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European Journal of Combinatorics 29 (2008) 1449–1466
European Journal of Combinatorics
www.elsevier.com/locate/ejc

Catalan and Motzkin numbers modulo 4 and 8

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Available online 20 August 2007

Dedicated to Professor Zhe-Xian Wan on the occasion of his 80th birthday

Abstract

In this paper, we compute the congruences of Catalan and Motzkin numbers modulo 4 and 8. In particular, we prove the conjecture proposed by Deutsch and Sagan that no Motzkin number is a multiple of 8.

1. Introduction

Congruences of several well-known combinatorial numbers have been attracting much research interest. The most famous as well as age-old one is the Pascal's fractal which is formed by the parities of binomial coefficients $\binom{n}{k}$ [14]. As a pioneer of this problem, Kummer formulated the maximum power of a prime number p dividing $\binom{m+n}{m}$, by counting the number of carries that occurs when $[m]_p$ and $[n]_p$ are added as p-adic notations, where $[m]_p := \langle m_r \dots m_1 m_0 \rangle_p$ denotes the sequence of digits representing n in base p [9]. Lucas, another pioneer, also used the p-adic notation to develop a useful tool such that $\binom{n}{k} \equiv_p \prod_i \binom{n_i}{r_i}$, where "$\equiv_p$" denotes congruence modulo p (a prime) [11]. A generalization of Lucas' Theorem for prime powers was established by Davis and Webb [2]. The classical problem on Pascal's triangle also has modulo 4 and modulo 8 versions [3,8]. Several other combinatorial numbers have been studied on their congruences, too; like the Apéry numbers [7,12] and the central Delannoy numbers [6], not to mention the Catalan numbers.

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doi:10.1016/j.ejc.2007.06.019

The sequence of the Catalan numbers, $\langle C_n\rangle_{n=0}^{\infty} = \langle 1, 1, 2, 5, 14, 42, 132, \ldots\rangle$, defined by

$$C_n := \frac{1}{n+1}\binom{2n}{n},$$

is one of the most important sequences in combinatorics for its ubiquitous appearances in numerous problems and areas. Closely related and also well known is the sequence of the Motzkin numbers, $\langle M_n\rangle_{n=0}^{\infty} = \langle 1, 1, 2, 4, 9, 21, 51, \ldots\rangle$, which can be defined in terms of the Catalan numbers by

$$M_n := \sum_{k\geq 0}\binom{n}{2k} C_k.$$

There are many ways to define M_n, but in order to calculate the congruences, we choose the above definition. Readers may refer to [5,13] for further information.

It is well known that C_n is odd if and only if $n = 2^k - 1$ for a nonnegative integer k. The rare appearance of odd Catalan numbers partitions even Catalan numbers into consecutive runs of length $b_i = 2^i - 1$. This fact was generalized by Alter and Kubota [1] who also investigated the corresponding problem of C_n modulo any prime p. They also studied the divisibility of Catalan numbers with respect to primes and prime powers. Deutsch and Sagan [4] took one step further and derived the formula for the highest power of 2 dividing C_n. However, there is a lack of studies on the nonzero congruences for C_n. Part of our paper is devoted to this.

On the other hand, the studies on the congruences of the Motzkin numbers M_n are few and were energized very recently. Luca and Klazar proved that the Motzkin numbers are never periodic modulo any prime [10]. It seems that Deutsch and Sagan started the first systematical study on the congruences for the Motzkin numbers [4]. The congruences of M_n modulo 2, 3 and 5 are computed exactly in their paper.

However, even in the light of [1], there are few *exact* results concerning the nonzero congruences of C_n and M_n modulo a prime power. This paper is our first attempt to compensate this situation. The congruences of C_n modulo 4 and 8 are fully investigated in this paper. As for M_n, all even congruences modulo 4 and 8 are studied and this result proves a conjecture stated as follows.

Conjecture 1.1. *We have* $M_n \equiv_4 0$ *if and only if*

$$n = (4i+1)4^{j+1} - 1 \quad or \quad n = (4i+3)4^{j+1} - 2,$$

where i and j are nonnegative integers. Furthermore we never have $M_n \equiv_8 0$.

This conjecture was first given by Deutsch and Sagan [4], and part of the conjecture was due to their personal communication with Amdeberhan.

Since C_n is constructed by factorials $2n!$ and $n!$, a full understanding of the congruences of factorials is crucial for solving that of C_n. Furthermore, via the defining-equality of Motzkin number $M_n = \sum_k \binom{n}{2k} C_k$ and our result on C_k, we settle the even congruences of M_n by computing $\binom{n}{2k}$ and dealing with the summation.

The paper is organized as follows. In Section 2, we develop the main tool $E_4(3, \cdot)$ and compute the congruences of Catalan numbers modulo 4. In Section 3, we prove the first part of Conjecture 1.1 (for modulo 4). Section 4 is devoted to Catalan numbers modulo 8. The similar tool $E_8(t, \cdot)$ is developed for $t = 3, 5, 7$. Finally, we prove the second part of Conjecture 1.1 by showing all even congruences of Motzkin numbers modulo 8 in Section 5.

2. Catalan numbers modulo 4

Define $[a, b] := \{a, a+1, \ldots, b\}$ for two positive integers a and b with $a \le b$. Given positive integers n and p, let $[n]_p := \langle n_r n_{r-1} \ldots n_1 n_0 \rangle_p$ denote the sequence of digits representing n in base p, i.e., $n = n_r p^r + n_{r-1} p^{r-1} + \cdots + n_1 p + n_0$ with $n_i \in [0, p-1]$ and $n_r \neq 0$ for some integer r. For convenience, we let $n_{r+1} = n_{r+2} = \cdots = 0$, but these digits do not belong to the sequence $[n]_p$. We can even define $[0]_p$ as an empty sequence, while $0_0 = 0_1 = \cdots = 0$. Reversely, given a sequence of nonnegative integers $\langle n_r n_{r-1} \ldots n_1 n_0 \rangle_p$ with $0 \le n_i \le p-1$ for $0 \le i \le r$, we define $|\langle n_r n_{r-1} \ldots n_1 n_0 \rangle_p| := n_r p^r + n_{r-1} p^{r-1} + \cdots + n_1 p + n_0$. We use $p = 2$ in the whole paper, so sometimes we will skip the subscript 2.

Now let $[n]_2 = \langle n_r n_{r-1} \ldots n_1 n_0 \rangle_2$. Define $d_k(n) := \sum_{i \ge k} n_i$, which counts the number of the digit 1's from n_k to n_r. We also let $d(n) = d_0(n)$ for it will be used frequently. For a statement S, we set $\chi(S) = 1$ if S is true, otherwise $\chi(S) = 0$. Let us define $c_2(n) := \sum_{i \ge 0} \chi(n_i = n_{i+1} = 1)$ as the number of the consecutive pairs of 1's in the sequence $[n]_2$, and $r(n)$ the number of runs of digit 1's in $\langle n \rangle_2$. Clearly, $c_2(n) = d(n) - r(n)$.

Given a positive integer n, let $\alpha(n)$ be the highest power index of base 2 such that $2^{\alpha(n)}$ divides n. Let $m_1, m_2, \ldots, m_k$ be positive integers. For the formal product $\prod_{i=1}^k m_i$, we define $E_4(3, \prod_{i=1}^k m_i) := \sum_{i=1}^k \chi(m_i / 2^{\alpha(m_i)} \equiv_4 3)$. For instance, $E_4(3, 3 \times 4 \times 6) = 2$ and $E_4(3, 72) = 0$ even though $3 \times 4 \times 6 = 72$. In the following two lemmas, we compute $\alpha(n!)$ and the parity of $E_4(3, n!)$.

Lemma 2.1. *Let $[n]_2 = \langle n_r n_{r-1} \ldots n_1 n_0 \rangle_2$ and $2^{\alpha(n!)}$ be the highest power of 2 which divides $n!$. The power index $\alpha(n!)$ equals $\sum_{k=1}^r (2^k - 1) n_k$, and also equals $n - d(n)$.*

Proof. Notice that $\alpha(n!) = \sum_{k=1}^r \lfloor n/2^k \rfloor = |\langle n_r n_{r-1} \ldots n_2 n_1 \rangle_2| + |\langle n_r n_{r-1} \ldots n_2 \rangle_2| + \cdots + |\langle n_r \rangle_2|$, for $\lfloor n/2^k \rfloor$ counts the number of integers in $[1, n]$ that are multiples of 2^k. From this equation, the total contribution of n_k to $\alpha(n!)$ is $(2^{k-1} + 2^{k-2} + \cdots + 1) n_k$; thus $\alpha(n!) = \sum_{k=1}^r (2^k - 1) n_k = n - d(n)$ and the proof follows. ∎

Lemma 2.2. *We have $E_4(3, n!) \equiv_2 d_2(n) + c_2(n)$; also $E_4(3, n!) \equiv_2 r(n) + n_0 + n_1$.*

Proof. Suppose $[n]_2 = \langle n_r n_{r-1} \ldots n_1 n_0 \rangle_2$. For those $m \in [1, n]$ with the same value of $\alpha(m)$, say i, the sum of their $\chi(m/2^{\alpha(m)} \equiv_4 3)$ equals $\lfloor (\lfloor \frac{n}{2^i} \rfloor + 1)/4 \rfloor$. Therefore, we have

$$
\begin{aligned}
E_4(3, n!) &= \sum_{i \ge 0} \left\lfloor \frac{\lfloor \frac{n}{2^i} \rfloor + 1}{4} \right\rfloor && (1) \\
&= \sum_{i \ge 2} (2^{i-1} - 1) n_i + \sum_{i \ge 0} \chi(n_i = n_{i+1} = 1) && (2) \\
&\equiv_2 \sum_{i \ge 2} n_i + \sum_{i \ge 0} \chi(n_i = n_{i+1} = 1) \\
&= d_2(n) + c_2(n),
\end{aligned}
$$

where the second summation in (2) counts the effect caused by the addend 1 on the right-hand side of (1), while the first summation in (2) is obtained by ignoring this 1.

Since $d_2(n) = d(n) - n_0 - n_1$ and $c_2(n) = d(n) - r(n)$, we get $d_2(n) + c_2(n) \equiv_2 r(n) + n_0 + n_1$, and then the second statement of this lemma is proved. ∎

For a formal quotient $\prod_{i=1}^{k} m_i / \prod_{j=1}^{l} q_j$, define $E_4(3, \prod_{i=1}^{k} m_i / \prod_{j=1}^{l} q_j) := E_4(3, \prod_{i=1}^{k} m_i) - E_4(3, \prod_{j=1}^{l} q_j)$. Let $mrr_1(n)$ or $mrr_1([n]_2)$ be the length of the *most right run* of 1's in the sequence $[n]_2$ starting at n_0; similarly, $mrr_0(n)$ or $mrr_0([n]_2)$ is defined for the *most right run* of 0's. For instance, $mrr_1(2n) = 0$ for $(2n)_0 = 0$, $mrr_0(2n+1) = 0$ for $(2n+1)_0 = 1$, and $mrr_0(0) = 0$ for $[0]_2$ is an empty sequence. We are now ready to compute the congruences of the Catalan numbers modulo 4.

Theorem 2.3. *Let C_n be the nth Catalan number. First of all, $C_n \not\equiv_4 3$ for any n. As for other congruences, we have*

$$C_n \equiv_4 \begin{cases} 1 & \text{if } n = 2^a - 1 \text{ for some } a \geq 0; \\ 2 & \text{if } n = 2^a + 2^b - 1 \text{ for some } b > a \geq 0; \\ 0 & \text{otherwise}. \end{cases}$$

Proof. We shall first consider the case that $C_n = \frac{1}{n+1}\binom{2n}{n}$ is an odd integer. This case occurs if and only if $\alpha(n+1) = \alpha\left(\binom{2n}{n}\right)$. By Lemma 2.1, we get

$$\begin{aligned} \alpha\left(\binom{2n}{n}\right) &= \alpha(2n!) - 2\alpha(n!) \\ &= 2n - d(n) - 2[n - d(n)] \\ &= d(n). \end{aligned} \tag{3}$$

On the other hand, $\alpha(n+1)$ is equal to $mrr_0(n+1)$, which is also equal to $mrr_1(n)$; so, we derive that $\alpha(n+1) = d(n)$ if and only if $n = 2^a - 1$ for some integer $a \geq 0$. Because $E_4(3, n+1) = E_4(3, 2^a) = 0$, the congruence of this odd C_n satisfies

$$C_n \equiv_4 (-1)^{E_4\left(3, \frac{1}{n+1}\binom{2n}{n}\right)} = (-1)^{E_4(3,(2n)!) - 2E_4(3,n!)} = (-1)^{E_4(3,(2^{a+1}-2)!)} = 1,$$

where the last equality is due to Lemma 2.2 provided that both $r(2^{a+1} - 2)$ and $(2^{a+1} - 2)_1$ are the same (they could be both 0) and $(2^{a+1} - 2)_0 = 0$. Now the proof of $C_n \not\equiv_4 3$ and the situation of $C_n \equiv_4 1$ follows.

The proof will be done after finishing the situation of $C_n \equiv_4 2$. Congruence 2 happens if and only if $\alpha(n+1) = \alpha\left(\binom{2n}{n}\right) - 1 = d(n) - 1$, while the parity of $E_4(3, C_n)$ is irrelevant here, because for $2 \times 3 \equiv_4 2$. Notice that $\alpha(n+1) = d(n) - 1$ holds if and only if $n = 2^a + 2^b - 1$ for some $b > a \geq 0$, i.e., $[n]_2$ is of the form $\langle 1, 0, 0, \ldots, 0, 1, 1, \ldots, 1\rangle_2$. The whole proof follows. ∎

In Theorem 2.3, we set the second condition as "$n = 2^a + 2^b - 1$, for some $b > a \geq 0$". This unusual form ($b > a \geq 0$ not $a > b \geq 0$) is more convenient for notational purposes in many proofs.

Remark. We use Lemma 2.1 to find $\alpha\left(\binom{2n}{n}\right)$ (see Eq. (3)). Actually there is a well-known formula for $\alpha\left(\binom{n}{k}\right)$ due to Kummer [9], namely it is the number of carries that occurs when $[n-k]_2$ and $[k]_2$ are added as binary notations. In the following, Eqs. (6) and (10) can also be derived by Kummer's formula. Part of Theorem 2.3 (as well as Theorem 4.2 given later) is also a previous work of Deutsch and Sagan (see Theorem 2.1 in [4]). They showed a very neat formula

$$\alpha(C_n) = d(n+1) - 1. \tag{4}$$

3. Motzkin numbers module 4

Let us define $S_4(i, C) = \{k \in \mathbb{N} \mid C_k \equiv_4 i\}$ for $i = 0, 1, 2, 3$, where $\mathbb{N}$ is the set of nonnegative integers. By Theorem 2.3,

$$S_4(1, C) = \{2^a - 1 \mid a \geq 0\},$$
$$S_4(2, C) = \{2^a + 2^b - 1 \mid b > a \geq 0\}, \quad \text{and}$$
$$S_4(3, C) = \emptyset.$$

Using the defining-equality of the Motzkin number, $M_n = \sum_k \binom{n}{2k} C_k$, we derive that

$$M_n \equiv_4 \sum_{k \in S_4(1,C)} \binom{n}{2k} + 2 \sum_{k \in S_4(2,C)} \binom{n}{2k}. \tag{5}$$

We will analyze the above two summations to verify the conjecture that $M_n \equiv_4 0$ if and only if $n = (4i+1)4^{j+1} - 1$ or $n = (4i+3)4^{j+1} - 2$ for integers $i, j \geq 0$. In short, let us define

$$f(n) := \sum_{k \in S_4(1,C)} \binom{n}{2k}.$$

Since the second summation of (5) is even, $M_n \equiv_4 0$ happens only if $f(n)$ is even. Now we claim the first lemma as follows. This lemma is our stepping stone for solving the even congruences of M_n.

Lemma 3.1. *The summation $f(n) := \sum_{k \in S_4(1,C)} \binom{n}{2k}$ is even if and only if $n = X \cdot 4^{j+1} - \delta$ for $X, j \in \mathbb{N}$ with X being odd and $\delta = 1$ or 2.*

Proof. We shall apply the Lucas' Theorem [11] that claims $\binom{n}{m} \equiv_p \prod_i \binom{n_i}{m_i}$, where $[n]_p = \langle \ldots n_1 n_0 \rangle_p$ and $[m]_p = \langle \ldots m_1 m_0 \rangle_p$. Here we take $p = 2$ and then $\binom{n}{2k} \equiv_2 1$ if and only if $n_{i+1} \geq k_i$ for all $i \geq 0$. We have either $k = 0$ or that $[k]_2$ is a sequence of all 1's for $k \in S_4(1, C)$. Therefore, $\binom{n}{2k} \equiv_2 1$ if and only if $mrr_1(\langle \ldots n_2 n_1 \rangle_2)$ is not less than the length of $[k]_2$. (Recall $mrr_1(\cdot)$ in the paragraph before Theorem 2.3.) And then, no matter what n_0 is, we have

$$f(n) \equiv_2 \left| \left\{ k \in S_4(1, C) \,\middle|\, \binom{n}{2k} \equiv_2 1 \right\} \right| = mrr_1(\langle \ldots n_2 n_1 \rangle_2) + 1.$$

Thus, $f(n)$ is even if and only if $mrr_1(\langle \ldots n_2 n_1 \rangle_2)$ is odd. Suppose $mrr_1(\langle \ldots n_2 n_1 \rangle_2) = 2j + 1$ for $j \in \mathbb{N}$ and we conclude that $f(n)$ is even if and only if $n = X \cdot 4^{j+1} - \delta$ for $X, j \in \mathbb{N}$ with X being odd and $\delta = 1$ or 2, where δ depends on n_0. ∎

As X is odd, we say $X = 4i + \varepsilon$ with $\varepsilon = 1$ or 3. Now we narrow down to only four types of n depending on ε and δ. The layout of the sequence $[n]_2 = [(4i + \varepsilon)4^{j+1} - \delta]_2$ is important for the rest of the paper. From left to right, the sequence $[n]_2$ has four parts (four subsequences): $A := [i]_2$, $B := \langle \frac{\varepsilon - 1}{2} 0 \rangle_2$, $C := \langle 11..1 \rangle_2$, where $\langle 11..1 \rangle_2$ is of length $2j + 1$, and the single digit $D := \langle 2 - \delta \rangle$.

To investigate the even congruences of M_n modulo 4, first we need to see the congruences of $\binom{n}{2k}$ for those $n = (4i + \varepsilon)4^{j+1} - \delta$ and $k = 2^a - 1$ and we still need the following two lemmas.

Lemma 3.2. *Given $n = (4i + \varepsilon)4^{j+1} - \delta$ and $k = 2^a - 1$ for $a, i, j \in \mathbb{N}$, $\varepsilon = 1, 3$ and $\delta = 1, 2$, we have*

$$\alpha\left(\binom{n}{2k}\right) = \begin{cases} 0 & \text{if } a \le 2j+1; \\ 1 & \text{if } a = 2j+2 \quad \text{and} \quad \varepsilon = 3; \\ 2 & \text{if } i \equiv_2 1 \text{ and either (i) } a = 2j+2 \text{ and } \varepsilon = 1 \text{ or (ii) } a = 2j+3 \end{cases}$$

otherwise $\alpha\left(\binom{n}{2k}\right) \ge 3$.

Proof. By Lemma 2.1, we have

$$\begin{aligned}\alpha\left(\binom{n}{2k}\right) &= [n - d(n)] - [2k - d(2k)] - [(n-2k) - d(n-2k)] \\ &= -d(n) + d(k) + d(n-2k) \\ &= -d(n) + a + d(n-2k). \qquad (6)\end{aligned}$$

We take more efforts on the value of $d(n-2k)$. Let us observe the change of A, B, and C as we subtract $2k$ from n, while the unchanged D is ignored. The discussion is listed as four cases below. This discussion can also be done by Kummer's formula. Notice that $k = 2^a - 1$ means $[k]_2$ is an all 1 sequence of length a, and so is C whose length is $2j+1$.

(a) When $a \le 2j+1$, there are a digits 1's in C eliminated by subtracting $2k$; so $d(n-2k) = d(n) - a$ and then $\alpha\left(\binom{n}{2k}\right) = 0$.

(b) When $a = 2j+2$ and $\varepsilon = 3$, not only all 1's in C are eliminated, but also $B = \langle 10\rangle_2$ turns into $\langle 01\rangle_2$. We find $d(n-2k) = d(n) - a + 1$ and then $\alpha\left(\binom{n}{2k}\right) = 1$.

(c) As for $a = 2j+2$ and $\varepsilon = 1$, we must have $i \ge 1$ otherwise $\binom{n}{2k} = 0$ for $n < 2k$. After subtracting $2k$, all 1's in C are eliminated, $B = \langle 00\rangle_2$ becomes $\langle 11\rangle_2$. Also A becomes a sequence with at least $d(i) - 1$ digit 1's, while this minimum happens if and only if i is odd. Therefore, if i is odd then $d(n-2k) = d(n) - a + 2$ and $\alpha\left(\binom{n}{2k}\right) = 2$; if i is even then $d(n-2k) \ge d(n) - a + 3$ and $\alpha\left(\binom{n}{2k}\right) \ge 3$.

(d) Finally, let us check the remaining case $a \ge 2j+3$. After subtracting $2k$, all 1's in C are eliminated and $B = \langle \frac{\varepsilon-1}{2}0\rangle_2$ turns into $\langle \frac{\varepsilon-1}{2}1\rangle_2$. Also A becomes $[i - (2^{a-2j-3} - 1) - 1]_2 = [i - 2^{a-2j-3}]_2$. Notice that $d(i - 2^{a-2j-3}) \ge d(i) - 1$ while the equality holds if and only if $i_{a-2j-3} = 1$. Therefore, we have $\alpha\left(\binom{n}{2k}\right) \ge a - 2j - 1$ in this case. We conclude that if $a = 2j+3$ and $i \equiv_2 1$ then $d(n-2k) = d(n) - a + 2$ and $\alpha\left(\binom{n}{2k}\right) = 2$; otherwise $d(n-2k) \ge d(n) - a + 3$ and $\alpha\left(\binom{n}{2k}\right) \ge 3$. ∎

Lemma 3.3. *Given $n = (4i + \varepsilon)4^{j+1} - \delta$ and $k = 2^a - 1$ for $a, i, j \in \mathbb{N}$, $\varepsilon = 1, 3$ and $\delta = 1, 2$, we have*

$$\binom{n}{2k} \equiv_4 \begin{cases} (-1)^{\chi(a\ge 1)\chi(\delta=2)+\chi(a=2j+1)} & \text{if } a \le 2j+1, \\ 2 & \text{if } a = 2j+2 \text{ and } \varepsilon = 3; \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\sum_{k\in S_4(1,C)} \binom{n}{2k} \equiv_4 2(j + \chi(\varepsilon = 3)) + (-1)^{\chi(\delta=1)} + 1.$$

Proof. For the first equivalence, the congruences 2 and 0 are direct consequences of Lemma 3.2; so we only need to calculate the nontrivial case when $a \le 2j+1$. In this condition,

$\binom{n}{2k} \equiv_4 (-1)^{E_4(3,\binom{n}{2k})}$. We have

$$\begin{aligned} E_4\left(3, \binom{n}{2k}\right) &\equiv_2 E_4(3, n!) + E_4(3, (2k)!) + E_4(3, (n-2k)!) \\ &\equiv_2 [r(n) + (2-\delta) + 1] + [r(2k) + 0 + k_0] \\ &\quad + [r(n-2k) + (2-\delta) + (1-k_0)] \quad (7) \\ &\equiv_2 r(n) + r(2k) + r(n-2k) \quad (8) \\ &\equiv_2 r(n) + \chi(a \geq 1) + [r(n) + \chi(a \geq 1)(2-\delta) - \chi(a = 2j+1)] \quad (9) \\ &\equiv_2 \chi(a \geq 1) + \chi(a \geq 1)(2-\delta) + \chi(a = 2j+1) \\ &= \chi(a \geq 1)(3-\delta) + \chi(a = 2j+1) \\ &\equiv_2 \chi(a \geq 1)\chi(\delta = 2) + \chi(a = 2j+1), \end{aligned}$$

as required. Among the above equivalences, (7) is obtained by Lemma 2.2 provided $n_1 = 1$. To derive the three terms inside the brackets of (9), we shall refer to case (a) in the last proof. In that case, a digit 1's are eliminated from C. Of course, nothing changes when $a = 0$. If $1 \leq a \leq 2j$ and $\delta = 1$, then the most right run of $[n]_2$ is bisected. The number of runs remains the same if $a = 2j+1$ and $\delta = 1$ or if $1 \leq a \leq 2j$ and $\delta = 2$. As for the last case $a = 2j+1$ and $\delta = 2$, the new number of runs becomes $r(n) - 1$. We obtain (9) after summarizing these conditions.

Applying the first equivalence of this lemma, we obtain

$$\begin{aligned} \sum_{k \in S_4(1,C)} \binom{n}{2k} &\equiv_4 \sum_{a=0}^{2j+2} \binom{n}{2 \times 2^a - 2} \\ &\equiv_4 1 + 2j(-1)^{\chi(\delta=2)} + (-1)^{\chi(\delta=2)+1} + 2\chi(\varepsilon = 3) \\ &\equiv_4 1 + (-1)^{\chi(\delta=1)} + 2(j + \chi(\varepsilon = 3)), \end{aligned}$$

where the second equivalence is obtained by considering $a = 0$, $1 \leq a \leq 2j$, $a = 2j+1$, and $a = 2j+2$. The proof is now complete. ■

Now we shall turn our attention to the necessary and sufficient condition for $\sum_{k \in S_4(2,C)} \binom{n}{2k} \equiv_2 1$ because the second term of Eq. (5) is $2\sum_{k \in S_4(2,C)} \binom{n}{2k}$.

Lemma 3.4. *Given $n = (4i+\varepsilon)4^{j+1} - \delta$ and $k = 2^a + 2^b - 1$ for $a, b, i, j \in \mathbb{N}$ with $b > a$, $\varepsilon = 1, 3$ and $\delta = 1, 2$, the value $\binom{n}{2k}$ is odd if only if $a \leq 2j+1$ and $n_{b+1} = 1$. Furthermore, $\sum_{k \in S_4(2,C)} \binom{n}{2k} \equiv_2 j$.*

Proof. Similar to Eq. (6), but this time we have

$$\alpha\left(\binom{n}{2k}\right) = -d(n) + (a+1) + d(n-2k), \quad (10)$$

because $k = 2^a + 2^b - 1$. For $d(n-2k)$, we can consider $n - 2k$ as n subtracted by $2(a^2 - 1)$ and then subtracted by $2(2^b)$. Before the second subtraction, the evaluation of $\alpha\left(\binom{n}{2k}\right)$ shall be as same as in the proof of Lemma 3.2. We refer to cases (a)–(d) in that proof and also compare (10) with (6); The value of α here is 1 greater than it is in each case there. For the value of $d(n-2k)$, the effect from the second subtraction is independent of that from the first one. Subtracting $2(2^b)$ can at most reduce the value of d by 1 (sometimes, it can even increase d), when the extreme case happens if and only if $n_{b+1} = 1$. Combining $n_{b+1} = 1$ and the case (a) in the proof of Lemma 3.2, which requires $a \leq 2j+1$, forms the necessary and sufficient condition for $\alpha\left(\binom{n}{2k}\right) = 0$, as well as for $\binom{n}{2k} \equiv_2 1$.

Now for the parity of $\sum_{k\in S_4(2,C)} \binom{n}{2k}$, let us refer to the four parts, A, B, C and D, of the sequence $[n]_2 = [(4i+\varepsilon)4^{j+1}-\delta]_2$. Given a fixed a, we count how many b's make $\binom{n}{2k} \equiv_2 1$. If $a = 2j+1$ then there are $d(i)+\chi(\varepsilon=3)$ such b's; if $0 \le a \le 2j$ then there are $d(i)+\chi(\varepsilon=3)+(2j-a)$. Therefore

$$\begin{aligned}\sum_{k\in S_4(2,C)} \binom{n}{2k} &\equiv_2 d(i)+\chi(\varepsilon=3)+\sum_{a=0}^{2j}[d(i)+\chi(\varepsilon=3)+(2j-a)]\\ &\equiv_2 (2j+2)(d(i)+\chi(\varepsilon=3))+\sum_{a=0}^{2j} a\\ &\equiv_2 j(2j+1)\\ &\equiv_2 j. \quad \blacksquare\end{aligned}$$

Proof for the first part of Conjecture 1.1. By Lemmas 3.3 and 3.4, we can simplify Eq. (5) as $M_n \equiv_4 [2(j+\chi(\varepsilon=3))+(-1)^{\chi(\delta=1)}+1]+2j$. We simply plug in the four types of $n=(4i+\varepsilon)4^{j+1}-\delta$ with $\varepsilon=1,3$ and $\delta=1,2$ and then obtain

$$\begin{aligned}&M_n \equiv_4 0 \quad \text{if } (\varepsilon,\delta)=(1,1) \text{ or } (3,2);\\ &M_n \equiv_4 2 \quad \text{if } (\varepsilon,\delta)=(1,2) \text{ or } (3,1).\end{aligned} \tag{11}$$

Notice that these four types of n are the necessary and sufficient conditions for M_n to be even; so the proof is complete. $\blacksquare$

We not only prove the first part of Conjecture 1.1, but also Eq. (11) in the above proof offers an auxiliary property as follows.

Theorem 3.5. *We have $M_n \equiv_4 2$ if and only if*

$$n=(4i+1)4^{j+1}-2 \quad \text{or} \quad n=(4i+3)4^{j+3}-1, \quad \text{where } i,j\in\mathbb{N}.$$

4. Factorials and Catalan numbers modulo 8

The process is almost the same. To verify the conjecture that $M_n \equiv_8 0$ never happens, we need first to take care of the congruences of the factorials and the Catalan numbers modulo 8. For a formal product $\prod_{i=1}^k m_i$ of positive integers, we define $E_8(t,\prod_{i=1}^k m_i) := \sum_{i=1}^k \chi(m_i/2^{\alpha(m_i)} \equiv_8 t)$ for $t=3,5,7$. Similarly, we define $E_8(t,\prod_{i=1}^k m_i/\prod_{j=1}^l q_j) := E_8(t,\prod_{i=1}^k m_i) - E_8(t,\prod_{j=1}^l q_j)$. For example, $E_8(3,7!)=2$ and $E_8(5,7!)=E_8(7,7!)=1$; however, $E_8(3,5040)=1$ and $E_8(5,5040)=E_8(7,5040)=0$. The difference between $E_8(t,7!)$ and $E_8(t,5040)$ is due to $5\times 7 \equiv_8 3$. The table for the product rules of $\mathbb{Z}_8$, as follows, is useful.

	3	5	7	2	4	6
3	1	7	5	6	4	2
5		1	3	2	4	6
7			1	6	4	2

From this table, we are again interested in the parity of $E_8(t,\prod_{i=1}^k m_i)$ because $3^2 \equiv_8 5^2 \equiv_8 7^2 \equiv_8 1$.

Suppose $[n]_2 = \langle n_r \dots n_1 n_0\rangle_2$. Some new notation helps us to evaluate the parity of $E_8(t, \prod_{i=1}^k m_i)$. In the bit sequence $[n]_2$, let $r_1(n)$ be the number of isolated 1's, $zr(n)$ the number of runs made by 0's, $zr_1(n)$ the number of isolated 0's. We compute the parity of $E_8(t, n!)$ in the following lemma.

Lemma 4.1. *We have*

1. $E_8(3, n!) \equiv_2 r_1(n) + zr(n) + n_0 + n_2$,
2. $E_8(5, n!) \equiv_2 r(n) + zr_1(n) + n_0 + n_2$, *and*
3. $E_8(7, n!) \equiv_2 r_1(n) + n_0 + n_1 + n_2$.

Proof. Suppose $[n]_2 = \langle n_r n_{r-1} \dots n_2 n_1 n_0\rangle_2$. For $t = 3, 5, 7$, we define $A_t := \{[x]_2 \mid t \le x \le 7\}$, for here $[3]_2 = \langle 011\rangle_2$ not $\langle 11\rangle_2$ to make all elements in A_t of length 3. The argument for Eqs. (1) and (2) still works here. So we obtain

$$\begin{aligned} E_8(t, n!) &= \sum_{i\ge 0} \left\lfloor \frac{\lfloor \frac{n}{2^i} \rfloor + (8-t)}{8} \right\rfloor \\ &= \sum_{i\ge 3} (2^{i-2} - 1) n_i + \sum_{i\ge 0} \chi(|\langle n_{i+2} n_{i+1} n_i\rangle_2| \ge t) \qquad (12) \\ &\equiv_2 d_3(n) + \sum_{i\ge 0} \chi(\langle n_{i+2} n_{i+1} n_i\rangle_2 \in A_t) \qquad (13) \end{aligned}$$

for $t = 3, 5, 7$. In the second summation of both (12) and (13), the upper limit of index i is $r - 1$ or $r - 2$, because $n_{r+k} = 0$ for $k \ge 1$ and both $\langle 00n_r\rangle$ and $\langle 000\rangle$ are irrelevant to the counting.

For the further precise evaluation, we need the following equations. They are easy to check and left to the reader. Again $d(n)$, $r(n)$, $r_1(n)$, $zr(n)$, $zr_1(n)$ are irrelevant to those $n_{r+k} = 0$ for $k \ge 1$ for they do not belong to $[n]_2$.

(a) $\sum_{i=0}^{r-1} \chi(\langle n_{i+2} n_{i+1} n_i\rangle = \langle 011\rangle) = r(n) - r_1(n)$.
(b) $\sum_{i=0}^{r-2} \chi(\langle n_{i+2} n_{i+1} n_i\rangle = \langle 100\rangle) = zr(n) - zr_1(n)$.
(c) $\sum_{i=0}^{r-2} \chi(\langle n_{i+2} n_{i+1} n_i\rangle = \langle 101\rangle) = zr_1(n) - n_1(1 - n_0)$.
(d) $\sum_{i=0}^{r-2} \chi(\langle n_{i+2} n_{i+1} n_i\rangle = \langle 110\rangle) = r(n) - r_1(n) - n_0 n_1$.
(e) $\sum_{i=0}^{r-2} \chi(\langle n_{i+2} n_{i+1} n_i\rangle = \langle 111\rangle) = c_3(n) = d(n) - 2r(n) + r_1(n)$.

By (13) and summing up (c), (d) and (e), we obtain

$$\begin{aligned} E_8(5, n!) &\equiv_2 d_3(n) + [zr_1(n) - n_1(1 - n_0)] + [r(n) - r_1(n) - n_0 n_1] \\ &\quad + [d(n) - 2r(n) + r_1(n)] \\ &\equiv_2 r(n) + zr_1(n) + n_0 + n_2. \end{aligned}$$

The checking of the other two is left to the reader. ■

With the help of Lemma 4.1, we can bisect each condition in Theorem 2.3 to form a new conditional equation as a refinement to evaluate C_n modulo 8.

Theorem 4.2. *Let C_n be the nth Catalan number. First of all, $C_n \not\equiv_8 3$ and $C_n \not\equiv_8 7$ for any n. As for other congruences, we have*

$$C_n \equiv_8 \begin{cases} 1 & \text{if } n = 0 \text{ or } 1; \\ 2 & \text{if } n = 2^a + 2^{a+1} - 1 \text{ for some } a \geq 0; \\ 4 & \text{if } n = 2^a + 2^b + 2^c - 1 \text{ for some } c > b > a \geq 0; \\ 5 & \text{if } n = 2^a - 1 \text{ for some } a \geq 2; \\ 6 & \text{if } n = 2^a + 2^b - 1 \text{ for some } b - 2 \geq a \geq 0; \\ 0 & \text{otherwise}. \end{cases}$$

Proof. Since $C_n \not\equiv_4 3$ for any n, we have $C_n \not\equiv_8 3$ and $C_n \not\equiv_8 7$. Now for the other congruences, by Theorem 2.3, $C_n \equiv_4 1$ if and only if $n = 2^a - 1$ for $a \geq 0$. Since $C_0 = C_1 = 1$, we need only to check that if $n = 2^a - 1$ for $a \geq 2$ then $C_n \equiv_8 5$. Let us apply $E_8(t, \cdot)$ on the formal quotient $C_n = \binom{2n}{n}/(n+1)$ with $n = 2^a - 1$ for $a \geq 2$, and obtain $E_8(t, C_n) = E_8(t, (2n)!) - 2E_8(t, n!) - E_8(t, n+1) \equiv_2 E_8(t, (2n)!)$. After deriving $E_8(3, C_n) \equiv_2 E_8(7, C_n) \equiv_2 0$ and $E_8(5, C_n) \equiv_2 1$ by Lemma 4.1, the cases of congruences 1 and 5 are done.

Now for the cases of congruences 2 and 6. By Theorem 2.3, we shall suppose $n = 2^a + 2^b - 1$ for $b > a \geq 0$. We need to find out the necessary and sufficient condition for $C_n \equiv_8 2$. In the ring $\mathbb{Z}_8$, we have four possible products to create congruence 2, but we sort them as two occasions: (i) 2 and $2 \times 3 \times 5 \times 7$, (ii) 2×5 and $2 \times 3 \times 7$. For $\alpha(C_{2^a+2^b-1}) = 1$ we shall consider the possible combination of the parities of $E_8(t, n!)$ for $t = 3, 5, 7$. Let $W_{tu} := E_8(t, C_n) + E_8(u, C_n)$. Clearly, $(W_{35} + 1)(W_{37} + 1) \equiv_2 1$ is the necessary and sufficient condition for the occasion (i). Similarly, $W_{35}(W_{37} + 1) \equiv_2 1$ is the condition for the occasion (ii). We conclude that

$$C_{2^a+2^b-1} \equiv_8 2 \iff W_{37} \equiv_2 0. \tag{14}$$

By Lemma 4.1 and the layout of $[2^a + 2^b - 1]_2$, we have

$$\begin{aligned} W_{37} &\equiv_2 [E_8(3, (2n)!) + E_8(7, (2n)!)] + [E_8(3, n+1) + E_8(7, n+1)] \\ &\equiv_2 [zr(2n) + (2n)_1] + [E_8(3, 2^{b-a} + 1) + E_8(7, 2^{b-a} + 1)] \\ &\equiv_2 [1 + \chi(a \geq 1) + \chi(a + 2 \leq b) + \chi(a \geq 1)] + [\chi(a + 1 = b) + \chi(a + 2 \leq b)] \\ &\equiv_2 1 + \chi(a + 1 = b). \end{aligned}$$

Therefore, $W_{37} \equiv_2 0$ if and only if $a + 1 = b$, and the proof of this case follows.

It requires $\alpha(C_n) = 2$ to get congruence 4. Referring to the proof of Theorem 2.3, it is the same to require $\alpha(n + 1) = d(n) - 2$. The checking for the necessary and sufficient condition, $n = 2^a + 2^b + 2^c - 1$ for $c > b > a \geq 0$, is left to the reader. The checking can also be done by using Deutsch and Sagan's formula (see (4) of this paper). ■

5. Motzkin numbers modulo 8

Now we are ready to prove that $M_n \equiv_8 0$ never happens. Actually, we focus on even congruences 0, 2, 4, and 6 modulo 8, and we shall suppose $n = (4i + \varepsilon)4^{j+1} - \delta$ for $i, j \in \mathbb{N}$, $\varepsilon = 1, 3$ and $\delta = 1, 2$.

We recall that the layout of $[(4i + \varepsilon)4^{j+1} - \delta]_2$, from left to right, consists of four subsequences: $A := [i]_2$, $B := \langle \frac{\varepsilon-1}{2} 0 \rangle_2$, $C := \langle 11 \ldots 1 \rangle_2$ of length $2j + 1$, and the single digit $D := \langle 2 - \delta \rangle_2$. In addition, we define $Y := 4i + \varepsilon - 1$ and $y := d(Y)$. The sequence $[Y]_2$ is the combination of the first two parts of $[n]_2$, namely A and B.

Let $S_8(i, C) := \{k \in \mathbb{N} \mid C_k \equiv_8 i\}$ for $i = 0, \ldots, 7$. By Theorem 4.2, we know $S_8(3, C) = S_8(7, C) = \emptyset$ and also the types of elements in the other $S_8(i, C)$. For $M_n = \sum_{k\geq 0} \binom{n}{2k} C_k$, we have

$$M_n \equiv_8 A_1(n) + 2A_2(n) + 4A_4(n) + 5A_5(n) + 6A_6(n),$$

where $A_t(n) = \sum_{k\in S_8(t,C)} \binom{n}{2k}$. The following lemma evaluates $4A_4(n)$. It is easy because we only need to check the parity of the corresponding summation.

Lemma 5.1. *Let $n = (4i + \varepsilon)4^{j+1} - \delta$ for $i, j \in \mathbb{N}$, $\varepsilon = 1, 3$ and $\delta = 1, 2$. Also let $S_8(4, C) = \{2^a + 2^b + 2^c - 1 \mid a, b, c \in \mathbb{N}$ with $c > b > a \geq 0\}$. We have*

$$4A_4 := 4 \sum_{k\in S_8(4,C)} \binom{n}{2k} \equiv_8 4jy + 4j.$$

Proof. Suppose $[n]_2 = \langle n_r \ldots n_1 n_0\rangle_2$. By the Lucas' Theorem, $\binom{n}{2k} \equiv_2 1$ if and only if $n_{l+1} \geq k_l$ for all $l \geq 0$. Notice that $[k]_2$ consists of $a + 2$ digit 1's with $\langle k_a k_{a-1} \ldots k_1 k_0\rangle = \langle 01 \ldots 11\rangle$. So $\binom{n}{2k} \equiv_2 1$ means $a \leq 2j + 1$ and $n_{b+1} = n_{c+1} = 1$ with $c > b > a \geq 0$. The following is obtained by counting the possible ordered pairs (b, c) with respect to a fixed $a \in [0, 2j + 1]$, where $y = d(Y)$.

$$\begin{aligned} A_4(n) &\equiv_2 \sum_{a=0}^{2j} \left(\binom{y}{2} + y(2j - a) + \binom{2j - a}{2}\right) + \binom{y}{2} \\ &\equiv_2 jy + \frac{1}{3} j(4j^2 - 1). \end{aligned}$$

We complete this proof after knowing that $\frac{1}{3} j(4j^2 - 1) \equiv_2 j$ and multiplying A_4 by 4 under modulo 8. ■

We abuse the notation W_{st}, once used in the proof of Theorem 4.2, and let $W_{st} := E_8(s, \binom{n}{2k}) + E_8(t, \binom{n}{2k})$. The congruence of $\binom{n}{2k}/2^{\alpha(\binom{n}{2k})}$ is crucial, and we can use the following properties to evaluate it.

$$\begin{aligned} &\binom{n}{2k}/2^{\alpha(\binom{n}{2k})} \equiv_8 1 \Leftrightarrow (W_{35} + 1)(W_{37} + 1) = W_{35}W_{37} + W_{57} + 1 \equiv_2 1, \\ &\binom{n}{2k}/2^{\alpha(\binom{n}{2k})} \equiv_8 3 \Leftrightarrow W_{35}(W_{57} + 1) \equiv_2 1, \quad \text{and} \\ &\binom{n}{2k}/2^{\alpha(\binom{n}{2k})} \equiv_8 1 \text{ or } 5 \Leftrightarrow W_{37} \equiv_2 0. \end{aligned}$$

Actually (14) was obtained by the last property and the first two properties were discussed in the same proof.

In the following we compute $A_1 + 5A_5$ and $2A_2 + 6A_6$ under modulo 8. Indeed, we can evaluate A_1, A_2, A_5, and A_6 independently, but we pair them up and derive simpler formulas.

Lemma 5.2. *We have*

$$A_1(n) + 5A_5(n) \equiv_8 \begin{cases} 2j + 4 & \text{if } \varepsilon = 1 \text{ and } \delta = 1; \\ 6j + 2 & \text{if } \varepsilon = 1 \text{ and } \delta = 2; \\ 2j + 6 & \text{if } \varepsilon = 3 \text{ and } \delta = 1; \\ 6j & \text{if } \varepsilon = 3 \text{ and } \delta = 2. \end{cases}$$

Proof. We shall look at those $n = (4i + \varepsilon)4^{j+1} - \delta$ and $k = 2^a - 1$. According to the value of $\alpha\left(\binom{n}{2k}\right)$ illustrated in Lemma 3.2, let us group the contribution of $\binom{n}{2k}$ into three classes as follows, while the case with $\alpha\left(\binom{n}{2k}\right) \geq 3$ is ignored because of modulo 8. In each class, $\binom{n}{2k}$ contributes to A_1 when $a = 0$ and 1; to A_5 when $a \geq 2$.

Class I. Suppose $\alpha\left(\binom{n}{2k}\right) = 2$ which is provided by $i \equiv_2 1$ (i.e., $i_0 = 1$) and either (i) $a = 2j+2$ and $\varepsilon = 1$ or (ii) $a = 2j+3$. Both situations make $a \geq 2$, so there is no contribution to A_1. The total contribution of $\binom{n}{2k}$ to A_5 is

$$4i_0(\chi(\varepsilon = 1) + 1) = 4i_0\chi(\varepsilon = 3) \pmod 8, \tag{15}$$

where 4 is due to $\alpha\left(\binom{n}{2k}\right) = 2$. For $5 \times 4 \equiv_8 4$, the same value contributes to $5A_5$ as well as $A_1 + 5A_5$ in this class.

Class II. Suppose $\alpha\left(\binom{n}{2k}\right) = 0$ which is provided by $a \leq 2j+1$. We will calculate W_{st} in this class, because they determine whether $\binom{n}{2k} \equiv_8 1, 3, 5$, or 7. But first we shall deal with r_1, r, zr_1, and zr by plugging in $n - 2k$:

$$\begin{aligned}
r_1(n-2k) &= r_1(n) + \chi(\delta = 1)\chi(a \geq 1) - \chi(\delta = 2)\chi(a = 1)\chi(j = 0) \\
&\quad + \chi(a = 2j)\chi(j \geq 1), \\
r(n-2k) &= r(n) + \chi(\delta = 1)\chi(1 \leq a \leq 2j)\chi(j \geq 1) - \chi(\delta = 2)\chi(a = 2j+1), \\
zr_1(n-2k) &= zr_1(n) + \chi(\delta = 1)\chi(a = 1)\chi(j \geq 1) - \chi(\delta = 2)\chi(a \geq 1) \\
&\quad - \chi(\varepsilon = 3)\chi(a = 2j+1), \\
zr(n-2k) &= zr(n) + \chi(\delta = 1)\chi(1 \leq a \leq 2j)\chi(j \geq 1) \\
&\quad - \chi(\delta = 2)\chi(a = 2j+1).
\end{aligned} \tag{16}$$

Basically, these four formulas are derived by observing the layout of the four subsequences, A, B, C, and D, of $[n]_2$ and $[2k]_2 = \langle 11\ldots10\rangle_2$, which is of length $a+1$. As a special case when $a = 0$, plugging in $n - 2k$ is as same as plugging in n. Also $a = 0, 1, 2j$ and $2j+1$ are special in some sense. Notice that $\chi(j \geq 1)$ can actually be removed from the formulas of $r(n-2k)$ and $zr(n-2k)$, but it was kept for the convenience of the following calculation.

In general, we define $(f+g)(n) := f(n) + g(n)$ for briefness. With the help of the above four formulas and by Lemma 4.1, we have

$$\begin{aligned}
W_{35} &\equiv_2 E_8\left(3, \binom{n}{2k}\right) + E_8\left(5, \binom{n}{2k}\right) \\
&\equiv_2 [(r_1 + r + zr_1 + zr)(n) - (r_1 + r + zr_1 + zr)(n-2k)] \\
&\quad - (r_1 + r + zr_1 + zr)(2k) \\
&\equiv_2 [\chi(\delta = 1)\chi(a = 1)\chi(j \geq 1) + \chi(\delta = 2)\chi(a = 1)\chi(j = 0) \\
&\quad + \chi(\varepsilon = 3)\chi(a = 2j+1) + \chi(a = 2j)\chi(j \geq 1) + \chi(a \geq 1)] - \chi(a \geq 2) \\
&\equiv_2 \chi(\delta = 1)\chi(a = 1)\chi(j = 0) + \chi(\delta = 2)\chi(a = 1)\chi(j \geq 1) \\
&\quad + \chi(\varepsilon = 3)\chi(a = 2j+1) + \chi(a = 2j)\chi(j \geq 1), \\
W_{37} &\equiv_2 [zr(n) - zr(n-2k)] - zr(2k) + [n_1 - (2k)_1 - (n-2k)_1] \qquad (17) \\
&\equiv_2 \chi(\delta = 1)\chi(1 \leq a \leq 2j)\chi(j \geq 1) + \chi(\delta = 2)\chi(a = 2j+1) + \chi(a \geq 1), \\
W_{57} &\equiv_2 W_{35} + W_{37} \\
&\equiv_2 \chi(\delta = 1)\chi(2 \leq a \leq 2j)\chi(j \geq 1) + \chi(\delta = 2)\chi(a = 2j+1)\chi(j \geq 1) \\
&\quad + \chi(\varepsilon = 3)\chi(a = 2j+1) + \chi(a = 2j)\chi(j \geq 1) + \chi(a \geq 2). \qquad (18)
\end{aligned}$$

Table 1
Contribution of $\binom{n}{2k}$ (mod 8) to $A_1 + 5a_5$ in Class II

	$j=0$, $\varepsilon=1$, $\delta=1$	$j=0$, $\varepsilon=1$, $\delta=2$	$j=0$, $\varepsilon=3$, $\delta=1$	$j=0$, $\varepsilon=3$, $\delta=2$	$j\geq 1$, $\varepsilon=1$, $\delta=1$	$j\geq 1$, $\varepsilon=1$, $\delta=2$	$j\geq 1$, $\varepsilon=3$, $\delta=1$	$j\geq 1$, $\varepsilon=3$, $\delta=2$	$\#\{a\}$
$a=0^*$	1	1	1	1	1	1	1	1	1
$a=1^*$	3	1	7	5	1	3	1	3	1
$2 \leq a \leq 2j-1$					1	7	1	7	$2j-2$
$a=2j\geq 2$					5	3	5	3	1
$a=2j+1\geq 3$					7	1	3	5	1
A_1^*	4	2	0	6	2	4	2	4	
A_5	0	0	0	0	$2j+2$	$6j+6$	$2j+6$	$6j+2$	
A_1+5A_5	4	2	0	6	$2j+4$	$6j+2$	$2j$	$6j+6$	

Notice that $n_1 - (2k)_1 - (n-2k)_1 = 0$ in (17). We can also use $W_{57} \equiv_2 [(r_1 + r + zr_1)(n) - (r_1 + r + zr_1)(n-2k)] - (r_1 + r + zr_1)(2k) + [n_1 - (2k)_1 - (n-2k)_1]$ to obtain (18).

Respectively, the necessary and sufficient condition for $\binom{n}{2k} \equiv_8 1, 3, 5, 7$ are the following four values being odd.

$$\begin{aligned}
W_{35}W_{37} + W_{57} + 1 \equiv_2 {} & \chi(\delta=1)[\chi(j=0, a=1) + \chi(j\geq 2, 2\leq a\leq 2j-1)] \\
& + \chi(\delta=2)[\chi(j\geq 1, a=1) + \chi(a=2j+1)(\chi(\varepsilon=3) \\
& + \chi(j\geq 1))] + \chi(a\geq 2) + 1, \\
W_{37}(W_{57}+1) \equiv_2 {} & \chi(\delta=1, a=2j+1)[\chi(j=0) + \chi(\varepsilon=3)] \\
& + \chi(\delta=2, j\geq 1)[\chi(a=2j) + (a=1)], \\
W_{35}(W_{37}+1) \equiv_2 {} & \chi(\delta=1)\chi(j\geq 1, a=2j) + \chi(\delta=2)\chi(\varepsilon=3, a=2j+1), \\
W_{37}(W_{35}+1) \equiv_2 {} & \chi(a\geq 1) + \chi(a=2j+1)[1 + \chi(\delta=1)(\chi(j=0) + \chi(\varepsilon=1))] \\
& + \chi(j\geq 1)[\chi(a=2j) + \chi(\delta=1, 1\leq a\leq 2j-1) \\
& + \chi(\delta=2, a=1)].
\end{aligned}$$

Now let us summarize the above criteria by Table 1 to evaluate $\binom{n}{2k}$ modulo 8, and also the contribution to A_1, A_5, and $A_1 + 5A_5$. Even though we have to bisect conditions $j = 0$ and $j \geq 1$ in this table, the final formulas of $A_1 + 5A_5$ when $j \geq 1$ work for the corresponding special cases when $j = 0$.

Class III. Suppose $\alpha\left(\binom{n}{2k}\right) = 1$ which is provided by $a = 2j + 2$ and $\varepsilon = 3$. Since $a \geq 2$ there is no contribution to A_1. The congruences of $\binom{n}{2k}$ in this class can only be 2 and 6. For $5 \times 2 \equiv_8 2$ and $5 \times 6 \equiv_8 6$. The contribution of $\binom{n}{2k}$ to A_5 is as same as that to $5A_5$.

Referring to the similar situation in (14), we know $\binom{n}{2k} \equiv_8 2$ if and only if $W_{37} \equiv_2 0$. Because $k = 2^{2j+2} - 1$ and $\varepsilon = 3$, we have

$$zr(n-2k) = zr(n) + i_0 - \chi(\delta=2)$$

and then obtain

$$\begin{aligned}
W_{37} &\equiv_2 [zr(n) - zr(n-2k)] - zr(2k) + [n_1 - (2k)_1 - (n-2k)_1] \\
&\equiv_2 i_0 + \chi(\delta=2) + 1.
\end{aligned}$$

Therefore, $\binom{n}{2k} \equiv_8 2$ if (i) $\delta = 1$ and $i_0 = 1$ or (ii) $\delta = 2$ and $i_0 = 0$; $\binom{n}{2k} \equiv_8 6$ otherwise. So the total contribution to A_5 as well as to $A_1 + 5A_5$ in this class is

$$\begin{aligned}&\chi(\varepsilon = 3)[2\chi(\delta = 1, i_0 = 1) + 2\chi(\delta = 2, i_0 = 0)\\&\qquad +6\chi(\delta = 1, i_0 = 0) + 2\chi(\delta = 2, i_0 = 1)]\\&\quad = \chi(\delta = 1, \varepsilon = 3)(6 + 4i_0) + \chi(\delta = 2, \varepsilon = 3)(2 + 4i_0) \pmod 8.\end{aligned} \tag{19}$$

Now the whole proof is done by summing up the contributions from the above three classes (see (15), the last line of Table 1 and (19)). For example, when $\varepsilon = 3$ and $\delta = 2$ we have

$$A_1 + 5A_5 = 4i_0 + (6j + 6) + (2 + 4i_0) = 6j.$$

The checking of the other three is left to the reader. ■

The next lemma is a preparation for evaluating $2A_2 + 6A_6$.

Lemma 5.3. *Given $n = (4i + \varepsilon)4^{j+1} - \delta$ and $k = 2^a + 2^b - 1$ for $a, b, i, j \in \mathbb{N}$, $b > a$, $\varepsilon = 1, 3$ and $\delta = 1, 2$, we have*

$$\alpha\left(\binom{n}{2k}\right) = \begin{cases} 0 & \text{if (1) } a \le 2j + 1 \text{ and } n_{b+1} = 1;\\ 1 & \text{if } \begin{array}{l}(2)\ a \le 2j + 1,\ n_{b+1} = 0, \text{ and } n_{b+2} = 1, \text{ or}\\ (3)\ a = 2j + 2,\ n_{b+1} = 1, \text{ and } \varepsilon = 3\end{array}\end{cases}$$

otherwise $\alpha\left(\binom{n}{2k}\right) \ge 2$. Moreover, given condition (1), we must have $b \ne 2j + 1$ and $\binom{n}{2k} \equiv_4 (-1)^E$ where

(a) $E = \chi(a \ge 1, \delta = 2) + \chi(b = 2j)$ *if* $b \le 2j$ *and*

(b) $E = \chi(a = 0) + \chi(1 \le a \le 2j, \delta = 1) + \chi(a = 2j + 1, \delta = 2)$
$\quad + \chi(\langle n_{b+2}n_{b+1}n_b\rangle = \langle 111\rangle \text{ or } \langle 010\rangle)$ *if* $b \ge 2j + 2$.

Proof. We skip the proof of the first part, because it is not only similar to the proof of Lemma 3.2 but also a follow-up of cases (a) and (b) of that proof. A precise proof needs the technique used in the first paragraph of the proof of Lemma 3.4.

As for the second part of the lemma, $b \ne 2j + 1$ is due to the fact that n_{2j+2} (the right digit of B) is always 0. Since $\binom{n}{2k} \equiv_4 (-1)^{E_4(3,\binom{n}{2k})}$, now we are going to show that $E_4(3, \binom{n}{2k}) \equiv_2 E$ which can be done by using the identity $E_4(3, \binom{n}{2k}) \equiv_2 r(n) + r(2k) + r(n - 2k)$ (see (8)). Clearly, $r(2k) = 1 + \chi(a \ge 1)$ because $k = 2^a + 2^b - 1$ and $b > a$. If $b \le 2j$, then we have

$$r(n - 2k) = r(n) + \chi(a \ge 1, \delta = 2) + \chi(b \le 2j - 1).$$

If $b \ge 2j + 2$, then we have

$$\begin{aligned}r(n - 2k) &= r(n) + \chi(1 \le a \le 2j, \delta = 1) - \chi(a = 2j + 1, \delta = 2)\\&\qquad - \chi(\langle n_{b+2}n_{b+1}n_b\rangle = \langle 111\rangle) + \chi(\langle n_{b+2}n_{b+1}n_b\rangle = \langle 010\rangle).\end{aligned}$$

The above two formulas can be checked by the layout of A, B, C and D. Now simplify $r(n) + r(2k) + r(n - 2k)$ and our proof follows. ■

Lemma 5.4. *We have $2A_2(n) + 6A_6(n)$ equal to*

$$4y\chi(\varepsilon = 3) + [2j + 4j\chi(\delta = 2)] + [4y(j + \chi(\delta = 2)) + 4\chi(\varepsilon = 3)] \pmod 8$$

Table 2
Contribution of $\binom{n}{2k}$ (mod 8) to $2A_2 + 6a_6$ in Subclass II(a)

			$j \geq 1$ $\delta = 1$	$j \geq 1$ $\delta = 2$	$\#\{(a, b)\}$
$b = 1$,	$a = 0^*$		1	1	1
$2 \leq b \leq 2j - 1$,	$a = 0$,		1	1	$2j - 2$
$2 \leq b \leq 2j - 1$,	$a \geq 1$,	$a + 2 \leq b$	1	3	$(2j - 3)(j - 1)$
$2 \leq b \leq 2j - 1$,	$a \geq 1$,	$a + 1 = b^*$	1	3	$2j - 2$
$b = 2j$,	$a = 0$		3	3	1
$b = 2j$,	$a \geq 1$,	$a + 2 \leq b$	3	1	$2j - 2$
$b = 2j$,	$a \geq 1$,	$a + 1 = b^*$	3	1	1
	A_2^*	(mod 4)	$2j + 2$	$2j$	
	A_6	(mod 4)	$j + 2$	$3j$	
	$2A_2 + 6A_6$	(mod 8)	$2j$	$6j$	

or

$$2A_2(n) + 6A_6(n) \equiv_8 \begin{cases} 2j + 4jy & \textit{if } \varepsilon = 1 \textit{ and } \delta = 1; \\ 6j + 4jy + 4y & \textit{if } \varepsilon = 1 \textit{ and } \delta = 2; \\ 2j + 4jy + 4y + 4 & \textit{if } \varepsilon = 3 \textit{ and } \delta = 1; \\ 6j + 4jy + 4 & \textit{if } \varepsilon = 3 \textit{ and } \delta = 2. \end{cases}$$

Proof. Notice that $S_8(2, C) \cup S_8(6, C) = \{2^a + 2^b - 1 \mid a, b \in \mathbb{N}, b > a\}$. According to the value of $\alpha\left(\binom{n}{2k}\right)$ given in Lemma 5.3, we discuss two classes as follows, while the third case with $\alpha\left(\binom{n}{2k}\right) \geq 2$ is irrelevant.

Class I. Suppose $\alpha\left(\binom{n}{2k}\right) = 1$. Because each $\binom{n}{2k}$ is even, so are $A_2(n)$ and $A_6(n)$. For $2A_2(n) + 6A_6(n) \equiv_8 4(A_2(n)/2 + A_6(n)/2)$, we only need to consider the parity of $A_2(n)/2 + A_6(n)/2$, which is exactly the number of $\binom{n}{2k}$'s involved in this class. The number of those k's satisfying $a \leq 2j + 1$, $n_{b+1} = 0$, and $n_{b+2} = 1$ is

$$r(Y) - \chi(\varepsilon = 3) + \sum_{a=0}^{2j} r(Y) \equiv_2 \chi(\varepsilon = 3),$$

because the digit $n_{b+2} = 1$ shall be the last 1 in any run of Y. But there is an unnecessary counting that happens when $\chi(\varepsilon = 3)$ and $a = b = 2j + 1$. Given $\varepsilon = 3$, the number of those k's satisfying $a = 2j + 2$ and $n_{b+1} = 1$ is

$$(y - 1)\chi(\varepsilon = 3).$$

Therefore, the total contribution to $2A_2 + 6A_6$ in this class is

$$4y\chi(\varepsilon = 3). \tag{20}$$

Class II. Suppose $\alpha\left(\binom{n}{2k}\right) = 0$ which is provided by $k = 2^a + 2^b - 1$ with $a \leq 2j + 1$ and $n_{b+1} = 1$. Actually, we shall evaluate A_2 and A_6 under modulo 4 not 8, and then multiply them by 2 and 6 respectively under modulo 8. According the second part of Lemma 5.3, we deal with the following two subclasses:

(a) Suppose $n_{b+1} = 1$ belongs to C which is provided by $a + 1 \leq b \leq 2j$. For $2j \geq a + 1$, we must have $j \geq 1$ in this subclass. Now let us refer to Lemma 5.3 and use Table 2 to evaluate

$\binom{n}{2k}$, the contribution to A_2, A_6 (mod 4), and $2A_2+6A_6$ (mod 8). In the table, we shall simplify $(2j-3)(j-1) = 2(j-2)(j-1) + j - 1 \equiv_4 j+3$. We conclude that the total contribution of this subclass is

$$(2 + 4\chi(\delta = 2))j\chi(j \geq 1) \equiv_8 2j + 4j\chi(\delta = 2). \tag{21}$$

Even though $j \geq 1$ is required in this subclass, the value of the above formula is 0 when $j = 0$; so the formula also works for $j = 0$.

(b) Suppose $n_{b+1} = 1$ belongs to Y which is provided by $b \geq 2j+2$ (not $b \geq 2j+1$ because n_{2j+2} is always 0). Lemma 5.3 will be used here again.

We shall first deal with $2A_2$. In this subclass, there is only one possible situation when $\binom{n}{2k}$ contributes to $2A_2$, namely $\varepsilon = 3$ and $a + 1 = b = 2j + 2$. The latter one is because $S_8(2, C) = \{2^a + 2^{a+1} - 1 \mid a \in \mathbb{N}\}$, $a \leq 2j + 1$ and $b \geq 2j + 2$. Now for $n_{2j+3} = 1$ (the left digit of B), we must have $\varepsilon = 3$. In addition if $i \equiv_2 0$ then we have $\langle n_{2n+4}n_{2n+3}n_{2n+2}\rangle = \langle 010\rangle$, and then $E \equiv_2 \chi(\delta = 2)+1$ and $\binom{n}{2k} \equiv_4 1+2\chi(\delta = 1)$. Similarly, if $i \equiv_2 1$ then $\binom{n}{2k} \equiv_4 1 + 2\chi(\delta = 2)$. Thus, the contribution to $2A_2$ is

$$\begin{aligned}&2\chi(\varepsilon = 3)[\chi(i \equiv_2 0)(1 + 2\chi(\delta = 1)) + \chi(i \equiv_2 1)(1 + 2\chi(\delta = 2))]\\ &\quad\equiv_4 2\chi(\varepsilon = 3)[1 + 2\chi(i \equiv_2 0)\chi(\delta = 1) + 2\chi(i \equiv_2 1)\chi(\delta = 2)].\end{aligned} \tag{22}$$

The required condition $\varepsilon = 3$ has already been implanted in this formula.

As for $6A_6$, we analyze E further. Notice that the choices of a and b are independent. This fact is also revealed by corresponding E in Lemma 5.3. Let us bisect E into two terms as follows:

$$\begin{aligned}&U(a) := \chi(a = 0) + \chi(1 \leq a \leq 2j, \delta = 1) + \chi(a = 2j + 1, \delta = 2) \quad \text{and}\\ &V(b) := \chi(\langle n_{b+2}n_{b+1}n_b\rangle = \langle 111\rangle \quad \text{or} \quad \langle 010\rangle).\end{aligned}$$

Now we deal with the total contribution of $\binom{n}{2k}$ to $6A_6$. In the following sum, we *assume* every $\binom{n}{2k}$ in this subclass goes to A_6 for convenience. Of course, this is a false assumption because the only situation causing contribution to A_2 might happen. Anyway, we will fix this false assumption later.

$$\sum_{a=0}^{2j+1} \sum_{\substack{b\geq 2j+2\\ n_{b+1}=1}} \binom{n}{2k} = \sum_{a=0}^{2j+1} \sum_{\substack{b\geq 2j+2\\ n_{b+1}=1}} (-1)^{U(a)+V(b)} \tag{23}$$

$$= \sum_{a=0}^{2j+1}(-1)^{U(a)} \sum_{b\geq 2j+2} \chi(n_{b+1} = 1)(-1)^{V(b)}$$

$$\equiv_4 \sum_{a=0}^{2j+1}(-1)^{U(a)} \sum_{b\geq 2j+2} \chi(n_{b+1} = 1)(-1)^{V(b)}. \tag{24}$$

Let us deal with the two summations (factors) in (24). The first one can be easily evaluated by a table of $(-1)^{U(a)}$. See Table 3. We conclude that

$$\sum_{a=0}^{2j+1}(-1)^{U(a)} \equiv_4 2(j + \chi(\delta = 2)), \tag{25}$$

whether $j \geq 1$ or not.

Table 3
Contribution of $(-1)^{U(a)}$ in Subclass II(b)

	$j=0$ $\delta=1$	$j=0$ $\delta=2$	$j\geq 1$ $\delta=1$	$j\geq 1$ $\delta=2$	$\#\{a\}$
$a=0$	-1	-1	-1	-1	1
$1\leq a\leq 2j$			-1	1	$2j$
$a=2j+1$	1	-1	1	-1	1
$\sum_{a=0}^{2j+1}(-1)^{U(a)}$	0	2	$2j$	$2j+2$	

The second summation is evaluated as follows. Notice that the fact $Y_0=0$ will be used several times in the following calculations.

$$\begin{aligned}
&\sum_{b\geq 2j+2}\chi(n_{b+1}=1)(-1)^{V(b)}\\
&\quad=\sum_{l\geq 1}\chi(\langle Y_{l+1}Y_lY_{l-1}\rangle=\langle 011\rangle \text{ or } \langle 110\rangle)-\sum_{l\geq 1}\chi(\langle Y_{l+1}Y_lY_{l-1}\rangle=\langle 111\rangle \text{ or } \langle 010\rangle)\\
&\quad=2\sum_{l\geq 1}\chi(\langle Y_{l+1}Y_lY_{l-1}\rangle=\langle 011\rangle \text{ or } \langle 110\rangle)-y\\
&\quad=2\sum_{l\geq 1}\left[\chi(\langle Y_{l+1}Y_l\rangle=\langle 01\rangle)+\chi(\langle Y_lY_{l-1}\rangle=\langle 10\rangle)-2\chi(\langle Y_{l+1}Y_lY_{l-1}\rangle=\langle 010\rangle)\right]-y\\
&\quad=2[2r(Y)-2r_1(Y)]-y\\
&\quad\equiv_4 -y.
\end{aligned}\tag{26}$$

By (24)–(26), the contribution to A_6 is $2y(j+\chi(\delta=2))(\text{mod }4)$ if we assume that every $\binom{n}{2k}$ contributes to A_6, and then the contribution to $6A_6$ is $4y(j+\chi(\delta=2))(\text{mod }8)$.

Now refer to (22) and return the extra value caused by the false assumption in the situation that $\varepsilon=3$ and $a+1=b=2j+2$. Therefore, the real contribution to $2A_2+6A_6$ in Subclass (b) is

$$\begin{aligned}
&4y(j+\chi(\delta=2))-(6-2)\chi(\varepsilon=3)[1+2\chi(i\equiv_2 0)\chi(\delta=1)+2\chi(i\equiv_2 1)\chi(\delta=2)]\\
&\quad\equiv_8 4y(j+\chi(\delta=2))+4\chi(\varepsilon=3).
\end{aligned}\tag{27}$$

Finally, sum up the contribution from Class I and Subclasses II(a) and II(b), i.e., (20), (21) and (27), to finish the proof. ■

With the values obtained from Lemmas 5.1, 5.2 and 5.4, our final main result is obtained as follows. This result proves that $M_n\equiv_8 0$ never happens (the second part of Conjecture 1.1).

Theorem 5.5. *The nth Motzkin number M_n is even if and only if $n=(4i+\varepsilon)4^{j+1}-\delta$ for $i,j\in\mathbb{N}$, $\varepsilon=1,3$ and $\delta=1,2$. Moreover, we have*

$$M_n\equiv_8\begin{cases}4 & \textit{if } (\varepsilon,\delta)=(1,1) \textit{ or } (3,2);\\ 4y+2 & \textit{if } (\varepsilon,\delta)=(1,2) \textit{ or } (3,1),\end{cases}$$

where y is the number of digit 1's in $[4i+\varepsilon-1]_2$.

Acknowledgements

The authors would like to thank Bruce Sagan for his valuable suggestions which led to a great improvement on the earlier version of this paper. S.-P. Eu was partially supported by NSC 95-

2115-M-390-005. S.-C. Liu was partially supported by NSC 95-2115-M-163-002 and Y.-N. Yeh was partially supported by NSC 95-2115-M-001-009.

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