessenberg matrices D and $\{D^{(i)}\}_{i=1}^{m}$ be as above. Consider the initial vector $v_0^{(i)} = (1, 0, 0, \ldots)^T$ for every $i = 1, \ldots, m$. Then the elements of the matrix $D = (d_{jk})_{j,k=0}^{\infty}$ associated with μ can be calculated recursively from the matrices $\{D^{(i)}\}_{i=1}^{m}$ using the following formulas for $n = 0, 1, 2, \ldots$*

$$d_{k,n} = \sum_{i=1}^{m} p_i \langle D^{(i)} v_n^{(i)}, v_k^{(i)} \rangle, \quad k = 0, \ldots, n \tag{2}$$

$$w_{n+1}^{(i)} = [D^{(i)} - d_{nn} I] v_n^{(i)} - \sum_{k=0}^{n-1} d_{k,n} v_k^{(i)}, \quad i = 1, \ldots, m \tag{3}$$

$$d_{n+1,n} = \sqrt{\sum_{i=1}^{m} p_i \langle w_{n+1}^{(i)}, w_{n+1}^{(i)} \rangle}, \tag{4}$$

$$v_{n+1}^{(i)} = \frac{w_{n+1}^{(i)}}{d_{n+1,n}}, \quad i = 1, \ldots, m. \tag{5}$$

1.2. Hermitian positive definite matrix associated with a Hessenberg matrix

An equivalent way to define a sequence of orthogonal polynomials is from the Hessenberg matrix.

Consider an infinite upper Hessenberg matrix D with subdiagonal strictly positive which defines a bounded, but not necessary subnormal, operator in ℓ^2.

Let $\{e_j\}_0^\infty$ be the canonical basis ℓ^2. Then, the infinite matrix $M = (c_{jk})_{j,k=0}^\infty$, given by

$$c_{jk} = \langle D^j e_0, D^k e_0 \rangle, \quad j, k \in \mathbb{N}_0 c_{0,0} = 1 \tag{6}$$

where $\langle\ .\ ,\ .\ \rangle$ is the usual inner product in ℓ^2, is a Hermitian positive definite matrix [14], which we call the Hermitian positive definite matrix associated with D.

The matrix M defines an inner product in the space $\mathcal{P}$ of polynomials. Applying the Gram–Schmidt process to z^n for $n = 0, 1, 2, \ldots$, we obtain a ONPS $\{P_n\}_{n=0}^\infty$ associated with this inner product.

The matrix D is the corresponding matrix representation of the multiplication by z operator in the Hilbert space $\mathcal{P}^2(M)$ with respect to the above ONPS.

In the next section we will need to use the matricial identities relating Hessenberg matrix and the associated Hermitian positive definite matrix, given in the following proposition.

Given an infinite Hermitian positive definite matrix $M = (c_{ij})_{i,j=0}^\infty$, we will denote by M' the matrix obtained after deleting the first column of the matrix M. M_n and M'_n will be the nth-sections of M and M' respectively, i.e., the submatrices formed by the first n rows and columns.

Proposition 1. *The Hermitian positive definite matrix M and its associated Hessenberg matrix D are related by the following formulas*

$$D = T^H S_R T^{-H} = T^{-1} M' T^{-H}, \tag{7}$$

where T is the infinite matrix whose nth-section is the lower triangular matrix, with real diagonal, obtained from the Cholesky factorization of the nth-section $M_n = T_n T_n^H$ of M and S_R is the usual shift-right matrix.

Proof. From the well known expression for monic orthogonal polynomials in terms of determinants (see for example [3])

$$\widetilde{P}_n(z) = \frac{1}{|M_n|} \begin{vmatrix} c_{0,0} & c_{1,0} & \cdots & c_{n,0} \\ c_{0,1} & c_{1,1} & \cdots & c_{n,1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & z & \ldots & z^n \end{vmatrix}$$

it can be proved that $\widetilde{P}_n(z) = |M_n^{-1} M'_n - z I_n|$.

We see now that $M_n^{-1} M'_n$ is the companion matrix F_n (also called Frobenius matrix, see for example [15]). A as consequence, it will be the companion matrix of the monic polynomial.

To see this, divide M_n in blocks as follows:

$$M_n = \left(\begin{array}{c|cccc} c_{0,0} & c_{1,0} & \cdots & c_{n-1,0} \\ \hline c_{0,1} & c_{1,1} & \cdots & c_{n-1,1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{0,n-1} & c_{1,n-1} & \cdots & c_{n-1,n-1} \end{array} \right) = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}.$$

Analogously, the matrices M'_n and M_n^{-1} can be written as

$$M'_n = \begin{pmatrix} Q_{12} & S \\ Q_{22} & T \end{pmatrix}, \qquad M_n^{-1} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix},$$

where $S = (c_{n,0})$ and $T = (c_{n,1}, c_{n,2}, \ldots, c_{n,n-1})^t$.

Since $M_n^{-1}M_n = I_n$, then

$$M_n^{-1}M_n = \begin{pmatrix} R_{11}Q_{11} + R_{12}Q_{21} & R_{11}Q_{12} + R_{12}Q_{22} \\ R_{21}Q_{11} + R_{22}Q_{21} & R_{21}Q_{12} + R_{22}Q_{22} \end{pmatrix} = \left(\begin{array}{c|ccc} 1 & 0 & \dots & 0 \\ \hline 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{array}\right)$$

and hence

$$M_n^{-1}M'_n = \begin{pmatrix} R_{11}Q_{12} + R_{12}Q_{22} & R_{11}S + R_{12}T \\ R_{21}Q_{12} + R_{22}Q_{22} & R_{21}S + R_{22}T \end{pmatrix} = \left(\begin{array}{cccc|c} 0 & 0 & \dots & 0 & a_1 \\ \hline 1 & 0 & \dots & 0 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & a_n \end{array}\right).$$

Therefore, $M_n^{-1}M'_n$ is a Frobenius matrix F_n and then, $D_n = T_n^H F_n T_n^{-H}$.

Taking into account the Cholesky factorization $M_n = T_n T_n^H$ of M_n, we obtain $M_n^{-1} = T_n^{-H}T_n^{-1}$. Then

$$D_n = T_n^H F_n T_n^{-H} = T_n^H M_n^{-1} M'_n T_n^{-H} = T_n^H T_n^{-H} T_n^{-1} M'_n T_n^{-H} = T_n^{-1} M'_n T_n^{-H}.$$

Finally, taking limits elementwise, we obtain the following identities between infinite matrices:

$$D = T^{-1}M'T^{-H} = T^H S_R T^{-H}.$$

Note that all the products are well defined since the matrices T, T^{-1}, T^H, T^{-H} are triangular. Observe also that the matrices D and S_R define bounded operators, however, if we consider the pairs of matrices T, T^{-1} and T^H, T^{-H}, in each pair just one of them defines a bounded operator. □

2. *m*-sum of Hessenberg matrices

In this section we define an algebraic operation between bounded Hessenberg matrices with subdiagonal real and positive.

Definition 1. Let $\{D^{(i)}\}_{i=1}^m$ be a family of bounded upper Hessenberg matrices with subdiagonal strictly positive and let $\{M^{(i)}\}_{i=1}^m$ be the associated family of Hermitian positive definite matrices. Consider the Hermitian positive definite sum matrix $M = \sum_{i=1}^m p_i \; M^{(i)}$ where $\sum_{i=1}^m p_i = 1$ and $p_i > 0$. We define the m-sum of the Hessenberg matrices $D^{(i)}$ with probabilities $(p_1, p_2, \dots, p_m)$ as the Hessenberg matrix D associated with M. We will denote it by

$$D = D_{p_1}^{(1)} \boxplus \cdots \boxplus D_{p_m}^{(m)}.$$

Theorem 2. *Let $\{D^{(i)}\}_{i=1}^m$ and the probabilities $(p_1, p_2, \dots, p_m)$ be as above. Let $D = D_{p_1}^{(1)} \boxplus \cdots \boxplus D_{p_m}^{(m)}$ be the m-sum. Then D can be expressed in the following way:*

$$D = \sum_{i=1}^m p_i \; [V^{(i)}]^H D^{(i)} V^{(i)},$$

with

$$V^{(i)} = [T^{(i)}]^H T^{-H}, \quad i = 1, \dots, m,$$

where $T^{(i)}$ and T are the Cholesky factors of $M^{(i)}$ and M, respectively.

Proof. Consider

$$M' = \sum_{i=1}^m p_i M'^{(i)}. \tag{8}$$

Since $V^{(i)} = [T^{(i)}]^H T^{-H}$, then $T^{-H} = [T^{(i)}]^{-H} V^{(i)}$ and $T^{-1} = [V^{(i)}]^H [T^{(i)}]^{-1}$. Multiplying (8) by T^{-1} and T^{-H} we have:

$$\begin{aligned} D &= T^{-1}M'T^{-H} = \sum_{i=1}^m p_i T^{-1}[M']^{(i)} T^{-H} \\ &= \sum_{i=1}^m p_i [V^{(i)}]^H [T^{(i)}]^{-1} [M']^{(i)} [T^{(i)}]^{-H} V^{(i)} \\ &= \sum_{i=1}^m p_i [V^{(i)}]^H D^{(i)} V^{(i)}. \end{aligned}$$

This ends the proof of the theorem. □

Proposition 2. *Let $\{D^{(i)}\}_{i=1}^{m}$ and the probabilities $(p_1, p_2, \dots, p_m)$ be as above. Let $D = D^{(1)}_{p_1} \boxplus \cdots \boxplus D^{(m)}_{p_m}$ be the m-sum. Then, $V_j^{(i)} = P_j(D^{(i)})e_0$, where $\{P_n\}$ is the* ONPS *associated with the m-sum D.*

Proof. Note that T^{-H} is the basis change from $\{P_k\}$ to $\{z^k\}$. Then, we have

$$T^{-H}e_j = \begin{pmatrix} a_{0j} \\ \vdots \\ a_{jj} \\ 0 \\ \vdots \end{pmatrix} = \sum_{k=0}^{j} a_{jk}\ e_k = \sum_{k=0}^{j} a_{jk}\ [S_R]^k e_0,$$

where $P_j(z) = \sum_{k=0}^{j} a_{kj} z^k$ and S_R is the shift-right matrix.

Since $D^{(i)} = [T^{(i)}]^H S_R [T^{(i)}]^{-H}$, then $[D^{(i)}]^k = [T^{(i)}]^H [S_R]^k [T^{(i)}]^{-H}$, for every $k = 0, 1, 2, \dots$, and hence $[D^{(i)}]^k [T^{(i)}]^H = [T^{(i)}]^H [S_R]^k$.

Therefore,

$$\begin{aligned} V_j^{(i)} &= [T^{(i)}]^H T^{-H} e_j = [T^{(i)}]^H \sum_{k=0}^{j} a_{jk} [S_R]^k e_0 \\ &= \sum_{k=0}^{j} a_{jk} ([T^{(i)}]^H [S_R]^k) e_0 = \sum_{k=0}^{j} a_{jk} ([D^{(i)}]^k [T^{(i)}]^H) e_0 \\ &= \sum_{k=0}^{j} a_{jk} [D^{(i)}]^k e_0 = P_j(D^{(i)}) e_0, \end{aligned}$$

where we have used the identity $[T^{(i)}]^H e_0 = e_0$. $\square$

Remark 1. The formulas in Theorem 1 still hold for the m-sum. These formulas allow us to define an algorithm to compute, in a recursive way, the finite n-sections D_n of the m-sum D from the n-sections of $D_n^{(i)}$.

Properties 1 (*Properties of the m-Sum*)**.** *The m-sum satisfies the following:*

(a) $D_p \boxplus D_{1-p} = D$.
(b) $D_p^{(1)} \boxplus D_q^{(2)} = D_q^{(2)} \boxplus D_p^{(1)}$ *(commutative).*
(c) $D^{(1)}_{p_1} \boxplus [D^{(2)}_{p_2} \boxplus D^{(3)}_{p_3}] = [D^{(1)}_{p_1} \boxplus D^{(2)}_{p_2}] \boxplus D^{(3)}_{p_3}$ *(associative).*
(d) *If $D^{(1)}$ and $D^{(2)}$ are subnormal matrices, then the m-sum is subnormal.*
(e) *If $D^{(1)}$ and $D^{(2)}$ are symmetric Jacobi matrices, then the m-sum is a symmetric Jacobi matrix.*

We now give some examples of computation of the m-sum of Hessenberg matrices.

Example 1. Consider $D^{(1)}$ and $D^{(2)}$ the Hessenberg matrices associated with the Hermitian positive definite matrices with entries in terms of binomial numbers

$$c_{i,j}^{(1)} = \binom{i+j-2}{i-1} \cdot i \cdot j, \quad \text{and} \quad c_{i,j}^{(2)} = \binom{i+j-2}{i-1} (i+j-1),$$

respectively. Then, the corresponding Hessenberg matrices are

$$D^{(1)} = \begin{pmatrix} 2 & -\frac{1}{2} & \frac{1}{3} & \cdots & \frac{(-1)^{n+1}}{n} & \cdots \\ 2 & 1 & 0 & \cdots & 0 & \cdots \\ 0 & \frac{3}{2} & 1 & \cdots & 0 & \cdots \\ 0 & 0 & \frac{4}{3} & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \\ 0 & 0 & 0 & \cdots & 1 & \cdots \\ 0 & 0 & 0 & \cdots & \frac{n+1}{n} & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \ddots \end{pmatrix}, \quad n \geq 2,$$

and

$$D^{(2)} = \begin{pmatrix} 2 & \frac{-\sqrt{2}}{2} & \frac{\sqrt{3}}{3} & \cdots & \frac{(-1)^n\sqrt{n}}{n} & \cdots \\ \sqrt{2} & 1 & 0 & \cdots & 0 & \cdots \\ 0 & \frac{\sqrt{2}\sqrt{3}}{2} & 1 & \cdots & 0 & \cdots \\ 0 & 0 & \frac{\sqrt{3}\sqrt{4}}{3} & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \\ 0 & 0 & 0 & \cdots & 1 & \cdots \\ 0 & 0 & 0 & \cdots & \frac{\sqrt{n(n+1)}}{n} & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \ddots \end{pmatrix}, \quad n \geq 2.$$

Then, the finite n-sections D_n of the m-sum Hessenberg matrix $D = D^{(1)}_{\frac{1}{2}} \boxplus D^{(2)}_{\frac{1}{2}}$ can be computed by the algorithm defined from Theorem 1 [9]. In particular, the 6th section of D is

$$\begin{pmatrix} 2 & -\frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{6} & -\frac{\sqrt{10}}{10} & \frac{\sqrt{15}}{15} & -\frac{\sqrt{21}}{21} \\ \sqrt{3} & 1 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{5}\sqrt{3}}{3} & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{3}\sqrt{2}}{2} & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{\sqrt{5}\sqrt{7}}{5} & 1 \end{pmatrix}.$$

It is easy to prove that the m-sum is the following infinite matrix

$$D = \begin{pmatrix} 2 & \frac{-\sqrt{3}}{3} & \frac{\sqrt{6}}{6} & \cdots & \frac{(-1)^n\sqrt{2n^2+6n+4}}{n^2+3n+2} & \cdots \\ \sqrt{3} & 1 & 0 & \cdots & 0 & \cdots \\ 0 & \frac{\sqrt{3}\sqrt{6}}{3} & 1 & \cdots & 0 & \cdots \\ 0 & 0 & \frac{\sqrt{10}\sqrt{6}}{6} & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \\ 0 & 0 & 0 & \cdots & 1 & \cdots \\ 0 & 0 & 0 & \cdots & \sqrt{\frac{2n^2+10n+12}{2n^2+6n+4}} & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \ddots \end{pmatrix} \quad n \geq 1.$$

Example 2. Consider $M^{(1)} = \frac{1}{2}I$, and

$$M^{(2)} = \begin{pmatrix} \frac{1}{2} & a & 0 & 0 & \cdots \\ a & a^2+\frac{1}{2} & a & 0 & \cdots \\ 0 & a & a^2+\frac{1}{2} & a & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then, $D^{(1)} = S_R$, and

$$D^{(2)} = \begin{pmatrix} a & -a^2 & a^3 & \dots & (-1)^{n+1}a^n & \dots \\ 1 & 0 & 0 & \dots & 0 & \dots \\ 0 & 1 & 0 & \dots & 0 & \dots \\ 0 & 0 & 1 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & & \vdots & \ddots \end{pmatrix}, \quad n \geq 1,$$

where $a < 1$, in order for $D^{(2)}$ to be bounded. Then, the m-sum is

$$D = D^{(1)}_{\frac{1}{2}} \boxplus D^{(2)}_{\frac{1}{2}} = \begin{pmatrix} 1 & a & 0 & 0 & \cdots \\ a & a^2+1 & a & 0 & \cdots \\ 0 & a & a^2+1 & a & \cdots \\ \vdots & \ddots & \ddots & \ddots & \end{pmatrix}.$$

We are interested in some properties of the m-sum for different types of matrix operators, in particular in subnormality and hyponormality. We will use some definitions and results of operator theory (see [4,11]), that concerns hyponormal and subnormal operators. An operator A is hyponormal if $A^*A - AA^* \geq 0$. Every subnormal operator is hyponormal.

In the first example, the matrices $D^{(1)}$ and $D^{(2)}$ both define operators which are not subnormal and it is easy to check that the m-sum is not hyponormal and therefore is not subnormal.

In the second example, the matrix $D^{(1)} = S_R$ defines a subnormal operator. In this case, the m-sum is not hyponormal and, therefore, the matrix $D^{(2)}$ is not subnormal.

In the next section we study another example to see which of the properties of subnormality and hyponormality are preserved under m-sum.

3. *m*-sum of weighted shifts

Consider the Hilbert space $\mathcal{P}^2$ with orthonormal basis $\{P_n\}_{n=0}^{\infty}$. Following Stampfli [10], we will call S a monotone shift if $SP_n = a_nP_{n+1}$, when the weight sequence $\{a_n\}$ is non-decreasing and bounded. Every monotone shift is hyponormal. Stampfli studied which weighted shifts are subnormal. We use Stampfli's results to construct our examples.

Proposition 3. *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two bounded sequences of positive real numbers. Consider D_a and D_b the following weighted shifts*

$$D_a = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_5 & 0 \end{pmatrix}, \qquad D_b = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ b_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_5 & 0 \end{pmatrix}.$$

Then the m-sum $D = D_{a,p} \boxplus D_{b,q}$ is a weighted shift with weight sequence given by the following formula

$$d_n = \frac{\sqrt{p\prod_{k=1}^{n} a_k^2 + q\prod_{k=1}^{n} b_k^2}}{\sqrt{p\prod_{k=1}^{n-1} a_k^2 + q\prod_{k=1}^{n-1} b_k^2}}, \quad \textit{for } n \geq 1, \tag{9}$$

where the product along the empty set is taken as 1.

Proof. The associated Hermitian positive definite matrix M_a for a weighted shift D_a is a diagonal matrix. From Eq. (6) we obtain the expression of the entries of this Hermitian positive definite matrix, which are given by

$$c_{nn} = \prod_{k=1}^{n} a_k^2 \quad \text{for } n \geq 1 \quad \text{and} \quad c_{00} = 1.$$

Then, it is easy to check that $pM_a + qM_b = M(D)$ where $M(D)$ is the Hermitian positive definite associated with the m-sum of D_a and D_b.

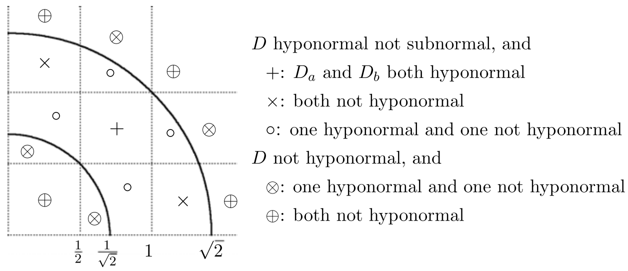

Fig. 1. Regions of parameters a and b for which D_a and D_b are subnormal or hyponormal.

In this case the auxiliary vectors which we construct in the process are

$$v_0^{(a)} = e_0 \quad \text{and} \quad v_n^{(a)} = \frac{\prod_{k=1}^{n} a_k}{\sqrt{p\prod_{k=1}^{n} a_k^2 + q\prod_{k=1}^{n} b_k^2}} e_n, \quad n = 1, 2, \ldots$$

where $\{e_n\}_{n=0}^{\infty}$ are the vectors of the canonical basis of ℓ^2. □

Example 3 (*A Subnormal Hessenberg Matrix Obtained as m-Sum of Two Hessenberg Matrices Which are not Subnormal*)**.** We consider the weighted shifts D_a and D_b with weight sequences $\{a_n \mid a_1 = \frac{1}{2}, a_2 = a, a_n = 1 \text{ for all } n \geq 3\}$ and $\{b_n \mid b_1 = \frac{1}{2},\ b_2 = b,\ b_n = 1 \text{ for all } n \geq 3\}$, respectively. Then, using (9) we have that the m-sum $D = D_{a,1/2} \boxplus D_{b,1/2}$ is a weighted shift with weight sequence

$$\left\{ d_n \mid d_1 = \frac{1}{2}, d_2 = \frac{\sqrt{a^2+b^2}}{\sqrt{2}},\ d_n = 1 \text{ for all } n \geq 3 \right\}.$$

We use the following result in [10]: let S_r be a monotone weighted shift with weight sequence $\{r_n \mid r_1 = \frac{1}{2}, r_2 = r; r_n = 1 \text{ for all } n \geq 1\}$. Then, S_r is subnormal if and only if $r_2 = 1$. Moreover, every monotone weighted shift is hyponormal.

Note that in this example the three weighted shifts D_a, D_b and D have the same kind of weight sequence $a_1 = b_1 = d_1 = \frac{1}{2}, a_2 = a, b_2 = b, d_2 = d$ and $a_n = b_n = d_n = 1$ for all $n \geq 3$. We will study three different cases for the m-sum D (subnormal, hyponormal but not subnormal, and not hyponormal), using Stampfli's result in order to take different values for the parameters a and b according to Fig. 1.

1. The m-sum $D = D_{a,\frac{1}{2}} \boxplus D_{b,\frac{1}{2}}$ is subnormal. The matrix D is subnormal if and only if $d_2 = 1$, which correspond to the circle

   $$a^2 + b^2 = 2.$$

   We consider two subcases:

   (a) A trivial case. Any of the matrices D_a or D_b is subnormal. In this example, if D_a is subnormal then $a = 1$ and this implies, in this case, that $b = 1$ and hence $D_b = D_a = D$ and both are subnormal.

   (b) The matrices D_a and D_b are both not subnormal. This is a surprising case which does not have an analog in the real case. We sum two Hermitian positive definite matrices which are not moment matrices (their Hessenberg matrices are not subnormal and hence there is not an associated measure [11,4]). However, we obtain as sum a moment matrix with an associated measure. In the real case, all bounded Jacobi matrices are self-adjoint and hence subnormal. Therefore there is always an associated measure.
   If we want D_a and D_b not to be subnormal, it suffices to take a and b, different from 1, in the circle of radius $\sqrt{2}$. Note that the case when D_a and D_b are both not hyponormal does not exist on this circle.
   In this example, the measure associated to the m-sum is the Lebesgue measure in the unit circle with an atom in the origin. The Hermitian positive definite matrix M associated with D is a diagonal matrix, whose elements are those of the sequence $1, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \ldots$. This matrix satisfies that $M = \frac{3}{4}M_1 + \frac{1}{4}M_2$ with M_1 and M_2 diagonal matrices with elements those of the sequence $1, 0, 0, \ldots$ and $1, 1, 1, \ldots$ respectively, where M_1 corresponds to an atom in the origin and M_2 corresponds to Lebesgue measure in the unit circle.
   On the other hand, the Hermitian positive definite matrix M_a associated with D_a is a diagonal matrix with elements $1, \frac{1}{4}, \frac{a^2}{4}, \frac{a^2}{4}, \frac{a^2}{4}, \ldots$ (the entries of M_b are analogous). Since $M = pM_a + (1-p)M_b$, it must be $pa^2 + (1-p)b^2 = 1$. Moreover, since (a, b) is in the circle of radius $\sqrt{2}$, for every p we have that (a, b) must be in the intersection of the circle and the ellipse, which is the point $(1, 1)$, except for $p = 1/2$, in which case the circle and the ellipse agree.

2. The m-sum $D = D_{a,\frac{1}{2}} \boxplus D_{b,\frac{1}{2}}$ is hyponormal but not subnormal.
   Since a monotone shift is hyponormal, we take for this case $\frac{1}{2} \le d_2 < 1$. Then we have the region between the two circles
   $$\frac{1}{2} \le a^2 + b^2 < 2.$$
   This region is marked in the figure with the symbols $+$, $\circ$ and $\times$, which correspond with the following different possibilities.
   (a) The matrices D_a and D_b are both hyponormal. This case is marked in the figure with $+$ and corresponds to the region
   $$\frac{1}{2} \le a \le 1 \quad \text{and} \quad \frac{1}{2} \le b \le 1.$$
   Moreover, taking $a = 1$, we have D_a subnormal and D_b not subnormal.
   (b) One of the matrices D_a or D_b is hyponormal, the other not. This case corresponds with the region marked with $\circ$ in the figure.
   (c) The matrices D_a and D_b are both not hyponormal. This case corresponds with the region $\times$ in the figure.
3. The m-sum $D = D_{a,\frac{1}{2}} \boxplus D_{b,\frac{1}{2}}$ is not hyponormal.
   This case corresponds with the values $d_2 < \frac{1}{2}$ or $d_2 > 1$, i.e. with the region
   $$a^2 + b^2 < \frac{1}{2} \quad \text{or} \quad a^2 + b^2 > 2.$$
   This region is marked in the figure with the symbols $\otimes$ and $\oplus$, which correspond with the following different possibilities.
   (a) One of the matrices D_a or D_b is hyponormal, the other not. This case corresponds with the region $\otimes$ in the figure.
   Note that in this regions, when we take $a = 1$ or $b = 1$, we have that D_a or D_b is subnormal.
   (b) The matrices D_a and D_b are both not hyponormal. This case corresponds with the region $\oplus$ in the figure.
   Note that in these regions, when $a = 1$ we have D_a is subnormal.

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