MODULAR FORMS AND L-FUNCTIONS WITH A PARTIAL EULER PRODUCT

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Abstract. It is believed that Dirichlet series with a functional equation and Euler product of a particular form are associated to holomorphic newforms on a Hecke congruence group. We explore this conjecture in two ways. First, we perform computer algebra experiments which find that in certain cases one can associate a kind of "higher order modular form" to such Dirichlet series. This suggests a possible approach to a proof of the conjecture. Second, we perform numerical experiments which directly check the conjecture. These experiments suggest that the conjecture is true.

1. Introduction

We investigate the relationship between L-functions and modular forms. We review some classical results on modular forms and then describe the conjecture which motivates our work. A good reference for this material is Iwaniec's book [9].

Let

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \text{ are integers, } ad - bc = 1, \text{ and } c \equiv 0 \bmod N \right\}$$

be the Hecke congruence group of level N, and suppose χ is a character mod N. The group $\Gamma_0(N)$ acts on functions $f : \mathcal{H} \to \mathbb{C}$ by $f \to f|\gamma$ where

$$f(z)| \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \chi(d)^{-1}(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right). \tag{1.1}$$

Here $\mathcal{H} = \{x + iy \in \mathbb{C} \ : \ y > 0\}$ is the upper half of the complex plane. The vector space of *cusp forms of weight k and character χ for* $\Gamma_0(N)$, denoted $S_k(\Gamma_0(N), \chi)$, is the set of holomorphic functions $f : \mathcal{H} \to \mathbb{C}$ which satisfy $f|\gamma = f$ for all $\gamma \in \Gamma_0(N)$ and which vanish at all cusps of $\Gamma_0(N)$. Since

$$T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma_0(N) \tag{1.2}$$

we have $f(z) = f(z+1)$, so there is a Fourier expansion of the form

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}. \tag{1.3}$$

In the case χ is the trivial character χ_0, the newforms in $S_k(\Gamma_0(N), \chi_0)$ have a distinguished basis of *Hecke eigenforms* which satisfy

$$f|H_n = \pm f \tag{1.4}$$

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and

$$f|T_p = a_p f \tag{1.5}$$

for prime p. Here

$$H_N = \begin{pmatrix} & -1 \\ N & \end{pmatrix}$$

is the Fricke involution. If ℓ is prime,

$$T_\ell = \chi(\ell)\begin{pmatrix} \ell & \\ & 1 \end{pmatrix} + \sum_{a=0}^{\ell-1} \begin{pmatrix} 1 & a \\ & \ell \end{pmatrix}, \tag{1.6}$$

is the Hecke operator. If $\ell | N$ then $\chi(\ell) = 0$ and T_ℓ is known as the Atkin-Lehner operator U_ℓ.

We will now state our motivating conjecture, and then explain its relevance to the theory of L-functions.

Conjecture 1.1. *If $f : \mathcal{H} \to \mathbb{C}$ is analytic, is periodic with period 1 (1.3), and satisfies the Fricke (1.4) and Hecke (1.5) relations with $\chi = \chi_0$, then $f \in S_k(\Gamma_0(N), \chi_0)$.*

Thus, the invariance property $f|\gamma = f$, which leads to the Fricke and Hecke relations, would actually follow from them.

We will rephrase the conjecture in terms of L-functions. Associated to a cusp form with Fourier expansion (1.3) is an L-function

$$L(s, f) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}. \tag{1.7}$$

Using the Mellin transform and its inverse, it can be shown that the Fricke relation (1.4) is equivalent to the functional equation

$$\xi_f(s) := \left(\frac{2\pi}{\sqrt{N}}\right)^{-s} \Gamma(s)\, L_f(s) = \pm(-1)^{k/2}\xi_f(k-s). \tag{1.8}$$

Also, the Hecke relations (1.5) are equivalent to $L(s, f)$ having an Euler product of the form

$$L(s, f) = \prod_p \left(1 - a_p p^{-s} + \chi(p) p^{k-1-2s}\right)^{-1}, \tag{1.9}$$

because both statements are equivalent to $a_{p^n m} = a_{p^n} a_m$ for $p \nmid m$ and

$$a_{p^{n+1}} = a_p a_{p^n} - \chi(p) p^{k-1} a_{p^{n-1}}. \tag{1.10}$$

Thus, Conjecture 1.1 is equivalent to

Conjecture 1.2. *If a Dirichlet series continues to an entire function of order one which is bounded in vertical strips and satisfies the functional equation (1.8) and the Euler product (1.9) with $\chi = \chi_0$, then the Dirichlet series equals $L(s, f)$ for some $f \in S_k(\Gamma_0(N), \chi_0)$.*

This conjecture should be viewed as part of the Langlands' program. Note that one does not require functional equations for twists of the L-function, as in Weil's converse theorem. As a special case, the L-function of a rational elliptic curve automatically has an Euler product of form (1.9) with $k = 2$ and $\chi = \chi_0$, so the modularity of a rational elliptic curve would be reduced to proving analytic continuation and a functional equation for one L-function.

Progress on the conjecture has been made only for small N, for the trivial character [2], and (appropriately modified) for almost the same cases for nontrivial character [6]. For $N \le 4$, Hecke's original converse theorem establishes the conjecture. This follows from the fact that the group generated by T and H_N contains $\Gamma_0(N)$ exactly when $N \le 4$. Note that this only uses the functional equation, not the Euler product. For larger N, one must use the Euler product in a nontrivial way. This possibility was introduced in [2], and examples were given for certain $N \le 23$.

In this paper we specialize to the case $N = 13$, for the simple reason that this is the first case which has not been solved. Our hope is to discover some structure which can be used to attack the general case. It turns out that the $N = 13$ case leads to relations reminiscent of "higher order modular forms," which are described in the next section. In Section 3 we describe prior work and then in Section 4 we apply those methods to the case $N = 13$. Finally, in Section 5 we do numerical calculations which directly check Conjecture 1.1. The calculations give evidence that the conjecture is true.

2. Higher order modular forms

Our discussion here is imprecise and will only convey the general flavor of this new subject. For details see [1, 3].

We first introduce some slightly simpler notation. If $f|\gamma = f$ then we have

$$\gamma \equiv 1 \bmod \Omega_f \tag{2.1}$$

where Ω_f is the right ideal in the group ring $\mathbb{C}[GL(2,\mathbb{R})]$ which annihilates f, the action of matrices on f being extended linearly. We will write $\gamma \equiv 1$ instead of $\gamma \equiv 1 \bmod \Omega_f$ throughout this paper. Thus, if f is a cusp form for the group Γ, then the invariance properties of f can be written as $f|(1-\gamma) = 0$ for all $\gamma \in \Gamma$, or equivalently, $1 - \gamma \equiv 0$. This notation will make it easier to describe the properties of higher order modular forms.

If f is a *second order* cusp form for the group Γ, then f satisfies the relation

$$(1-\gamma_1)(1-\gamma_2) \equiv 0 \tag{2.2}$$

for all $\gamma_1, \gamma_2 \in \Gamma$. Similarly, third order modular forms satisfy

$$(1-\gamma_1)(1-\gamma_2)(1-\gamma_3) \equiv 0, \tag{2.3}$$

and so on. Roughly speaking, if f is an nth order modular form then $f|(1-\gamma)$ is an $(n-1)$st order modular form. There are additional conditions involving the cusps and the parabolic elements of Γ, but our goal here is just to introduce the general idea. Indeed, it is nontrivial to determine the proper technical conditions, see [1, 3].

In connection with our exploration of Conjecture 1.1, a condition of form (2.2) will arise where γ_1 and γ_2 come from *different* groups. This first appeared in the original work of Weil on the converse theorem involving functional equations for twists. Specifically, the relation (2.2) arose where γ_2 was *elliptic of infinite order*. The following lemma applies:

Lemma 2.1. *Suppose f is holomorphic in $\mathcal{H}$ and $\varepsilon \in GL_2(\mathbb{R})^+$ is elliptic. If $f|_k\varepsilon = f$, then either ε has finite order, or f is constant.*

This is Proposition 3 from [10]. See also the discussion in Section 7.4 of Iwaniec's book [9]. By the lemma, if γ_2 is elliptic of infinite order then (2.2) implies that actually $1 - \gamma_1 \equiv 0$, which is the conclusion Weil sought.

Denote by $S_k(\Gamma_1, \Gamma_2)$ the set of analytic functions (with appropriate technical conditions) satisfying (2.2) for all $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$. The above lemma says that if Γ_2 contains an elliptic element of infinite order then $S_k(\Gamma_1, \Gamma_2) = S_k(\Gamma_1)$. Note that the analyticity of f is necessary, and an analogue of Weil's converse theorem for Maass form L-functions has not been proven in classical language.

In Section 4 we will see that our assumptions on the Fricke involution and the Hecke operators lead to condition (2.2) with $\gamma_1 \in \Gamma_0(13)$ and γ_2 in some other discrete group. We also obtain higher order conditions (2.3) where each γ_j comes from a different group. This suggests the following question:

Question 2.2. *What conditions on Γ_1 and Γ_2 ensure that $S_k(\Gamma_1, \Gamma_2)$ is finite dimensional? What conditions imply that $S_k(\Gamma_1, \Gamma_2) = S_k(\Gamma_1)$?*

Part of the problem is determining the appropriate technical conditions to incorporate into the definition of $S_k(\Gamma_1, \Gamma_2)$. Even when $\Gamma_1 = \Gamma_2$ this is nontrivial. See [1, 3].

3. Manipulating the Hecke Operators

In [2] results were obtained for various N up to $N = 23$. The idea is to manipulate the relations $T \equiv 1$, $H_N \equiv \pm 1$ and $T_n \equiv a_n$ to obtain $\gamma \equiv 1$ for all γ in a generating set for $\Gamma_0(N)$. We will describe the cases of $N = 5, 7, 9, 11$ from [2], and then the remainder of the paper will concern the interesting relationships that arose in our exploration of the case $N = 13$.

We have the following generating sets:

$$\begin{aligned} \Gamma_0(N) &= \left\langle T,\ W_N,\ \begin{pmatrix} 2 & -1 \\ -N & \frac{N+1}{2} \end{pmatrix} \right\rangle, \qquad N = 5,7,9, \\ \Gamma_0(11) &= \left\langle T,\ W_{11},\ \begin{pmatrix} 2 & -1 \\ -11 & 6 \end{pmatrix}, \begin{pmatrix} 3 & -1 \\ -11 & 4 \end{pmatrix} \right\rangle, \\ \Gamma_0(13) &= \left\langle T,\ W_{13},\ \begin{pmatrix} 2 & -1 \\ -13 & 7 \end{pmatrix}, \begin{pmatrix} -3 & -1 \\ 13 & 4 \end{pmatrix}, \begin{pmatrix} 3 & -1 \\ 13 & -4 \end{pmatrix} \right\rangle, \end{aligned} \tag{3.1}$$

where

$$T = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \qquad \text{and} \qquad W_N = \begin{pmatrix} 1 & \\ N & 1 \end{pmatrix}. \tag{3.2}$$

The generator T is for free because we have assumed a Fourier expansion. The generator W_N now follows from the Fricke relation, because $W_N = H_N T H_N$. So for these groups we have two of the generators. Note that this uses the functional equation, but not the Euler product.

In the next section we repeat the calculations from [2] in the cases $N = 5, 7, 9, 11$, and in the following sections we treat the case $N = 13$.

3.1. **Levels 5, 7, 9, and 11.** For every N we obtain a new generator from T_2. This will resolve the cases $N = 5$, 7, and 9.

Lemma 3.1 (Lemma 2 of [2])**.** *If $H_N \equiv \pm 1$ and $T_2 \equiv a_2$ then*

$$\begin{pmatrix} 2 & -1 \\ -N & \frac{N+1}{2} \end{pmatrix} \equiv 1.$$

Proof. Note that

$$H_N^{-1}T_2H_N = \begin{pmatrix}1 & \\ & 2\end{pmatrix} + \begin{pmatrix}2 & \\ & 1\end{pmatrix} + \begin{pmatrix}2 & \\ -N & 1\end{pmatrix}.$$

Since $H_N^{-1}T_2H_N \equiv a_2H_N^{-1}H_N \equiv a_2 \equiv T_2$, we have:

$$\begin{pmatrix}1 & \\ & 2\end{pmatrix} + \begin{pmatrix}2 & \\ & 1\end{pmatrix} + \begin{pmatrix}2 & \\ -N & 1\end{pmatrix} \equiv \begin{pmatrix}2 & \\ & 1\end{pmatrix} + \begin{pmatrix}1 & \\ & 2\end{pmatrix} + \begin{pmatrix}1 & 1\\ & 2\end{pmatrix}.$$

Canceling common terms from both sides we are left with

$$\begin{pmatrix}2 & \\ -N & 1\end{pmatrix} \equiv \begin{pmatrix}1 & 1\\ & 2\end{pmatrix}.$$

Right multiplying by $\begin{pmatrix}1 & 1\\ & 2\end{pmatrix}^{-1}$ we have

$$M_2 := \begin{pmatrix}2 & -1\\ -N & \frac{N+1}{2}\end{pmatrix} \equiv 1.$$

□

The lemma provides the final generator for $\Gamma_0(5)$, $\Gamma_0(7)$, and $\Gamma_0(9)$.

To obtain the final generator for $\Gamma_0(11)$ we will combine the Hecke operators T_3 and T_4 For T_3 we have

$$\begin{aligned}
0 &\equiv H_N(T_3-a_3)H_N-(T_3-a_3)\\
&= \begin{pmatrix}1 & 1\\ & 3\end{pmatrix} + \begin{pmatrix}1 & 2\\ & 3\end{pmatrix} + \begin{pmatrix}3 & \\ -N & 1\end{pmatrix} + \begin{pmatrix}3 & \\ -2N & 1\end{pmatrix}\\
&\equiv -\begin{pmatrix}1 & 1\\ & 3\end{pmatrix} - \begin{pmatrix}1 & -1\\ & 3\end{pmatrix} + \begin{pmatrix}3 & \\ -N & 1\end{pmatrix} + \begin{pmatrix}3 & \\ N & 1\end{pmatrix},
\end{aligned} \tag{3.3}$$

where the second step used

$$\begin{pmatrix}1 & -1\\ & 1\end{pmatrix} \equiv 1 \qquad \text{and} \qquad \begin{pmatrix}1 & \\ N & 1\end{pmatrix} \equiv 1. \tag{3.4}$$

We can combine the terms in pairs using

$$\begin{pmatrix}1 & a\\ & p\end{pmatrix} - \begin{pmatrix}p & \\ Nb & 1\end{pmatrix} = \left(1 - \begin{pmatrix}p & -a\\ Nb & \frac{-Nab+1}{p}\end{pmatrix}\right)\begin{pmatrix}1 & a\\ & p\end{pmatrix}$$

to get

$$\left(1 - \begin{pmatrix}3 & -1\\ -11 & 4\end{pmatrix}\right)\beta(1/3) + \left(1 - \begin{pmatrix}3 & 1\\ 11 & 4\end{pmatrix}\right)\beta(-1/3) \equiv 0, \tag{3.5}$$

where $\beta(x) = \begin{pmatrix}1 & x\\ & 1\end{pmatrix}$. We will combine this with a relation obtained from T_4

Since T_4 and T_2 are not independent, there is more than one way to proceed. The calculation which seems most natural to us begins with

$$\begin{aligned}
0 \equiv\ & H_N(T_4-a_4)H_N-(T_4-a_4)\\
&- [H_N(T_2-a_2)H_N-(T_2-a_2)]\begin{pmatrix}2 & \\ & 1\end{pmatrix}\\
&- [H_N(T_2-a_2)H_N-(T_2-a_2)]\begin{pmatrix}1 & \\ & 2\end{pmatrix}
\end{aligned}$$

$$= -\begin{pmatrix} 1 & 1 \\ & 4 \end{pmatrix} + \begin{pmatrix} 4 & \\ -3N & 1 \end{pmatrix} - \begin{pmatrix} 1 & 3 \\ & 4 \end{pmatrix} + \begin{pmatrix} 4 & \\ -N & 1 \end{pmatrix}. \tag{3.6}$$

Combining terms as in the T_3 case gives

$$\left(1 - \begin{pmatrix} 4 & -1 \\ -11 & 3 \end{pmatrix}\right) \beta(1/4) + \left(1 - \begin{pmatrix} 4 & 1 \\ 11 & 3 \end{pmatrix}\right) \beta(-1/4) \equiv 0. \tag{3.7}$$

Combining (3.5) and (3.7) we obtain

$$\begin{aligned}
1 - \begin{pmatrix} 3 & -1 \\ -11 & 4 \end{pmatrix} &\equiv -\left(1 - \begin{pmatrix} 3 & 1 \\ 11 & 4 \end{pmatrix}\right) \beta\left(-\frac{2}{3}\right) \\
&= \left(1 - \begin{pmatrix} 4 & -1 \\ -11 & 3 \end{pmatrix}\right) \begin{pmatrix} 3 & 1 \\ 11 & 4 \end{pmatrix} \beta\left(-\frac{2}{3}\right) \\
&\equiv -\left(1 - \begin{pmatrix} 4 & 1 \\ 11 & 3 \end{pmatrix}\right) \beta\left(-\frac{2}{4}\right) \begin{pmatrix} 3 & 1 \\ 11 & 4 \end{pmatrix} \beta\left(-\frac{2}{3}\right) \\
&= \left(1 - \begin{pmatrix} 3 & -1 \\ -11 & 4 \end{pmatrix}\right) \begin{pmatrix} 4 & 1 \\ 11 & 3 \end{pmatrix} \beta\left(-\frac{2}{4}\right) \begin{pmatrix} 3 & 1 \\ 11 & 4 \end{pmatrix} \beta\left(-\frac{2}{3}\right). \qquad (3.8)
\end{aligned}$$

However,

$$\begin{pmatrix} 4 & 1 \\ 11 & 3 \end{pmatrix} \beta\left(-\frac{2}{4}\right) \begin{pmatrix} 3 & 1 \\ 11 & 4 \end{pmatrix} \beta\left(-\frac{2}{3}\right) = \begin{pmatrix} 1 & -2/3 \\ 11/2 & -8/3 \end{pmatrix}$$

is elliptic but not of finite order. So by Lemma 2.1,

$$\begin{pmatrix} 3 & -1 \\ -11 & 4 \end{pmatrix} \equiv 1.$$

This is the final generator for $\Gamma_0(11)$.

4. Level 13, mimic previous methods

We will mimic the method used for $\Gamma_0(11)$ for $\Gamma_0(13)$, but things will not work out as nicely. What will arise is an expression of the form (2.2) that appears in the definition of second order modular form.

4.1. **The case of T_3.**

From T_3 we obtain the following expression, which is analogous to (3.5),

$$\left(1 - \begin{pmatrix} 3 & 1 \\ -13 & -4 \end{pmatrix}\right) \beta(1/3) + \left(1 - \begin{pmatrix} 3 & -1 \\ 13 & -4 \end{pmatrix}\right) \beta(-1/3) \equiv 0. \tag{4.1}$$

We manipulate this similarly to the example for $\Gamma_0(11)$:

$$\begin{aligned}
1 - \begin{pmatrix} 3 & 1 \\ -13 & -4 \end{pmatrix} &\equiv -\left(1 - \begin{pmatrix} 3 & -1 \\ 13 & -4 \end{pmatrix}\right) \beta\left(-\frac{2}{3}\right) \\
&= \left(1 - \begin{pmatrix} -4 & 1 \\ -13 & 3 \end{pmatrix}\right) \begin{pmatrix} 3 & -1 \\ 13 & -4 \end{pmatrix} \beta\left(-\frac{2}{3}\right) \\
&\equiv \left(1 - H_{13} \begin{pmatrix} -4 & 1 \\ -13 & 3 \end{pmatrix} H_{13}\right) H_{13} \begin{pmatrix} 3 & -1 \\ 13 & -4 \end{pmatrix} \beta\left(-\frac{2}{3}\right) \\
&= \left(1 - \begin{pmatrix} 3 & 1 \\ -13 & -4 \end{pmatrix}\right) H_{13} \begin{pmatrix} 3 & -1 \\ 13 & -4 \end{pmatrix} \beta\left(-\frac{2}{3}\right). \qquad (4.2)
\end{aligned}$$

So,

$$\left(1-\begin{pmatrix}3 & 1\\ -13 & -4\end{pmatrix}\right)(1-\varepsilon_1)\equiv 0 \tag{4.3}$$

where

$$\varepsilon_1 = H_{13}\begin{pmatrix}3 & -1\\ 13 & -4\end{pmatrix}\beta\left(-\frac{2}{3}\right) = \begin{pmatrix}\sqrt{13} & \frac{14}{3\sqrt{13}}\\ -3\sqrt{13} & -\sqrt{13}\end{pmatrix}. \tag{4.4}$$

Since ε_1 is elliptic of order 2 we cannot obtain anything from Lemma 2.1. However, we do have an expression of the form (2.2) which looks like the definition of a second order modular form.

4.2. **The case of T_4.** From T_4, again proceeding as in the $\Gamma_0(11)$ example, we first have

$$\left(1-\begin{pmatrix}4 & -1\\ 13 & -3\end{pmatrix}\right)\beta(1/4)+\left(1-\begin{pmatrix}4 & 1\\ -13 & -3\end{pmatrix}\right)\beta(-1/4)\equiv 0. \tag{4.5}$$

Continuing exactly as above, this leads to

$$\left(1-\begin{pmatrix}3 & 1\\ -13 & -4\end{pmatrix}\right)(1-\varepsilon_2)\equiv 0 \tag{4.6}$$

where

$$\varepsilon_2 = \begin{pmatrix}-\sqrt{13} & \frac{-4}{\sqrt{13}}\\ \frac{7\sqrt{13}}{2} & \sqrt{13}\end{pmatrix}. \tag{4.7}$$

Again ε_2 is elliptic of order 2.

4.3. **Combining T_3 and T_4.** We can combine the two relationships to obtain

$$0\equiv\left[1-\begin{pmatrix}3 & 1\\ -13 & -4\end{pmatrix}\right](1-\varepsilon) \tag{4.8}$$

for any ε in the group generated by ε_1 and ε_2, and perhaps one of those elements will be elliptic of infinite order? Unfortunately, this is not the case. Note that

$$\varepsilon_1\varepsilon_2 = \begin{pmatrix}\frac{10}{3} & \frac{2}{3}\\ -\frac{13}{2} & -1\end{pmatrix},$$

which is parabolic. Since ε_1 and ε_2 have order 2, the group they generate contains only the elements $(\varepsilon_1\varepsilon_2)^n$ and $\varepsilon_2(\varepsilon_1\varepsilon_2)^n$, so that group is discrete.

Although T_3 and T_4 were not sufficient to obtain the missing generator, there are an infinite number of other Hecke operators to try.

4.4. **The case of T_6.** We now proceed with similar calculations with T_6. We have

$$\begin{aligned}
0 \equiv\;& H_{13}(T_6-a_6)H_{13}-(T_6-a_6)\\
&-[H_{13}(T_2-a_2)H_{13}-(T_2-a_2)]\begin{pmatrix}3 & \\ & 1\end{pmatrix}-[H_{13}(T_2-a_2)H_{13}-(T_2-a_2)]\begin{pmatrix}1 & \\ & 3\end{pmatrix}\\
&-[H_{13}(T_3-a_3)H_{13}-(T_3-a_3)]\begin{pmatrix}2 & \\ & 1\end{pmatrix}-[H_{13}(T_3-a_3)H_{13}-(T_3-a_3)]\begin{pmatrix}1 & \\ & 2\end{pmatrix}\\
\text{(4.9)}\equiv\;& -\begin{pmatrix}1 & 1\\ & 6\end{pmatrix}+\begin{pmatrix}6 & \\ -65 & 1\end{pmatrix}-\begin{pmatrix}1 & 5\\ & 6\end{pmatrix}+\begin{pmatrix}6 & \\ -13 & 1\end{pmatrix}.
\end{aligned}$$

Using manipulations similar to those above gives

$$\begin{aligned} 0 &\equiv \left[-1+\begin{pmatrix} 6 & -1 \\ 65 & 11 \end{pmatrix}\right]\begin{pmatrix} 1 & 1 \\ & 6 \end{pmatrix} + \left[-1+\begin{pmatrix} 6 & -5 \\ -13 & 11 \end{pmatrix}\right]\begin{pmatrix} 1 & 5 \\ & 6 \end{pmatrix} \\ &\equiv \left[-1+\begin{pmatrix} 6 & -1 \\ 65 & 11 \end{pmatrix}\right]\begin{pmatrix} 1 & 1 \\ & 6 \end{pmatrix}, \end{aligned}$$

because $\begin{pmatrix} 6 & -5 \\ -13 & 11 \end{pmatrix} = M_2^{-1}H_{13}T^{-1}H_{13}T^{-1}$ so the second term on the first line is $\equiv 0$. So we have

$$\begin{pmatrix} 1 & 1 \\ & 6 \end{pmatrix} + \begin{pmatrix} 6 & \\ -65 & 1 \end{pmatrix} \equiv 0,$$

so

$$\begin{pmatrix} 6 & -1 \\ -65 & 11 \end{pmatrix} \equiv 1$$

This is not a new matrix because $\begin{pmatrix} 6 & -1 \\ -65 & 11 \end{pmatrix} = H_{13}TH_{13}TH_{13}M_2H_{13}$. That is, the above manipulations with T_6 produce results that can be obtained from T_2.

4.5. **Computer manipulation of Hecke operators.** The explicit manipulation of Hecke operators described in this paper are quite tedious to do by hand, so we decided to make use of a computer. We modified Mathematica to do calculations in the group ring $\mathbb{C}[SL(2,\mathbb{R})]$, made functions for the Hecke operators, automated manipulations that occur repeatedly (such as the first step in every example in the previous section of this paper), and implemented some crude simplifications procedures.

For the simplification procedures, we sought to automate the discovery, for example, that if $T \equiv 1$, $H_{13} \equiv \pm 1$, and $M_2 \equiv 1$, then

$$-1+\begin{pmatrix} 6 & -1 \\ 65 & 11 \end{pmatrix} \equiv 0, \tag{4.11}$$

as we saw at the end of the previous section. Our approach was to put all of the matrices in each expression in "simplest form" by considering all products (on the left) with, for example, fewer than 6 matrices where are known to be $\equiv 1$, and then keeping the representative which has the smallest entries. This idea worked surprisingly well.

We also implemented a "factorization" function which would do the (trivial) calculation to check such things as whether $1-\gamma_1-\gamma_2+\gamma_3$ was of the form $(1-\gamma_1)(1-\gamma_2)$ or $(1-\gamma_2)(1-\gamma_1)$.

4.6. **The case of T_7.** Calculations with T_7 yield interesting results. We have

$$\begin{aligned} 0 &\equiv H_{13}(T_7-a_7)H_{13}-(T_7-a_7) \\ &\equiv -\begin{pmatrix} 1 & 2 \\ & 7 \end{pmatrix} + \begin{pmatrix} 7 & \\ -52 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 3 \\ & 7 \end{pmatrix} + \begin{pmatrix} 7 & \\ -65 & 1 \end{pmatrix} \\ &\quad -\begin{pmatrix} 1 & 4 \\ & 7 \end{pmatrix} + \begin{pmatrix} 7 & \\ -26 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 5 \\ & 7 \end{pmatrix} + \begin{pmatrix} 7 & \\ -39 & 1 \end{pmatrix}. \end{aligned}$$

Note that the expression on the right consists of 4 pair of matrices, as opposed to the 6 pair that one would expect to obtain from T_7. This is because two pair canceled during simplification.

It turns out that the right side of the above expression factors as

$$
\begin{aligned}
&\left[-1+\begin{pmatrix}-3 & 1\\ -13 & 4\end{pmatrix}\right]\begin{pmatrix}1 & 2\\ & 7\end{pmatrix}+\left[-1+\begin{pmatrix}7 & 4\\ -65 & -37\end{pmatrix}\right]\begin{pmatrix}1 & -4\\ & 7\end{pmatrix}\\
&\quad+\left[-1+\begin{pmatrix}7 & -4\\ -26 & 15\end{pmatrix}\right]\begin{pmatrix}1 & 4\\ & 7\end{pmatrix}+\left[-1+\begin{pmatrix}3 & 1\\ -13 & -4\end{pmatrix}\right]\begin{pmatrix}1 & -2\\ & 7\end{pmatrix}\\
&=\left[-\begin{pmatrix}3 & 1\\ -13 & -4\end{pmatrix}+1\right]\begin{pmatrix}3 & 1\\ -13 & -4\end{pmatrix}^{-1}H_{13}\begin{pmatrix}1 & 2\\ & 7\end{pmatrix}\\
&\quad+\left[-\begin{pmatrix}3 & 1\\ -13 & -4\end{pmatrix}+1\right]\begin{pmatrix}3 & 1\\ -13 & -4\end{pmatrix}^{-1}H_{13}\begin{pmatrix}7 & -1\\ -13 & 2\end{pmatrix}^{-1}\begin{pmatrix}1 & -4\\ & 7\end{pmatrix}\\
&\quad+\left[-1+\begin{pmatrix}3 & 1\\ -13 & -4\end{pmatrix}\right]\begin{pmatrix}7 & 1\\ 13 & 2\end{pmatrix}^{-1}\begin{pmatrix}1 & 4\\ & 7\end{pmatrix}+\left[-1+\begin{pmatrix}3 & 1\\ -13 & -4\end{pmatrix}\right]\begin{pmatrix}1 & -2\\ & 7\end{pmatrix}\\
&=\left[-1+\begin{pmatrix}3 & 1\\ -13 & -4\end{pmatrix}\right]\\
&\qquad\left(-\begin{pmatrix}-\sqrt{13} & \frac{2}{\sqrt{13}}\\ 3\sqrt{13} & -\sqrt{13}\end{pmatrix}-\begin{pmatrix}2\sqrt{13} & \frac{1}{\sqrt{13}}\\ -7\sqrt{13} & \end{pmatrix}+\begin{pmatrix}2 & 1\\ -13 & -3\end{pmatrix}+\begin{pmatrix}1 & -2\\ & 7\end{pmatrix}\right).
\end{aligned}
$$

We can right multiply by the inverse of any of the four matrices in the second factor to rewrite this in the form $(1-\gamma)(1+A-B-C)$. For no good reason we choose the first term, giving

$$
\begin{aligned}
0 \;&\equiv\; \left[1-\begin{pmatrix}3 & 1\\ -13 & -4\end{pmatrix}\right]\\
&\qquad\times\left(1+\begin{pmatrix}\frac{29}{7} & \frac{5}{7}\\ -13 & -2\end{pmatrix}-\begin{pmatrix}\frac{5\sqrt{13}}{7} & \frac{17}{7\sqrt{13}}\\ \frac{-22\sqrt{13}}{7} & \frac{-5\sqrt{13}}{7}\end{pmatrix}-\begin{pmatrix}\frac{5\sqrt{13}}{7} & \frac{24}{7\sqrt{13}}\\ -3\sqrt{13} & -\sqrt{13}\end{pmatrix}\right)\\
&=\; \left[1-\begin{pmatrix}3 & 1\\ -13 & -4\end{pmatrix}\right](1+A-B-C)\,, \qquad (4.14)
\end{aligned}
$$

say. This expression factors further. Specifically, one can check that $A=CB$, so we have

$$
0\equiv\left[1-\begin{pmatrix}3 & 1\\ -13 & -4\end{pmatrix}\right](1-C)(1-B) \qquad (4.15)
$$

Unfortunately, $B^2=1$, so we cannot immediately cancel the final factor to reduce to a second-order type expression. It would be good if that happened, because we would have another matrix to combine with the ε_1 and ε_2 from Sections 4.1 and 4.2.

However, there is a curious benefit to having $B^2=1$, for we also have $AB=C$, so

$$
0\equiv\left[1-\begin{pmatrix}3 & 1\\ -13 & -4\end{pmatrix}\right](1-A)(1-B). \qquad (4.16)
$$

Note that if $B^2=1$, independent of any conditions on A and C, then $(1+A-B-C)(1+B)=(1-CA^{-1})(1+ABA^{-1})A$, so

$$
0\equiv\left[1-\begin{pmatrix}3 & 1\\ -13 & -4\end{pmatrix}\right](1-CA^{-1})(1+ABA^{-1}), \qquad (4.17)
$$

which is almost a third-order condition. Such expressions arise whenever we have an order-2 matrix, so some types of factorization are not a surprise. In the particular case at hand,

$CA^{-1} = ABA^{-1}$, which has order 2, so $(1 - CA^{-1})(1 + ABA^{-1}) = 0$ and (4.17) contains absolutely no information. Perhaps one should think that if $B^2 = 1$ then there always is some factorization, for either (4.17) is nontrivial, or the expression factors nontrivially in another way.

4.7. **A few other cases.** From T_{10} we get

$$\begin{aligned}
0 &\equiv \left[1 - \begin{pmatrix} 3 & 1 \\ -13 & -4 \end{pmatrix}\right] \\
&\qquad \times \left(1 + \begin{pmatrix} \frac{21}{5} & \frac{2}{5} \\ -13 & -1 \end{pmatrix} - \begin{pmatrix} \frac{2\sqrt{13}}{5} & \frac{7}{5\sqrt{13}} \\ \frac{-11\sqrt{13}}{5} & \frac{-2\sqrt{13}}{5} \end{pmatrix} - \begin{pmatrix} \frac{4\sqrt{13}}{5} & \frac{19}{5\sqrt{13}} \\ -3\sqrt{13} & -\sqrt{13} \end{pmatrix}\right) \\
&= \left[1 - \begin{pmatrix} 3 & 1 \\ -13 & -4 \end{pmatrix}\right](1 + A - B - C), \qquad (4.18)
\end{aligned}$$

say. Again $A = CB$ and $B^2 = 1$, so we obtain two factorizations.

From T_{15} we get

$$\begin{aligned}
0 &\equiv \left[1 - \begin{pmatrix} 3 & 1 \\ -13 & -4 \end{pmatrix}\right] \\
&\qquad \times \left(1 + \begin{pmatrix} \frac{16}{5} & 1 \\ -\frac{117}{5} & -7 \end{pmatrix} - \begin{pmatrix} 4\sqrt{13} & \frac{15}{\sqrt{13}} \\ \frac{-209\sqrt{13}}{15} & -4\sqrt{13} \end{pmatrix} - \begin{pmatrix} \frac{17\sqrt{13}}{15} & \frac{4}{\sqrt{13}} \\ \frac{-59\sqrt{13}}{15} & -\sqrt{13} \end{pmatrix}\right),
\end{aligned}$$

which again factors in the same two ways.

From T_9 we get

$$\begin{aligned}
0 &\equiv \left[1 - \begin{pmatrix} 3 & 1 \\ -13 & -4 \end{pmatrix}\right] \\
&\qquad \times \left(1 + \begin{pmatrix} \frac{10}{3} & 1 \\ -\frac{13}{3} & -1 \end{pmatrix} - \begin{pmatrix} 2\sqrt{13} & \frac{9}{\sqrt{13}} \\ \frac{-53\sqrt{13}}{9} & -2\sqrt{13} \end{pmatrix} - \begin{pmatrix} \frac{7\sqrt{13}}{9} & \frac{4}{\sqrt{13}} \\ \frac{-25\sqrt{13}}{9} & -\sqrt{13} \end{pmatrix}\right).
\end{aligned}$$

which again factors in the same two ways.

It would be helpful to understand the underlying reason why these expressions factor.

More time on the computer should produce more relations, but it is not clear how they will combine to produce the desired result. It would be interesting if the relations could build to the point where one could reduce higher order relations to lower order ones, which could then combine with previously found relations to cause additional cancellation, and so on, reducing down to the one missing generator for $\Gamma_0(13)$. It would be more satisfying if one could find manipulations which produce any specific matrix, as one does in the proof of Weil's converse theorem.

Our approach here is to look for factorizations $(1 - \gamma)(1 - \delta)(1 - \varepsilon) \equiv 0$ in the hopes of eliminating the last factor, perhaps because ε is elliptic of infinite order. In the case of expressions that do not factor, it would be interesting to know if there are cancellation laws beyond those implied by Weil's lemma. That is, are there conditions on A, B, C such that $f|(1 + A - B - C) = 0$ implies some apparently stronger condition on f, beyond those cases where $1 + A - B - C$ factors and Weil's lemma applies?

4.8. **A curiosity.** All the manipulations in this paper involve "pairing up" the terms in a linear combination of matrices. Usually there is a natural way to do this, for one is hoping to produce matrices in $\Gamma_0(N)$. However, it is possible to pair the matrices in different ways, and one would like some justification for the choices and to know the consequences of making the right (or wrong) choices. This is discussed extensively in [5].

We now give an example by repeating the analysis of Section 3 making the wrong choices. From (3.3) with $N = 11$ we have

$$\left(1 - \begin{pmatrix} 3 & -1 \\ 11 & -\frac{10}{3} \end{pmatrix}\right) \beta(1/3) + \left(1 - \begin{pmatrix} 3 & 1 \\ -11 & -\frac{10}{3} \end{pmatrix}\right) \beta(-1/3) \equiv 0, \tag{4.21}$$

where $\beta(x) = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$. Now doing manipulations exactly as in Section 4.1 we obtain

$$0 \equiv \left(1 - \begin{pmatrix} 3 & -1 \\ 11 & -\frac{10}{3} \end{pmatrix}\right)(1 - \varepsilon), \tag{4.22}$$

where

$$\varepsilon = H_{11} \begin{pmatrix} 3 & 1 \\ -11 & -10/3 \end{pmatrix} \beta(-2/3) = \begin{pmatrix} \sqrt{11} & -\frac{4}{\sqrt{11}} \\ 3\sqrt{11} & -\sqrt{11} \end{pmatrix}, \tag{4.23}$$

which has order 2.

Note that the above manipulations cannot lead to $\begin{pmatrix} 3 & -1 \\ 11 & -\frac{10}{3} \end{pmatrix} \equiv 1$. Indeed, if p is prime, the group generated by $\Gamma_0(p)$ and H_p is a maximal discrete subgroup of $SL(2, \mathbb{R})$. So no manipulation can lead to a new matrix which is $\equiv 1$. Yet, we do obtain additional second order modular form type properties for newforms in $S_k(\Gamma_0(11))$. It is not clear what mechanism will lead to the production of new matrices for $N = 13$, yet not produce a contradiction when $N = 11$.

5. Direct tests of the conjecture

In the previous section we used computer algebra experiments to search for a proof of Conjecture 1.1 in the case $N = 13$. In this section we describe numerical experiments which directly test the conjecture. We rephrase the conjecture as a question that can be explored with a computer program.

Question 5.1. *Suppose N is prime, k is a positive even integer, and $\mathcal{P} = \{p_1, \ldots, p_m\}$ is a set of primes. Are there only finitely many m-tuples of real numbers $(a_{p_1}, \ldots, a_{p_m})$, with $|a_{p_j}| \le 2p^{\frac{k-1}{2}}$, such that there exists a function*

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z} \tag{5.1}$$

that satisfies

$$f|_k H_N = \varepsilon f \tag{5.2}$$

with $\varepsilon = \pm 1$, and

$$f|_k T_{p_j} = a_{p_j} f, \tag{5.3}$$

for all $p_j \in \mathcal{P}$?

Question 5.2. *If the answer to Question 5.1 is 'yes', are all such f in $S_k(\Gamma_0(N))$?*

Note that we assumed N is prime because the assumptions in Question 5.1 can force f to vanish only at the cusps 0 and ∞ of $\Gamma_0(N)$. So, for non-prime N it may be that the answer to Question 5.2 generally is 'no'. However, there is a straightforward extension to the case of square-free level using the Atkin-Lehner operators U_q for $q|N$. Alternatively, one can put an assumption on the growth of the coefficients a_n.

A more subtle situation in which the answer to Question 5.2 is 'no' concerns arithmetic properties of p_j mod N. An example will illustrate the difficulty. Consider the case $N = 17$ with $\mathcal{P} = \{2\}$. If χ_{17} is the primitive quadratic character modulo 17, then $\chi_{17}(2) = 1$. Thus, the Hecke operator T_2 for $S_k(\Gamma_0(17), \chi)$, given in (1.6) is identical in form to the Hecke operator T_2 for $S_k(\Gamma_0(17))$. Thus, such an assumption cannot lead to a 'yes' answer to Question 5.2, and the best one can hope for is $f \in S_k(\Gamma)$ where $\Gamma_1(17) \leq \Gamma < \Gamma_0(17)$. This shortcoming is discussed in [5].

5.1. **Tests of Question 5.1.** For $N = 13$ and various cases of k and $\mathcal{P}$, we performed numerical calculations which suggest answers to Question 5.1. The idea behind the calculation is that the Fricke relation (5.2) can be modeled by forcing

$$f(z_j) = N^{-k/2} z_j^{-k} f\left(\frac{-1}{N z_j}\right) \tag{5.4}$$

for a sufficiently large number of z_j. One can view (5.4) as an equation in the coefficients a_n. More specifically, given numerical values for $(a_{p_1}, \ldots, a_{p_m})$ and using the Hecke relations (5.3), the only unknown coefficients in the expansion of f are a_n where $(n, p_1 \cdots p_m) = 1$. Then (5.4) is a *linear* equation in those unknown coefficients. We truncate the Fourier expansion, and choose enough points z_j in order to have an overdetermined system. That system should be far from consistent unless there is an actual function $f(z)$ which is an eigenfunction of the Fricke involution and the Hecke operators. Thus, we test various m-tuples $(a_{p_1}, \ldots, a_{p_m})$, searching for those which lead to a consistent overdetermined linear system.

In order to truncate the Fourier series, we must first choose a $PRECISION$ to which we will work, and a lower bound Y_{MIN} such that $\Im(z_j) \geq Y_{MIN}$ for all j. That is,

$$f(z) = \sum_{n=1}^{NTERMS} a_n e^{2\pi i n z} + err, \tag{5.5}$$

where $|err| < 10^{-PRECISION}$ whenever $\Im(z) \geq Y_{MIN}$. We express the overdetermined system in matrix form $Mx = b$, and find the least squares solution $\bar{x}$. Let $F = \|M\bar{x} - b\|_\infty$ and set $E = F \times 10^{-PRECISION}$. We refer to E as the *(scaled) consistency measure* of the overdetermined system. Note that $E < 1$ if the overdetermined system is consistent to our chosen precision.

In the remainder of this section we give some example results. Details of our calculations can be found in Section 6.

5.2. **Sample results.** Figure 5.1 shows E vs. $a(2) = a_2 2^{-5/2}$ for $(N, k, \varepsilon, \mathcal{P}) = (13, 6, -1, 2)$. We have $(PRECISION, Y_{MIN}) = (75, 0.45/\sqrt{13})$. The calculation was done with 250 digits of extra precision. The vertical scale of 10^{63} suggests that there are at most 4 values of $a(2)$ which lead to a consistent overdetermined system. Thus, the answer to Question 5.1 appears to be 'yes' in the case $(N, k, \varepsilon, \mathcal{P}) = (13, 6, -1, \{2\})$. Also see Table 5.3.

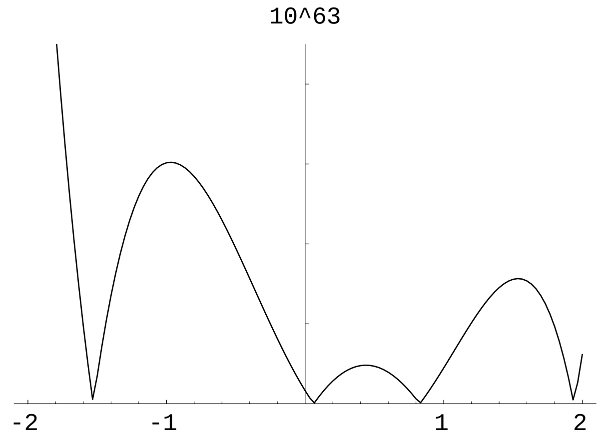

FIGURE 5.1. E vs. $a(2)$ for $(N, k, \varepsilon, \mathcal{P}) = (13, 6, -1, \{2\})$.

Figure 5.2 shows E vs. $a(3) = a_3 3^{-5/2}$ for $\mathcal{P} = \{3\}$, with the other parameters the same as in Figure 5.1. The vertical scale of 10^{-16} suggests that, to $PRECISION = 75$, all values of $a(3)$ which lead to a consistent overdetermined system. Thus, the answer to Question 5.1 may be 'no' in the case $(N, k, \varepsilon, \mathcal{P}) = (13, 6, -1, \{3\})$. Also see Table 5.3.

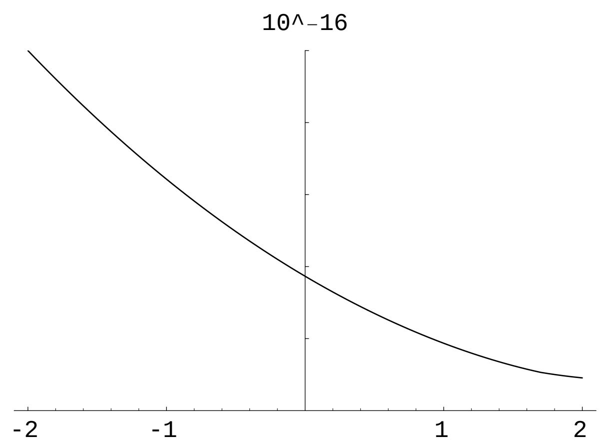

FIGURE 5.2. E vs. $a(3)$ for $(N, k, \varepsilon, \mathcal{P}) = (13, 6, -1, \{3\})$.

Figure 5.3 shows E vs. $a(3) = a_3 3^{-5/2}$ for $\mathcal{P} = \{3, 5, 7\}$, with the other parameters the same as in Figures 5.1 and 5.2. We set $a(5) \approx -0.17733$ and $a(7) \approx 0.94038$. The vertical scale of 10^{75} suggests that there are no values of $a(3)$, given our choices of $a(5)$ and $a(7)$, which lead to a consistent overdetermined system. This is what one would expect if there were only finitely many triples (a_3, a_5, a_7) which led to a consistent system, so the answer to Question 5.1 appears to be 'yes' in the case $(N, k, \varepsilon, \mathcal{P}) = (13, 6, -1, \{3, 5, 7\})$.

5.3. **Summary of results.** In Table 5.3 we give some representative results of our calculations. Let $E_{PRECISION} = \log_{10}(E)$, where E is the consistency measure described in Section 5.1. Thus, $E_{PRECISION} < 0$ corresponds to a consistent system. In Table 5.3 we report the value of $E_{PRECISION}$ for a single run of our Mathematica code.

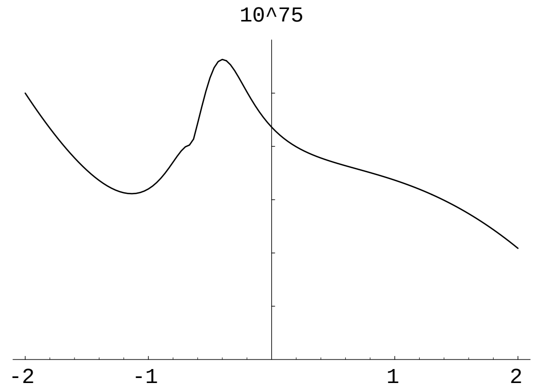

FIGURE 5.3. E vs. $a(3)$ for $(N, k, \varepsilon, \mathcal{P}) = (13, 6, -1, \{3, 5, 7\})$. We set $a_5 \approx -0.17733$ and $a_7 \approx 0.94038$.

Some trends in the table are expected, such as the increase of E with decreasing Y_{MIN}. When $E_{PRECISION}$ is positive, it is not surprising that increasing $PRECISION$ makes $E_{PRECISION}$ even larger.

One puzzling feature is that in some cases, such as $\mathcal{P} = \{3\}$, we have E significantly smaller than 1. Also surprising is the fact that in this case the error becomes smaller with increasing precision.

Another interesting feature is that the sign of $\varepsilon(-1)^{k/2}$, which is the sign of the functional equation of the associated L-function, has a noticeable effect on the consistency of the system. In particular, the error seems to be larger for systems corresponding to L-functions with an odd functional equation.

Our algorithm makes random choices to the z_j, and to create the data in the table we also must make a choice of a_p for $p \in \mathcal{P}$. Our experiments with various random choices showed a range of 6 in $E_{PRECISION}$ due to the randomness. To minimize the effect of these choices, we make the same choice for each example $\mathcal{P}$. Additionally, the same choice of points z_j were used for each instance of $(PRECISION, Y_{MIN})$.

5.4. **Another curiosity.** Immediately following Question 5.2 we described some scenarios in which the answer to Question 5.2 can be 'no'. That discussion describes all situations in which we have observed that the answer to Question 5.2 is 'no', however, there have been some surprisingly close calls.

For example, the values of $a(2) = a_2 2^{-5/2}$ in the -1 space of $S_6(\Gamma_0(13))$ are

$$a(2) \in \{-1.52887,\ 0.827455,\ 1.93885\}. \tag{5.6}$$

However, in Plot 5.1 there also appears to be a value of $a(2)$ near 0.062 which leads to a consistent overdetermined system. Could it be that there is a "fake" L-function having a functional equation and an Euler factor at $p = 2$, but which does not arise from a modular form? Figure 5.4 suggests that there does not exist such a solution with $a(2) \approx 0.062$. Note that if one could only work to, say, 8 decimal places, it would be quite difficult to determine if there was a solution with $a(2) \approx 0.062$.

It would be interesting to determine the underlying cause for this phenomenon. One is reminded of Hejhal's explanation [8] for the surprising (and incorrect) appearance of zeros of the Riemann ζ-function on an early table of eigenvalues of Maass forms for $SL(2, \mathbb{Z})$.

$\mathcal{P}$	k	ε	$\sqrt{13}\,Y_{MIN}$	E_{25}	E_{50}	E_{75}	E_{100}
2	4	1	0.85	-0.6	-0.4	6.0	10.2
			0.55	11.5	29.8	47.3	64.3
			0.45	17.5	41.5	65.8	86.5
			0.35	23.3	48.1	72.0	97.3
		-1	0.85	6.0	7.2	14.1	18.8
			0.55	18.7	38.0	55.9	73.5
			0.45	24.4	49.9	73.4	96.6
			0.35	24.3	49.7	73.8	99.3
	10	1	0.85	-6.0	-5.5	-4.6	0.4
			0.55	8.0	24.8	40.0	58.0
			0.45	14.8	37.3	60.9	79.6
		-1	0.55	4.5	20.3	36.0	52.6
			0.45	10.9	32.4	55.2	73.8
	16	1	0.55	-0.3	12.8	28.0	43.0
			0.45	5.5	24.9	47.8	67.0
2, 3	16	1	0.55	29.2	53.9	79.0	104.3
			0.45	30.1	54.8	79.5	104.8
3	10	1	0.55	-10.8	-15.5	-23.0	-27.8
			0.45	-8.5	-14.4	-17.7	-24.4
			0.35	-7.9	-10.8	-15.1	-19.1
3, 5	10	1	0.55	-1.9	3.0	9.4	16.5
			0.45	0.1	9.1	20.8	30.2
			0.35	6.5	20.4	36.8	52.8
		-1	0.55	-2.6	-1.2	3.9	10.3
			0.45	-1.6	4.1	14.6	23.5
			0.35	2.6	14.4	30.0	44.9
	12	1	0.55	-2.0	-3.7	-0.9	5.0
			0.45	-2.1	-0.1	10.2	18.3
			0.35	0.0	10.2	25.5	39.5
		-1	0.55	-1.9	-1.1	4.0	10.8
			0.45	-2.3	4.4	15.8	24.7
			0.35	3.2	15.7	32.0	46.9
3, 7	10	1	0.55	-3.6	-6.7	-8.2	-11.3
			0.35	-2.9	-3.6	-3.9	-6.4
3, 7, 11	8	1	0.55	2.4	11.4	21.9	34.8
			0.35	14.2	35.5	59.0	80.8
		-1	0.55	7.4	18.9	30.1	43.3
			0.35	19.8	43.5	67.3	91.8
	16	1	0.55	-3.6	0.1	6.7	18.8
			0.35	5.4	19.7	44.2	63.8
5, 7, 11, 17, 19	8	-1	0.55	4.2	8.3	8.2	5.8
			0.35	12.2	15.1	15.3	14.2

TABLE 5.1. $E_{PRECISION} = \log_{10}(E)$ for various cases of $(\mathcal{P}, k, \varepsilon)$ and Y_{MIN}. For each $\mathcal{P}$ the same random choice was made for each $p \in \mathcal{P}$. And for each choice of Y_{MIN} the same random choice was made for the z_j.

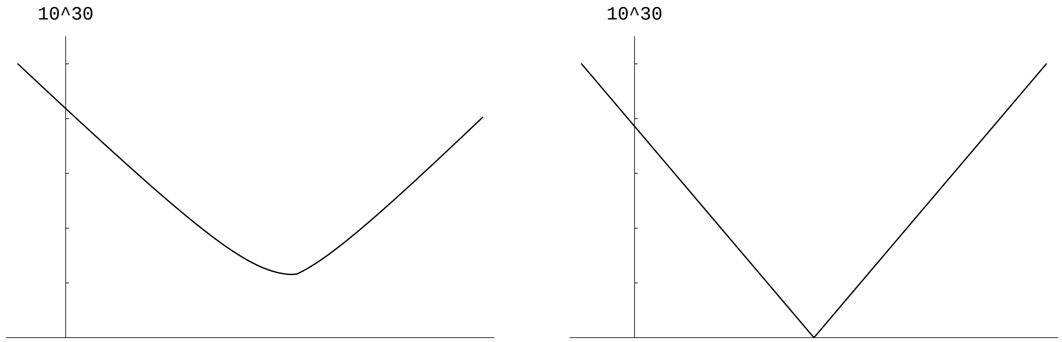

FIGURE 5.4. Close-ups of the minima near 0.062 and 0.827 in Plot 5.1 showing that there is not a "fake" L-function with $a_2 \approx 0.062$. In both plots the horizontal scale has width $\frac{1}{2} \times 10^{-9}$ and the vertical scale is of size 10^{30}. We have $(PRECISION, Y_{MIN}) = (50, 0.45/\sqrt{13})$, which is slightly less $PRECISION$ than in Plot 5.1.

5.5. **Other groups and further work.** A separate paper containing more detailed calculations for other groups is in preparation. For example, those calculations suggest that for $\mathcal{P} = \{2\}$ the answer to Question 5.1 is 'yes' for odd $N \leq 15$, and for $\mathcal{P} = \{2, 3\}$ the answer is 'yes' for $N \leq 31$ with $(N, 6) = 1$. Also, there are good prospects for answering similar questions for higher rank L-functions.

6. The Algorithm

In this section we describe some additional ideas behind our method and then give a complete description of our algorithm. The algorithm is quite similar to early methods of locating Maass waveforms [7], and it is closely based on the method of Farmer and Lemurell [4] for studying Maass forms.

6.1. **Parameters in the algorithm.** The Fourier expansion (5.1) must be truncated in order to deal with it computationally. We will be choosing pairs of points $(z_j, H_N z_j)$ in the upper half-plane $\mathcal{H}$. Since H_N switches the interior and exterior of the circle $|z| = 1/\sqrt{N}$, we may assume $|z_j| < 1/\sqrt{N}$. Note that this implies $\Im(H_N z_j) > \Im(z_j)$. If we fix $Y_{MIN} > 0$ and always choose $\Im(z_j) \geq Y_{MIN}$, then it is possible to precisely determine the error caused by truncating the Fourier series (5.1). Note that this requires a bound on the coefficients a_n. See Table 6.1 for some representative cases, of the number of terms, $NTERMS$, in the truncated Fourier expansion.

	25	50	75	100
0.85	52	92	132	172
0.55	82	145	206	268
0.45	101	178	253	328
0.35	132	231	327	424

TABLE 6.1. The number of terms in our truncated Fourier expansion, N_{TERMS}, for $PRECISION \in \{25, 50, 75, 100\}$ and various Y_{MIN}, for $N = 13$ and $k = 10$.

Thus, the input to our algorithm is the data from Question 5.1, along with Y_{MIN} (typically around $1/2\sqrt{N}$) and the desired precision (typically 50 or 100 digits).

6.2. **The algorithm.** Given $(N, \varepsilon, k, \mathcal{P})$, we use the following algorithm to suggest an answer to Question 5.1. First we choose a desired $PRECISION$ and a height Y_{MIN} for the imaginary part of z_j. Since we will be producing an overdetermined linear system, we must choose the factor by which the system is overdetermined. Generally, we use 30% more equations than unknowns.

Step 1. Choose test values for $a_{p_1}, \ldots, a_{p_m}$.

Step 2. Use Y_{MIN}, $PRECISION$, and the assumed bound $a_n \le d(n)n^{\frac{k-1}{2}}$, to determine $NTERMS$ such that

$$\tilde{f}(z) = \sum_{n=1}^{NTERMS} a_n e^{2\pi i n z} \tag{6.1}$$

satisfies

$$|\tilde{f}(z) - f(z)| < 10^{-PRECISION} \tag{6.2}$$

for $\Im z > Y_{MIN}$.

Step 3. Use the Hecke relations (5.3) and the values of $a_{p_1}, \ldots, a_{p_m}$ to rewrite $\tilde{f}$ as

$$\tilde{f}(z) = \sum_{\substack{n=1 \\ (n, p_1 \cdots p_m)=1}}^{NTERMS} b_n a_n e^{2\pi i n z}, \tag{6.3}$$

where the b_n are specific numerical values.

The a_n in (6.3) with $n > 1$ are the *unknowns* in the linear system we will produce. Let N_{UNK} denote the number of unknowns.

Step 4. Randomly choose approximately $N_{EQNS} \approx 1.3\, N_{UNK}$ points $z_j = x_j + i y_j \in \mathcal{H}$ with $y_j = Y_{MIN}$ and $|z_j| < 1/\sqrt{N}$. Let $z_j^* = H_N z_j$. Form the system of equations

$$\left\{ \tilde{f}(z_j) = N^{-k/2} z_j^{-k} \tilde{f}(z_j^*) \right\}_{j=1}^{N_{EQNS}}. \tag{6.4}$$

Step 5. Expressing (6.4) in matrix form $Mx = b$, find the least squares solution $\bar{x}$. Let $F = \|M\bar{x} - b\|_\infty$ and set $E = F \times 10^{-PRECISION}$. We refer to E as the *(scaled) consistency measure* of the overdetermined system.

Since $E < 1$ means that the overdetermined system is consistent to within the error from truncating the Fourier series, we interpret the output of the procedure as follows:

- If $E < 1$, it is possible that $(a_{p_1}, \ldots, a_{p_m})$ corresponds to a function satisfying (5.2) and (5.3). If E is much larger than 1, then $(a_{p_1}, \ldots, a_{p_m})$ probably does not correspond to such a function.
- If E is much smaller than 1, say by a factor of more than 10000, the calculation suggests that to within our $PRECISION$, almost all m-tuples $(a_{p_1}, \ldots, a_{p_m})$ correspond to such a function. Indeed, if there was only a finite number of such functions, and $(a_{p_1}, \ldots, a_{p_m})$ corresponded to one of them, then the consistency measure E should be about the same size as the error due to truncating the Fourier series. That is, $E \approx 1$. It would be unusual if a large number of Fourier coefficients immediately following a_{NTERMS} all happened to very small compared to their expected size.

- If $(a_{p_1},\ldots,a_{p_m})$ was chosen randomly, then if E is much larger than 1 the calculation suggests that only finitely many m-tuples $(a_{p_1},\ldots,a_{p_m})$ correspond to a function f satisfying (5.2) and (5.3). Indeed, based on the example of Hecke's original converse theorem, we expect that if the answer to Question 5.1 is 'no', then every choice of $(a_{p_1},\ldots,a_{p_m})$ should lead to a consistent system.

The above points are illustrated by the plots in Section 5.2.

6.3. **Implementation issues.** Several issues must be addressed before the algorithm can be implemented.

An unavoidable difficulty is that the system of linear equations (6.4) is horribly ill-conditioned. This is due to the rapid decay of $e^{-2\pi ny}$ as n grows. However, choosing all points $z_j = x_j + iy_j$ with $y_j = Y_{MIN}$ results in all elements in any given column of M being of the same magnitude. One will then get a well-conditioned equivalent system by normalizing each column of M to have say maximal absolute value equal to 1. Hence this is not as serious an issue as one might suspect. However, one should bear in mind that the precision of the computed Fourier coefficients will decrease as n gets larger.

The coefficients of the Fourier expansion grows like $n^{\frac{k-1}{2}}$. If the size of the coefficient at the point of truncation is roughly 10^c, then the smallest values in the matrix M will be of size $10^{-PRECISION-c}$. Hence you need at least $PRECISION + c$ digits of precision in the computation for the last coefficients to be taken into account. In order to avoid computational errors you will need even more digits of precision. For the calculations we present here, with a precision of 50 we typically work with at least 100 extra digits in the calculations, and with a precision of 100 we typically make use of at least 250 extra digits.

The choice of Y_{MIN} strongly affects the number of terms required in the Fourier expansion, so there may seem to be some benefit in choosing Y_{MIN} as large as possible to get a short Fourier expansion. However, if Y_{MIN} is chosen too large, then the points z_j and z_j^* are all located in a small region of the upper half-plane. Examining the function f in only a small region will require calculations to extremely high precision in order to reveal the global behavior. Thus, one cannot avoid using a fairly large number of Fourier coefficients, and if Y_{MIN} is chosen too large it can misleadingly give a system that is consistent to within the chosen precision. For example, choosing $Y_{MIN} = 0.85/\sqrt{N}$ does not appear to give satisfactory results, as shown in Table 5.3.

6.4. **Implementation in Mathematica.** We implemented the algorithm in Mathematica. In Step 1 the test values a_{p_i} were chosen as exact numbers satisfying $| a_{p_i} | \leq 2p_i^{\frac{k-1}{2}}$. For Step 2, Y_{MIN} was also chosen as an exact (rational) number. The truncation error was approximated by the maximal absolute value of the first excluded term and we used the approximate bound $| a_n | \leq 2n^{\frac{k-1}{2}}$. (This is true for n prime and not too far off for non-primes.) In Step 4, the points z_j (z_j^*) were chosen (computed) with exact rational coordinates and then set to the desired precision. Then Mathematica will use the necessary precision when computing the exponential function etc when forming the system (6.4). So in Step 5, the matrix M has the desired precision and when finding the least squares solution Mathematica will use the necessary precision. We use the QR-factorization in order to compute the least squares solution. If you don't use enough precision, sometimes the residual $M\bar{x} - b$ will be so small that Mathematica regard it as zero. One way to detect that the precision is not large enough is that the last coefficients of the least squares solution $\bar{x}$ will be zero. Once precision is large enough so that all coefficients of $\bar{x}$ are different from zero, raising the precision doesn't seem to affect the solution significantly.

We tried the program on known modular forms with precision up to 200 digits and it gave the correct Fourier coefficients to the desired precision.

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