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Letter Section

# New generating functions for Gegenbauer polynomials

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**Abstract**

A family of new generating functions for the Gegenbauer polynomials is presented. This work is based upon the elementary manipulation of series and is motivated by the recent appearance of these polynomials in certain aspects of applied mathematics.

*Keywords:* Gegenbauer polynomial; Generating function; Ultraspherical

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The Gegenbauer or ultraspherical polynomials have been the subject of investigations by many authors, in particular on account of their relation to Legendre functions [1, p. 179]. Further recent interest has also arisen in connection with other topics, such as theories of neutron transport and radiative transfer [3] and the quantum relativistic harmonic oscillator [5] and [4]. This provides some motivation for deducing the new generating functions discussed in this study.

An approach to this matter is based upon the elementary manipulation of series. Consider the supposed absolutely convergent series

$$S = \sum \frac{D(m+n+p+q)\, V(2m+2n+p+q)\, (x^2y)^m (s^2t)^n x^p s^q}{m!\, n!\, p!\, q!}, \tag{1}$$

where it is assumed throughout that all indices of summation run over all of the nonnegative integers and that any values of parameters leading to results which do not make sense are tacitly excluded. The generalised coefficients $D$ and $V$ will be suitably specialised subsequently.

The series (1) is now rearranged by replacing $p$ and $q$, respectively, by $M-2m$ and $N-2n$, when we see that

$$S = \sum \frac{D(M+N-m-n)\, V(M+N) x^M y^m s^N t^n}{m!\, n!\, (M-2m)!\, (N-2n)!}$$


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$$= \sum \frac{x^M s^N}{M!N!} V(M+N)$$

$$\times \sum \frac{D(M+N-m-n)(-\frac{1}{2}M, m)(\frac{1}{2}-\frac{1}{2}M, m)(-\frac{1}{2}N, n)(\frac{1}{2}-\frac{1}{2}N, n) y^m t^n 4^{m+n}}{m!n!}. \tag{2}$$

As usual, the Pochhammer symbol $(a, n)$ is given by

$$(a, n) = a(a+1)(a+2)\cdots(a+n-1) = \Gamma(a+n)/\Gamma(a), \quad (a, 0) = 1. \tag{3}$$

The two forms of $S$, (1) and (2) may be equated, and by a further simple rearrangement of (1) and a slight modification of the notation, we obtain a general generating function, namely,

$$\sum \frac{(x^2 y)^m (s^2 t)^n}{m!n} \sum \frac{D(m+n+p+q) V(2m+2n+p+q) x^p s^q}{p!q!}$$

$$= \sum \frac{x^m s^n V(m+n)}{m!n!}$$

$$\times \sum \frac{D(m+n-p-q)(-\frac{1}{2}m, p)(\frac{1}{2}-\frac{1}{2}m, p)(-\frac{1}{2}n, q)(\frac{1}{2}-\frac{1}{2}n, q) y^p t^q 4^{p+q}}{p!q!}. \tag{4}$$

As an elementary consequence of the binomial theorem, the left-hand member of (4) can be written as

$$\sum \frac{(x^2 y)^m (s^2 t)^n}{m!n} \sum \frac{D(m+n+p) V(2m+2n+p)(x+s)^p}{p!}. \tag{5}$$

Making use of an idea employed by Exton [2], put $s = -x$, when the inner series of (5) reduces to $D(m+n) V(2m+2n)$, so that (4) becomes

$$\sum \frac{x^{m+n}(-1)^n V(m+n)}{m!n!}$$

$$\times \sum \frac{D(m+n-p-q)(-\frac{1}{2}m, p)(\frac{1}{2}-\frac{1}{2}m, p)(-\frac{1}{2}n, q)(\frac{1}{2}-\frac{1}{2}n, q) y^p t^q 4^{p+q}}{p!q!}$$

$$= \sum \frac{(x^2 y)^m (x^2 t)^n D(m+n) V(2m+2n)}{m!n!}. \tag{6}$$

Further, on letting $t = 0$, we have the general result in its final form:

$$\sum \frac{x^{m+n}(-1)^n V(m+n)}{m!n!}$$

$$\times \sum \frac{D(m+n-p)(-\frac{1}{2}m, p)(\frac{1}{2}-\frac{1}{2}m, p) y^{-2p}(-1)^p}{p!}$$

$$= \sum \frac{(-\frac{1}{4}x^2 y^{-2})^m D(m) V(2m)}{m!}, \tag{7}$$

where, for convenience, $y$ has been replaced by $-\frac{1}{4}y^{-2}$.

If we let

$$D(N) = (d, N), \tag{8}$$

the polynomial generated by (7) takes the form

$$\sum \frac{(d, m+n-p)(-\frac{1}{2}m, p)(\frac{1}{2}-\frac{1}{2}m, p)(-y^{-2})^p}{p!}$$
$$= (d, m+n)\,{}_2F_1(-\tfrac{1}{2}m, \tfrac{1}{2}-\tfrac{1}{2}m; 1-d-m-n; y^{-2}). \tag{9}$$

This last expression can be written as a Gegenbauer polynomial

$$m!(d, n)(2y)^{-m} C_m^{d+n}(y) \tag{10}$$

by means of a result quoted by Erdélyi [1, p. 176].

Hence, we obtain the family of generating functions of the Gegenbauer polynomials sought, that is

$$\sum \frac{x^{m+n}(-1)^n V(m+n)(d, n)(2y)^{-m}}{n!} C_m^{d+n}(y) = \sum \frac{(-\frac{1}{4}x^2y^{-2})^m (d, m) V(2m)}{m!}. \tag{11}$$

Since the coefficient $V(N)$ is disposable, some degree of flexibility of (11) is furnished, so that various new generating functions of the Gegenbauer polynomials can be written down. For example, if we put $V(N)$ equal to $\Gamma(d)/\Gamma(d+\frac{1}{2}N)$ and $1/(2d, n)$, respectively, we have

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{x^{m+n}(-1)^n \Gamma(d)(d, n)(2y)^{-m}}{n!\,\Gamma(d+\frac{1}{2}m+\frac{1}{2}n)} C_m^{d+n}(y) = \exp(-\tfrac{1}{4}x^2y^{-2}) \tag{12}$$

and

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{x^{m+n}(-1)^n (d, n)(2y)^{-m}}{n!\,(2d, m+n)} C_m^{d+n}(y)$$
$$= {}_0F_1(-; d+\tfrac{1}{2}; -\tfrac{1}{4}x^2y^{-2}) = \left(\frac{\frac{1}{2}x}{y}\right)^{-d+1/2} \Gamma(d+\tfrac{1}{2}) J_{d-1/2}\left(\frac{x}{y}\right). \tag{13}$$

It is interesting to note that these generating relations involve both the order and degree of the polynomials generated. Many other cases can be constructed.

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