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AN IDENTITY IN THE GENERALIZED FIBONACCI NUMBERS AND ITS APPLICATIONS

Mohammad Farrokhi D. G.
Department of Mathematics, Ferdowsi University of Mashhad, Mashhad, Iran
m.farrokhi.d.g@gmail.com

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Abstract

We first generalize an identity involving the generalized Fibonacci numbers and then apply it to establish some general identities concerning special sums. We also give a sufficient condition on a generalized Fibonacci sequence $\{U_n\}$ such that U_n is divisible by an arbitrary prime r for some $2 < n \leq r-2$.

1. Preliminaries

The generalized Fibonacci and Lucas numbers are defined, respectively, by Binet's formula, as follows

$$U_n(p,q) = \frac{\alpha^n - \beta^n}{\alpha - \beta},\ V_n(p,q) = \alpha^n + \beta^n,$$

where $\alpha = \frac{1}{2}(p + \sqrt{p^2-4q})$ and $\beta = \frac{1}{2}(p - \sqrt{p^2-4q})$. The numbers $U_n(p,q)$ and $V_n(p,q)$ can be defined recursively by

$$\begin{aligned} U_n(p,q) &= -qU_{n-2}(p,q) + pU_{n-1}(p,q), \\ V_n(p,q) &= -qV_{n-2}(p,q) + pV_{n-1}(p,q), \end{aligned}$$

for all integers n, where $U_0 = 0$, $U_1 = 1$, $V_0 = 2$ and $V_1 = p$. Throughout the paper, p and q denote the real numbers, U_n and V_n stand for $U_n(p,q)$ and $V_n(p,q)$, respectively, and $\Delta = p^2 - 4q$.

A sequence $\{G_n\}$ is said to be a (p,q)-sequence if G_n satisfies the recursive relation

$$G_n = -qG_{n-2} + pG_{n-1},$$

for all integers n. Clearly, the (p,q)-sequences, which are identified at two consecutive indices should be equal.

It is known that the formula

$$U_{a+b} = -qU_{a-1}U_b + U_aU_{b+1}$$

is valid for any generalized Fibonacci sequence $\{U_n(p,q)\}$ and all integers a, b.

We intend to present a generalization of this identity and derive some of its applications, which are the general solutions of some solved and unsolved problems concerning the generalized Fibonacci numbers. In Section 2, we prove our claim and give its generalization. In Section 3, we apply our identity to evaluate some summations involving that of Mansour [4] (see also [5]), the sum of powers of the generalized Fibonacci numbers and etc. In final section, we use our identity to get a divisibility property of the generalized Fibonacci numbers.

Remark 1. In the sequel we shall frequently use the fact that if a finite rational expression P contains some terms of a generalized Fibonacci sequence, which is not identically zero but it vanishes with respect to a special sequence $\{U_n\}$, we can always choose a sequence of the generalized Fibonacci sequences $\{U_n^m\}_{m=1}^{\infty}$ such that P does not vanishes over these sequences, while $\{U_n^m\}_{m=1}^{\infty}$ tends to $\{U_n\}$. Without loss of generality, we may assume that all the sequences under the consideration do not vanish over the expressions, which might appear in the denominators.

2. Main Results

It is well-known that if $\{U_n(p,q)\}$ is a generalized Fibonacci sequence, then

$$U_{a+b} = -qU_{a-1}U_b + U_aU_{b+1},$$

for all integers a, b. It is also proved in [2, Lemma 2.1(c)] that the identity

$$F_{a+b+c-3} = F_aF_bF_c + F_{a-1}F_{b-1}F_{c-1} - F_{a-2}F_{b-2}F_{c-2}$$

is valid, for all integers a, b, c. We give a generalization of these identities in terms of the generalized Fibonacci sequences.

Theorem 2. *If $\{U_n\}$ is a generalized Fibonacci sequence, then for all natural numbers m,*

$$U_{a_1+\cdots+a_m-\binom{m+1}{2}} = \sum_{i=1}^{m} \left\{ {m \atop i} \right\} U_{a_1-i}\cdots U_{a_m-i}, \tag{1}$$

where $a_1, \ldots, a_m$ are integers and

$$\left\{ {m \atop i} \right\} = \left(\prod_{\substack{j=1 \\ j\neq i}}^{m} U_{j-i} \right)^{-1}.$$

Proof. First suppose that $a_1, \ldots, a_m$ are equal to $1, 2, \ldots, i, i+2, i+3, \ldots, m+1$, in some order. Without loss of generality, we may assume that $a_1 = 1, \ldots, a_i = i$,

$a_{i+1} = i+2, \ldots, a_m = m+1$. If $j \neq i+1$, then $U_{a_1-j} \cdots U_{a_m-j} = 0$ holds and if $j = i+1$, then $U_{a_1-j} \cdots U_{a_m-j} = U_{-i} \cdots U_{-1} U_1 \cdots U_{m-i}$ so that $\left\{ {m \atop j} \right\} U_{a_1-j} \cdots U_{a_m-j} = U_{m-i}$. On the other hand, $U_{a_1+\cdots+a_m-\binom{m+1}{2}} = U_{m-i}$ and in this case the equality holds. If $U_n = U_n(p,q)$, then clearly the both sides of (1) are (p,q)-sequences with respect to each a_i and also they identify on the cube $(1,2,\ldots,m)+\{0,1\}^m$. Hence the both sides of (1) should be equal over all the integer values of $a_1, \ldots, a_m$. The proof is now complete. $\square$

Theorem 2 can be generalized in the following manner.

Theorem 3. *If $\{U_n\}$ is a generalized Fibonacci sequence, then for each natural numbers m and n (with the same parity),*

$$U_{a_1+\cdots+a_n-\binom{m+1}{2}} = \frac{1}{\Delta^{\frac{1}{2}(m-n)}} \sum_{i=1}^{m} \left\{ {m \atop i} \right\} U_{a_1-m_1 i} \cdots U_{a_n-m_n i},$$

where $m = m_1 + \cdots + m_n$ and $m_1, \ldots, m_n$ are odd natural numbers.

Proof. Let $U_k = U_k(p,q)$ and $m, m_1, \ldots, m_n$ be natural numbers such that $m = m_1 + \cdots + m_n$ and $m_1, \ldots, m_n$ are odd. By putting $a_{m-m_n+1} = \cdots = a_m = k$ in Theorem 2, we get

$$U_{a_1+\cdots+a_{m-m_n}+m_n k-\binom{m+1}{2}} = \sum_{i=1}^{m} \left\{ {m \atop i} \right\} U_{a_1-i} \cdots U_{a_{m-m_n}-i} U_{k-i}^{m_n}. \tag{2}$$

By definition, $U_k = (\alpha^k - \beta^k)/(\alpha-\beta)$, where $\alpha = (p+\sqrt{\Delta})/2$ and $\beta = (p-\sqrt{\Delta})/2$. Hence if $\Delta > 0$, then $\beta < \alpha$ and so $\lim_{k\to\infty} U_k/\alpha^k = 1/(\alpha-\beta)$, from which together with (2) we obtain

$$(\alpha-\beta)^{m_n-1} \alpha^{a_1+\cdots+a_{m-m_n}-\binom{m+1}{2}} = \sum_{i=1}^{m} \left\{ {m \atop i} \right\} U_{a_1-i} \cdots U_{a_{m-m_n}-i} \alpha^{-m_n i}. \tag{3}$$

A simple computation shows that $\alpha^k = -qU_{k-1} + U_k \alpha$, for each integer k. Suppose that p and q are rational but α is irrational. This together with (3) yields

$$(\alpha-\beta)^{m_n-1} U_{a_1+\cdots+a_{m-m_n}+\binom{m+1}{2}-1} = \sum_{i=1}^{m} \left\{ {m \atop 2} \right\} U_{a_1-i} \cdots U_{a_{m-m_n}-i} U_{-m_n i-1}$$

and

$$(\alpha-\beta)^{m_n-1} U_{a_1+\cdots+a_{m-m_n}+\binom{m+1}{2}} = \sum_{i=1}^{m} \left\{ {m \atop 2} \right\} U_{a_1-i} \cdots U_{a_{m-m_n}-i} U_{-m_n i},$$

from which we obtain

$$(\alpha-\beta)^{m_n-1}U_{a_1+\cdots+a_{m-m_n}+b_n+\binom{m+1}{2}} = \sum_{i=1}^{m} \left\{ {m \atop 2} \right\} U_{a_1-i}\cdots U_{a_{m-m_n}-i}U_{b_n-m_n i}, \quad (4)$$

for all integers b_n. Now, since the left and right hand sides of (4) are of the forms $P(p,q)+Q(p,q)\Delta$ and $P'(p,q)+Q'(p,q)\Delta$, respectively, where P, Q, P' and Q' are polynomials, the equation (4) should be held for all real values of p and q. The desired claim will be obtained by repeating the above procedure $(n-1)$ times. □

3. Applications

In this section, we use Theorem 2 to obtain some identities involving the generalized Fibonacci numbers. Our approach provides an alternative proof of Mansour's results [4] and general solutions of some summations, which are partially known.

Definition. Let m and k be any natural numbers. Then for any $n \geq 1$,

$$S_m(n;p,q;k) := \sum_{a_1+\cdots+a_m=n} U_{ka_1}\cdots U_{ka_m}.$$

In terms of the Fibonacci numbers, Vajda [6, Identity 98] and Dunlap [1, Identity 55] proved that

$$S_2(n;1,-1;1) = \sum_{a+b=n} F_aF_b = \frac{1}{5}\left(nL_n - F_n\right),$$

for each $n \geq 1$. Following the Vajda's and Dunlap's results, Zhang in [7] obtained $S_m(n;1,-1;1)$, when $m \leq 4$. Recently, Zhao and Wang [8] have proved Zhang's results in terms of the generalized Fibonacci numbers. Mansour in [4] has also obtained the following generalization of Zhao's and Wang's results, when m is an arbitrary natural number. For $n \geq m$,

$$\sum_{i=0}^{m}\left[(4q^k)^{m-i}\left(\sum_{j=0}^{i}(-1)^j\binom{i}{j}(i+1-j)^m\right)\left(\frac{V_k^2(p,q)-4q^k}{U_k(p,q)}\right)^i \times S_{i+1}(n+i-m;p,q;k)\right]$$

$$= \sum_{i=1}^{m}\left[\frac{(-1)^{m-1}(2q^k)^{m-i}}{(i-1)!}\left(\sum_{j=0}^{i-1}(-1)^j\binom{i-1}{j}(j+1)^{m-1}\right) \times \left(\sum_{j=0}^{i} v_{m,i,j}U_{(n+i-m-j)k}(p,q)\binom{i}{j}\right)\right],$$

where $v_{m,i,j} = (-2q^k)^j V_k^{i-j}(p,q)\prod_{l=1}^{i}(n+i+m-j-l)$.

Now, in the following, we give a different identity for $S_m(n;p,q;k)$, by using Theorem 2.

Theorem 4. *Let m and k be natural numbers. Then for any $n \geq 1$,*

$$\frac{S_m(n;p,q;k)}{U_k(p,q)^m} = \frac{(b_m + pa_m)\delta_{m,n+m^2} - a_m\delta_{m,n+m^2+1} + a_m^2 S_{m-1}(n+1;\bar{p},\bar{q};1)}{b_m^2 + pa_m b_m + qb_m^2},$$

where $(\bar{p},\bar{q}) = (V_k(p,q), q^k)$,

$$\begin{aligned}
\delta_{m,n} &= \alpha_{m,n} - \sum_{i=1}^{m-1}(a_i\gamma_{m,n,i} + b_i\beta_{m,n,i}),\\
\gamma_{m,n,i} &= -qU_m(\bar{p},\bar{q})\sum_{i=1}^{m} U_{i-m-1}(\bar{p},\bar{q})S_{m-1}(n-i;\bar{p},\bar{q};1)\\
&\quad + U_{m+1}(\bar{p},\bar{q})\sum_{i=1}^{m-1} U_{i-m}(\bar{p},\bar{q})S_{m-1}(n-i;\bar{p},\bar{q};1),\\
\beta_{m,n,i} &= -qU_{m-1}(\bar{p},\bar{q})\sum_{i=1}^{m-1} U_{i-m}(\bar{p},\bar{q})S_{m-1}(n-i;\bar{p},\bar{q};1)\\
&\quad + U_m(\bar{p},\bar{q})\sum_{i=1}^{m-2} U_{i-m+1}(\bar{p},\bar{q})S_{m-1}(n-i;\bar{p},\bar{q};1),\\
\alpha_{m,n} &= \binom{n-1}{m-1}U_{n-\binom{m+1}{2}}(\bar{p},\bar{q}) - \sum_{i=1}^{m}\left\{ m \atop i \right\}\sum_{j=1}^{m-1}\binom{m}{j}\\
&\quad \times \sum_{k=1}^{n-(i+1)(m-j)}\left(\sum_{\substack{a_1,\ldots,a_j<i\\ a_1+\cdots+a_j=k}} U_{a_1-i}(\bar{p},\bar{q})\cdots U_{a_j-i}(\bar{p},\bar{q})\right)\\
&\quad \times S_{m-j}(n-k-i(m-j));\bar{p},\bar{q};1)
\end{aligned}$$

and

$$\begin{bmatrix} a_0 \\ b_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ \begin{bmatrix} a_i \\ b_i \end{bmatrix} = \begin{bmatrix} U_{m+1} & U_m \\ -qU_m & -qU_{m-1} \end{bmatrix}\begin{bmatrix} a_{i-1} \\ b_{i-1} \end{bmatrix} + \begin{bmatrix} 0 \\ \left\{ m \atop i \right\} \end{bmatrix},$$

for $i = 1, \ldots, m$.

Proof. Let m and k be natural numbers. Since

$$\begin{aligned} U_n(p,q) &= \frac{\alpha^{kn}-\beta^{kn}}{\alpha-\beta} \\ &= \frac{\alpha^k-\beta^k}{\alpha-\beta}\cdot\frac{\left(\alpha^k\right)^n-\left(\beta^k\right)^n}{\alpha^k-\beta^k} = U_k(p,q)U_n\left(V_k(p,q),q^k\right), \end{aligned}$$

we have

$$S_m(n;p,q;k) = U_k(p,q)^m S_m(n;V_k(p,q),q^k;1).$$

Thus we may assume that $k=1$. Let $U_n = U_n(p,q)$ and $S_m(n) = S_m(n;p,q;1)$. By Theorem 2,

$$\begin{aligned} \binom{n-1}{m-1}U_{n-\binom{m+1}{2}} &= \sum_{a_1+\cdots+a_m=n} U_{a_1+\cdots+a_m-\binom{m+1}{2}} \\ &= \sum_{i=1}^{m}\left\{{m \atop i}\right\}\sum_{a_1+\cdots+a_m=n} U_{a_1-i}\cdots U_{a_m-i} \\ &= \sum_{i=1}^{m}\left\{{m \atop i}\right\}\left(S_m(n-im)+\sum_{\substack{a_1+\cdots+a_m=n\\ \exists j:a_j<i}} U_{a_1-i}\cdots U_{a_m-i}\right). \end{aligned}$$

Hence

$$\left\{{m \atop 1}\right\}S_m(n-m)+\cdots+\left\{{m \atop m}\right\}S_m(n-m^2) = \alpha_{m,n}, \tag{5}$$

where

$$\begin{aligned} \alpha_{m,n} = \binom{n-1}{m-1}U_{n-\binom{m+1}{2}} &- \sum_{i=1}^{m}\left\{{m \atop i}\right\}\sum_{j=1}^{m-1}\binom{m}{j} \\ &\times \sum_{k=1}^{n-k-(i+1)(m-j)}\left(\sum_{\substack{a_1,\ldots,a_j<i\\ a_1+\cdots+a_j=k}} U_{a_1-i}\cdots U_{a_j-i}\right) \\ &\qquad \times S_{m-j}(n-k-i(m-j)). \end{aligned}$$

Applying Theorem 2, when $m=2$, gives

$$U_{a+b} = -qU_{a-1}U_b + U_aU_{b+1},$$

from which we obtain

$$\begin{aligned} S_m(n) &= \sum_{a_1+\cdots+a_m=n} U_{a_1}\cdots U_{a_m} \\ &= -qU_{m-1}\sum_{a_1+\cdots+a_m=n} U_{a_1-m}U_{a_2}\cdots U_{a_m} \\ &\quad +U_m \sum_{a_1+\cdots+a_m=n} U_{a_1-m+1}U_{a_2}\cdots U_{a_m} \\ &= -qU_{m-1}S_{n-m}+U_mS_{n-m+1} \\ &\quad -qU_{m-1}\sum_{i=1}^{m-1} U_{i-m}S_{m-1}(n-i)+U_m\sum_{i=1}^{m-2} U_{i-m+1}S_{m-1}(n-i). \end{aligned}$$

Hence for each $0 \le i < \left[\frac{n}{m}\right]$,

$$S_m(n-im) = -qU_{m-1}S_m(n-(i+1)m)+U_mS_m(n-(i+1)m+1)+\beta_{m,n,i}, \quad (6)$$

where

$$\beta_{m,n,i} = -qU_{m-1}\sum_{i=1}^{m-1} U_{i-m}S_{m-1}(n-i)+U_m\sum_{i=1}^{m-2} U_{i-m+1}S_{m-1}(n-i).$$

Similarly, it can be shown that for each $0 \le i < \left[\frac{n}{m}\right]$,

$$S_m(n-im+1) = -qU_mS_m(n-(i+1)m)+U_{m+1}S_m(n-(i+1)m+1)+\gamma_{m,n,i}, \quad (7)$$

where

$$\gamma_{m,n,i} = -qU_m\sum_{i=1}^{m} U_{i-m-1}S_{m-1}(n-i)+U_{m+1}\sum_{i=1}^{m-1} U_{i-m}S_{m-1}(n-i).$$

Now, let

$$\begin{bmatrix} a_0 \\ b_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} a_i \\ b_i \end{bmatrix} = \begin{bmatrix} U_{m+1} & U_m \\ -qU_m & -qU_{m-1} \end{bmatrix} \begin{bmatrix} a_{i-1} \\ b_{i-1} \end{bmatrix} + \begin{bmatrix} 0 \\ \left\{ {m \atop i} \right\} \end{bmatrix},$$

for $i = 1, \ldots, m$. Then, by using (5), (6) and (7), it can be easily shown that for $i = 1, \ldots, m$,

$$\begin{aligned} a_iS_m(n-im+1)+b_iS_m(n-im)+\sum_{j=i+1}^{m} \left\{ {m \atop j} \right\} S_m(n-jm) \\ = \alpha_{m,n} - \sum_{j=1}^{i-1}(a_j\gamma_{m,n,j}+b_j\beta_{m,n,j}). \end{aligned} \quad (8)$$

Replacing i by m in (8), we get

$$a_m S_m(n - m^2 + 1) + b_m S_m(n - m^2) = \delta_{m,n}, \tag{9}$$

where

$$\delta_{m,n} = \alpha_{m,n} - \sum_{i=1}^{m-1} (a_i \gamma_{m,n,i} + b_i \beta_{m,n,i}).$$

By (9) we have

$$\begin{aligned} \delta_{m,n+1} &= a_m S_m(n - m^2 + 2) + b_m S_m(n - m^2 + 1) \\ &= a_m(-q S_m(n - m^2) + p S_m(n - m^2 + 1) + S_{m-1}(n - m^2 + 1)) \\ &\quad + b_m S_m(n - m^2 + 1), \end{aligned}$$

that is

$$\begin{aligned} &-q a_m S_m(n - m^2) + (b_m + p a_m) S_m(n - m^2 + 1) \\ &\qquad = \delta_{m,n+1} - a_m S_{m-1}(n - m^2 +). \end{aligned} \tag{10}$$

Solving the equations (9) and (10) we obtain

$$S_m(n - m^2) = \frac{(b_m + p a_m)\delta_{m,n} - a_m(\delta_{m,n+1} - a_m S_{m-1}(n - m^2 + 1))}{b_m^2 + p a_m b_m + q a_m^2},$$

whence the result follows. $\square$

Now, suppose that $\{U_n(p,q)\}$ is a generalized Fibonacci sequence. Then, by utilizing Binet's formulas,

$$\sum_{x=1}^{n} U_{ax+b} = \frac{q^a U_{na+b} - U_{(n+1)a+b} - q^a U_{b-a} + U_b}{1 + q^a - V_a} - U_b \tag{11}$$

and

$$\begin{aligned} \sum_{x+y=n} U_{ax+b} U_{cy+d} = \big[&(q^c V_a - q^a V_c) U_d U_{(n+1)a+b} - q^{2c} U_{d-c} U_{(n+1)a+b} \\ &+ q^{2a} U_d U_{na+b} - q^{2a+c} U_{d-c} U_{(n-1)a+b} + q^{a+2c} U_{d-2c} U_{na+b} \\ &+ (q^a V_c - q^c V_a) U_b U_{(n+1)c+d} - q^{2a} U_{b-a} U_{(n+1)c+d} \\ &+ q^{2c} U_b U_{nc+d} - q^{a+2c} U_{b-a} U_{(n-1)c+d} + q^{2a+c} U_{b-2a} U_{nc+d} \big] \\ / \big[&(q^a + q^c)^2 + q^a(V_{2c} - V_{a+c} - q^a V_{c-a}) + q^c(V_{2a} - V_{a+c} - q^c V_{a-c}) \big] \\ &- U_b U_{nc+d} - U_{na+b} U_d. \end{aligned} \tag{12}$$

Note that, the identity (12) generalizes the case $m = 2$ in Theorem 4. In the remainder of this section we use identities (11) and (12) to evaluate some summations involving the generalized Fibonacci numbers.

Theorem 5. *Let $\{U_n\}$ be a generalized Fibonacci sequence and let m be a natural number. Then for any $n \geq 1$,*

$$\sum_{i=1}^{n} U_i^m = \frac{1}{\sum_{i=1}^{m} \left\{ {m \atop i} \right\}} \left(\sum_{i=1}^{n} U_{im-\binom{m+1}{2}} + \sum_{i=1}^{m} \left\{ {m \atop i} \right\} \sum_{j=1}^{i} (U_{n-i+j}^m - U_{j-i}^m) \right).$$

Proof. Using Theorem 2, when $a_1 = \cdots = a_m$, we get

$$\begin{aligned} \sum_{i=1}^{n} U_{im-\binom{m+1}{2}} &= \sum_{j=1}^{m} \left\{ {m \atop j} \right\} \sum_{i=1}^{n} U_{i-j}^m \\ &= \sum_{j=1}^{m} \left\{ {m \atop j} \right\} \left(\sum_{i=1}^{n} U_i^m + \sum_{i=1}^{j} (U_{i-j}^m - U_{n-j+i}^m) \right) \\ &= \left(\sum_{i=1}^{m} \left\{ {m \atop i} \right\} \right) \sum_{i=1}^{n} U_i^m + \sum_{i=1}^{m} \left\{ {m \atop i} \right\} \sum_{j=1}^{i} (U_{j-i}^m - U_{n-i+j}^m), \end{aligned}$$

which gives the result. $\square$

It is known than the sequence of Fibonacci numbers satisfies the property that $F_{-n} = (-1)^{n-1} F_n$, for all integers n. In general, if $\{U_n\}$ is a generalized Fibonacci sequence then $U_{-n} = -U_n/q^n$, for all integers n and if $q = 1$ and m is even, or $q = -1$ and $4 \mid m$, then

$$\left\{ {m \atop i} \right\} + \left\{ {m \atop j} \right\} = 0,$$

when $i + j = m$. Hence $\sum_{i=1}^{m} \left\{ {m \atop i} \right\} = 0$.

Now, from the proof of Theorem 5 we may deduce the following.

Corollary 6. *If $\sum_{i=1}^{m} \left\{ {m \atop i} \right\} = 0$, then*

$$\sum_{i=1}^{n} U_{im-\binom{m+1}{2}} = \sum_{i=1}^{m} \left\{ {m \atop i} \right\} \sum_{j=1}^{i} (U_{j-i}^m - U_{n-i+j}^m).$$

In particular, the equality holds when $4 \mid m$ and $\{U_n\}$ is the sequence of Fibonacci numbers.

Definition. Let $m \geq 1$. Then for any $n \geq 1$,

$$T_m(n;p,q) := \sum_{i=1}^{n-1} U_{n-i} U_i^m.$$

By the above definitions, $T_1(n;p,q) = S_2(n;p,q;1)$, which is known from [8], [4] and Theorem 4. Using a group theoretical tool, the author in [3] proves that for Fibonacci numbers

$$T_2(n;1,-1) = \binom{F_{n+1}}{2} - \binom{F_n}{2}$$

and utilizing this tool once more, it can be proved that

$$T_3(n;1,-1) = \frac{1}{2} F_{n-1} F_n F_{n+1} - \binom{F_n+1}{3},$$

for all integers n.

Now, we extend the above results by computing $T_m(n;p,q)$, for each natural number m.

Theorem 7. *Let m be a natural number. Then for any $n \geq 1$,*

$$T_m(n) = \frac{q a_{m+1} \beta_{m,n+2m+1} + b_{m+1} \beta_{m,n+2m+2} - a_{m+1} b_{m+1} U_n^m}{b_{m+1}^2 + p a_{m+1} b_{m+1} + q a_{m+1}^2},$$

where

$$\begin{aligned} \beta_{m,n} &= \alpha_{m,n} - \sum_{j=1}^{m} (a_j U_{n-2j}^m + b_j U_{n-2j-1}^m), \\ \alpha_{m,n} &= \sum_{i=1}^{n-1} U_{(m-1)i+n-\binom{m+2}{2}} \\ &\qquad - \sum_{j=1}^{m+1} \left\{ {m+1 \atop j} \right\} \sum_{i=1}^{j-1} \left(U_{n-j-i} U_{i-j}^m + U_{i-j} U_{n-j-i}^m \right), \end{aligned}$$

and

$$\begin{bmatrix} a_0 \\ b_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} a_i \\ b_i \end{bmatrix} = \begin{bmatrix} -q & p \\ p & -q \end{bmatrix} \begin{bmatrix} a_{i-1} \\ b_{i-1} \end{bmatrix} + \begin{bmatrix} 0 \\ \left\{ {m+1 \atop i} \right\} \end{bmatrix},$$

for $i = 1, \ldots, m+1$. Also

$$T_m(n) = \frac{q a_{m-1} \beta_{m,n+m-1} + b_{m-1} \beta_{m,n+m} - a_{m-1} b_{m-1} U_n^m}{b_{m-1}^2 + p a_{m-1} b_{m-1} + q a_{m-1}^2},$$

where

$$\begin{aligned}\beta_{m,n} &= \alpha_{m,n} - \sum_{i=1}^{m-1} a_i U_{n-i-1}^m, \\ \alpha_{m,n} &= \sum_{i=1}^{n-1} U_{n-i} U_{mi-\binom{m+1}{2}} - \sum_{j=1}^{m} \left\{ {m \atop j} \right\} \sum_{i=1}^{j-1} U_{n-i} U_{i-j}^m\end{aligned}$$

and

$$\begin{bmatrix} a_{-1} \\ b_{-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ \begin{bmatrix} a_i \\ b_i \end{bmatrix} = \begin{bmatrix} p & 1 \\ -q & 0 \end{bmatrix} \begin{bmatrix} a_{i-1} \\ b_{i-1} \end{bmatrix} + \begin{bmatrix} 0 \\ \left\{ {m \atop i+1} \right\} \end{bmatrix},$$

for $i = 0, 1, \ldots, m$.

Proof. By Theorem 2,

$$U_{n-x+mx-\binom{m+2}{2}} = \sum_{i=1}^{m+1} \left\{ {m+1 \atop i} \right\} U_{n-x-i} U_{x-i}^m$$

so that

$$\begin{aligned}\sum_{i=1}^{n-1} U_{(m-1)i+n-\binom{m+2}{2}} &= \sum_{j=1}^{m+1} \left\{ {m+1 \atop j} \right\} \sum_{i=1}^{n-1} U_{n-j-i} U_{i-j}^m \\ &= \sum_{j=1}^{m+1} \left\{ {m+1 \atop j} \right\} \left(\sum_{i=j+1}^{n-j-1} U_{n-j-i} U_{i-j}^m + \sum_{i=1}^{j} U_{n-j-i} U_{i-j}^m \right. \\ &\qquad \left. + \sum_{i=n-j}^{n-1} U_{n-j-i} U_{i-j}^m \right) \\ &= \sum_{j=1}^{m+1} \left\{ {m+1 \atop j} \right\} \left(\sum_{i=1}^{n-2j-1} U_{n-2j-i} U_i^m \right. \\ &\qquad \left. + \sum_{i=1}^{j-1} \left(U_{n-j-i} U_{i-j}^m + U_{i-j} U_{n-j-i}^m \right) \right).\end{aligned}$$

Hence

$$\left\{ {m+1 \atop 1} \right\} T_m(n-2) + \cdots + \left\{ {m+1 \atop m+1} \right\} T_m(n-2m-2) = \alpha_{m,n}, \tag{13}$$

where

$$\begin{aligned}\alpha_{m,n} &= \sum_{i=1}^{n-1} U_{(m-1)i+n-\binom{m+2}{2}} \\ &\qquad - \sum_{j=1}^{m+1} \left\{ {m+1 \atop j} \right\} \sum_{i=1}^{j-1} \left(U_{n-j-i} U_{i-j}^m + U_{i-j} U_{n-j-i}^m \right).\end{aligned}$$

We have

$$\begin{aligned}T_m(n) &= \sum_{i=1}^{n-1} U_{n-i} U_i^m \\ &= -q \sum_{i=1}^{n-1} U_{n-2-i} U_i^m + p \sum_{i=1}^{n-1} U_{n-1-i} U_i^m \\ &= -qT_m(n-2) + pT_m(n-1) + U_{n-1}^m.\end{aligned} \tag{14}$$

Now, put

$$\begin{bmatrix} a_0 \\ b_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} a_i \\ b_i \end{bmatrix} = \begin{bmatrix} -q & p \\ p & -q \end{bmatrix} \begin{bmatrix} a_{i-1} \\ b_{i-1} \end{bmatrix} + \begin{bmatrix} 0 \\ \left\{ {m+1 \atop i} \right\} \end{bmatrix},$$

for $i = 1, \ldots, m+1$. Using (13) and (14), it can be easily verified that

$$\begin{aligned} a_i T_m(n-2i+1) + b_i T_m(n-2i) + \sum_{j=i+1}^{m+1} \left\{ {m+1 \atop j} \right\} T_m(n-2j) \\ = \alpha_{m,n} - \sum_{j=1}^{i-1} (a_j U_{n-2j}^m + b_j U_{n-2j-1}^m).\end{aligned} \tag{15}$$

Replacing i by $m+1$ in (15), we get

$$a_{m+1} T_m(n-2m-1) + b_{m+1} T_m(n-2m-2) = \beta_{m,n}, \tag{16}$$

where

$$\beta_{m,n} = \alpha_{m,n} - \sum_{j=1}^{m} (a_j U_{n-2j}^m + b_j U_{n-2j-1}^m).$$

By (16),

$$\begin{aligned}\beta_{m,n+1} &= a_{m+1} T_m(n-2m) + b_{m+1} T_m(n-2m-1) \\ &= a_{m+1}(-qT_m(n-2m-2) + pT_m(n-2m-1) + U_{n-2m-1}^m) \\ &\qquad + b_{m+1} T_m(n-2m-1),\end{aligned}$$

which gives

$$(b_{m+1} + pa_{m+1})T_m(n-2m-1) - qa_{m+1}T_m(n-2m-2) \\ = \beta_{m,n+1} - a_{m+1}U^m_{n-2m-1}. \tag{17}$$

Now, by solving the system of equations (16) and (17), we obtain

$$T_m(n-2m-1) = \frac{qa_{m+1}\beta_{m,n} + b_{m+1}(\beta_{m,n+1} - a_{m+1}U^m_{n-2m-1})}{b^2_{m+1} + pa_{m+1}b_{m+1} + qa^2_{m+1}},$$

which proves the result.

To prove the second identity we proceed in a similar way. By Theorem 2,

$$U_{mx-\binom{m+1}{2}} = \sum_{i=1}^{m} \left\{ {m \atop i} \right\} U^m_{x-i},$$

from which

$$\begin{aligned} \sum_{i=1}^{n-1} U_{n-i}U_{mi-\binom{m+1}{2}} &= \sum_{j=1}^{m} \left\{ {m \atop j} \right\} \sum_{i=1}^{n-1} U_{n-i}U^m_{i-j} \\ &= \sum_{j=1}^{m} \left\{ {m \atop j} \right\} \left(\sum_{i=1}^{n-j-1} U_{n-j-i}U^m_i + \sum_{i=1}^{j-1} U_{n-i}U^m_{i-j} \right). \end{aligned}$$

Hence

$$\left\{ {m \atop 1} \right\} T_m(n-1) + \cdots + \left\{ {m \atop m} \right\} T_m(n-m) = \alpha_{m,n}, \tag{18}$$

where

$$\alpha_{m,n} = \sum_{i=1}^{n-1} U_{n-i}U_{mi-\binom{m+1}{2}} - \sum_{j=1}^{m} \left\{ {m \atop j} \right\} \sum_{i=1}^{j-1} U_{n-i}U^m_{i-j}.$$

Now, put

$$\begin{bmatrix} a_{-1} \\ b_{-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} a_i \\ b_i \end{bmatrix} = \begin{bmatrix} p & 1 \\ -q & 0 \end{bmatrix} \begin{bmatrix} a_{i-1} \\ b_{i-1} \end{bmatrix} + \begin{bmatrix} 0 \\ \left\{ {m \atop i+1} \right\} \end{bmatrix},$$

for $i = 0, 1, \ldots, m$. Using (18) and (14), it can be easily verified that

$$a_iT_m(n-i) + b_iT_m(n-i-1) + \sum_{j=i+2}^{m} T_m(n-j) = \alpha_{m,n} - \sum_{j=1}^{i-1} a_iU^m_{n-j-1}. \tag{19}$$

Replacing $m-1$ by i in (19), we obtain

$$a_{m-1}T_m(n-m+1)+b_{m-1}T_m(n-m)=\beta_{m,n}, \tag{20}$$

where

$$\beta_{m,n}=\alpha_{m,n}-\sum_{i=1}^{m-1}a_iU^m_{n-i-1}.$$

By (20),

$$\begin{aligned}\beta_{m,n+1} &= a_{m-1}T_m(n-m+2)+b_{m-1}T_m(n-m+1)\\ &= a_{m-1}(-qT_m(n-m)+pT_m(n-m+1)+U^m_{n-m+1})\\ &\quad +b_{m-1}T_m(n-m+1),\end{aligned}$$

from which we get

$$\begin{aligned}&(b_{m-1}+pa_{m-1})T_m(n-m+1)-qa_{m-1}T_m(n-m+1)\\ &\quad=\beta_{m,n+1}-a_{m-1}U^m_{n-m+1}.\end{aligned} \tag{21}$$

Now, by solving the system of equations (20) and (21), we obtain

$$T_m(n-m+1)=\frac{qa_{m-1}\beta_{m,n}+b_{m-1}(\beta_{m,n+1}-a_{m-1}U^m_{n-m+1})}{b^2_{m-1}+pa_{m-1}b_{m-1}+qa^2_{m-1}},$$

from which the result follows. □

4. A Divisibility Property

In section 3, we gave some applications of Theorem 2 consisting some summations involving the generalized Fibonacci numbers. Now, we apply Theorem 2 to prove that if p,q are integers and r is an odd prime such that $r\nmid p,q,\Delta$, then r divides U_i, for some $2<i\le r-2$, when $q=1$, or $q=-1$ and $4|r-1$. To do this, we first obtain some properties of the generalized Fibonacci numbers.

Theorem 8. *Let r be an odd prime. Then for any $m\ge 0$,*

(i) $U_n(p,q)=\sum_{i=0}^m p^i(-q)^{m-i}\binom{m}{i}U_{n-m-i}(p,q)$;

and if p,q are integers, then

(ii) $U_n(p,q)\overset{r}{\equiv}-qU_{n-2r^m}(p,q)+pU_{n-r^m}(p,q)$;

(iii) $U_{rn}(p,q)\overset{r}{\equiv}U_r(p,q)U_n(p,q)$;

(iv) $U_{r+1}(p,q)\overset{r}{\equiv}\frac{1}{2}p\left(\left(\frac{\Delta}{r}\right)+1\right)$, $U_r(p,q)\overset{r}{\equiv}\left(\frac{\Delta}{r}\right)$ *and if* $r\nmid q$, *then* $U_{r-1}(p,q)\overset{r}{\equiv}\frac{p}{2q}\left(\left(\frac{\Delta}{r}\right)-1\right)$, *where* $\left(\frac{a}{b}\right)$ *denotes the Legendre's symbol.*

Proof. First suppose that d and k are integers and $\{G_n\}$ is a recursive sequence satisfying $G_n \stackrel{k}{\equiv} aG_{n-2d} + bG_{n-d}$, for all integers n. Then for each $m \geq 1$,

$$G_n \stackrel{k}{\equiv} \sum_{i=0}^{m} b^i a^{m-i} \binom{m}{i} G_{n-(m+i)d}. \tag{22}$$

In fact, if (22) holds for m, then we have

$$\begin{aligned} G_n &\stackrel{k}{\equiv} \sum_{i=0}^{m} b^i a^{m-i} \binom{m}{i} G_{n-(m+i)d} \\ &\stackrel{k}{\equiv} \sum_{i=0}^{m} b^i a^{m-i} \binom{m}{i} (aG_{n-(m+i+2)d} + bG_{n-(m+i+1)d}) \\ &\stackrel{k}{\equiv} \sum_{i=0}^{m+1} b^i a^{m+1-i} \binom{m+1}{i} G_{n-(m+1+i)d}. \end{aligned}$$

(i) it is obvious by (22).

(ii) Clearly, it is true when $m = 0$ and we may assume the result for m. Then, by (22),

$$\begin{aligned} U_n(p,q) &\stackrel{r}{\equiv} \sum_{i=0}^{r} p^i (-q)^{r-i} \binom{r}{i} U_{n-(r+i)r^m} \\ &\stackrel{r}{\equiv} -qU_{n-2r^{m+1}} + pU_{n-r^{m+1}}. \end{aligned}$$

(iii) By the definition, we have $U_0(p,q) \stackrel{r}{\equiv} U_r(p,q)U_0(p,q)$ and $U_r(p,q) \stackrel{r}{\equiv} U_r(p,q)$ $U_1(p,q)$. Now, suppose that the result holds for $n-2$ and $n-1$. Then, by part (ii),

$$\begin{aligned} U_{rn}(p,q) &\stackrel{r}{\equiv} -qU_{r(n-2)}(p,q) + pU_{r(n-1)}(p,q) \\ &\stackrel{r}{\equiv} -qU_r(p,q)U_{n-2}(p,q) + pU_r(p,q)U_{n-1}(p,q) \\ &\stackrel{r}{\equiv} U_r(p,q)U_n(p,q). \end{aligned}$$

(iv) Using Binet's formula, we obtain

$$\begin{aligned} U_n(p,q) &= \frac{\alpha^n - \beta^n}{\alpha - \beta} \qquad (23) \\ &= \frac{1}{2^{n-1}} \left(\binom{n}{1} p^{n-1} + \binom{n}{3} p^{n-3}\Delta + \binom{n}{5} p^{n-5}\Delta^2 + \cdots \right). \end{aligned}$$

Now, by putting $n = r-1$, r and $r+1$, respectively, in (23), we obtain the desired results. □

Theorem 9. *Let p and q be integers and r be an odd prime such that $r \nmid p, q, \Delta$. If $q = 1$, or $q = -1$ and $4 \mid r-1$, then $r \mid U_i$ for some $2 < i \leq r-2$. If $r \mid U_{r-1}$, then $r \mid U_{\frac{r-1}{2}}$.*

Proof. We first prove that $r \mid U_i$, for some $i = 1, \ldots, r-1$. If $r \nmid U_i$, where $i = 1 \ldots, r-2$, then we may apply identity (1) modulo r to get

$$U_{(r-1)a-\binom{r}{2}} \overset{r}{\equiv} \sum_{i=1}^{r-1} \begin{Bmatrix} r-1 \\ i \end{Bmatrix} U_{a-i}^{r-1}. \tag{24}$$

Moreover, if $r \nmid U_{r-1}$, then by putting $a = r$ in (24), we obtain

$$U_{\binom{r}{2}} \overset{r}{\equiv} \sum_{i=1}^{r-1} \begin{Bmatrix} r-1 \\ i \end{Bmatrix} U_{r-i}^{r-1} \overset{r}{\equiv} \sum_{i=1}^{r-1} \begin{Bmatrix} r-1 \\ i \end{Bmatrix} \overset{r}{\equiv} 0.$$

But, by Theorem 8(iii), $U_{\binom{r}{2}} \overset{r}{\equiv} U_r U_{\frac{r-1}{2}}$ and by Theorem 8(iv), $U_r \overset{r}{\equiv} \pm 1$. Hence $U_{\frac{r-1}{2}} \overset{r}{\equiv} 0$, which is a contradiction.

Clearly, the result holds if $r \nmid U_{r-1}$. Thus we can assume that $r \mid U_{r-1}$. Since $r \mid U_{r-1}$, by Theorem 8(iv), $\left(\frac{\Delta}{r}\right) = 1$ and so $U_r \overset{r}{\equiv} 1$. Hence for all integers n,

$$U_{n+r-1} = -qU_{n-1}U_{r-1} + U_n U_r \overset{r}{\equiv} U_n,$$

from which we get

$$U_{\frac{r-1}{2}} = U_{r-1-\frac{r-1}{2}} \overset{r}{\equiv} U_{-\frac{r-1}{2}} = -\frac{U_{\frac{r-1}{2}}}{q^{\frac{r-1}{2}}} = -U_{\frac{r-1}{2}}.$$

Therefore $r \mid U_{\frac{r-1}{2}}$ and the proof is complete. □

5. Open Problems

We strongly believe that the following conjecture is true.

Conjecture. Let $\{U_n\}$ be a generalized Fibonacci sequence with $\sum_{i=1}^{m} \begin{Bmatrix} m \\ i \end{Bmatrix} = 0$, for some m. Then

$$\sum_{i=1}^{m+n} \begin{Bmatrix} m+n \\ i \end{Bmatrix} U_{x_1 - n_1 i} \cdots U_{x_k - n_k i} = 0,$$

where $n \geq 0$, $n_1 + \cdots + n_k = n$ and $x_1, \ldots, x_k$ are arbitrary integers.

Note that, if the above conjecture is true for $n_1 = \cdots = n_k = 1$, then by applying a similar technique as in the proof of Theorem 3, it remains true for arbitrary $n_1, \ldots, n_k$.

References

[1] R.A. Dunlap, *The Golden Ratio and Fibonacci Numbers*, World Scientific Press, 1997.

[2] M. Farrokhi D.G., Some remarks on the equation $F_n = kF_m$ in Fibonacci numbers, *J. Integer Seq.* **10** (2007), no. 5 , Article 07.5.7, 9 pp.

[3] M. Farrokhi D.G., An identity generator: basic commutators, *Electron. J. Combin.* **15** (2008), no. 1, Note 15, 6 pp.

[4] T. Mansour, Generalizations of some identities involving the Fibonacci numbers, *Fibonacci Quart.* **43** (2005), 307–315.

[5] T. Mansour, Squaring the terms of an ℓ-th order linear recurrence, *Australas. J. Combin.* **31** (2005), 15–20.

[6] S. Vajda, *Fibonacci and Lucas numbers, and the Golden Section: Theory and Applications*, Halsted Press, 1989.

[7] W. Zhang, Some identities involving Fibonacci numbers, *Fibonacci Quart.* **35** (1997), 225-229.

[8] F. Zhao and T. Wang, Generalizations of some identities involving the Fibonacci numbers, *Fibonacci Quart.* **39** (2001), 165-167.