LINEAR ALGEBRA AND ITS APPLICATIONS

ELSEVIER Linear Algebra and its Applications 306 (2000) 155–163
www.elsevier.com/locate/laa

Spectral clustering properties of block multilevel Hankel matrices

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Received 16 September 1998; accepted 1 November 1999

Submitted by G. Heinig

Abstract

By means of recent results concerning spectral distributions of Toeplitz matrices, we show that the singular values of a sequence of block p-level Hankel matrices $H_n(\mu)$, generated by a p-variate, matrix-valued measure μ whose singular part is finitely supported, are always clustered at zero, thus extending a result known when $p = 1$ and μ is real valued and Lipschitz continuous. The theorems hold for both eigenvalues and singular values; in the case of singular values, we allow the involved matrices to be rectangular. © 2000 Elsevier Science Inc. All rights reserved.

Keywords: Hankel matrices; Asymptotic spectral distribution

1. Introduction

The theory of the asymptotic spectral distribution of a sequence of matrices dates back to the works of Szegő (see [6] and the references therein), who first solved the problem of showing the existence and computing the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} F(\lambda_j(T_n)), \tag{1}$$

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where $T_n = T_n(f)$ is the $n \times n$ Toeplitz matrix generated by the Fourier coefficients of a bounded real valued function *f*, the λ_js are its eigenvalues and F is a function continuous on the compact interval $[\inf f, \sup f]$; indeed, Szegö showed that the above limit exists and is equal to the integral

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F(f(x))\,\mathrm{d}x.$$

This result is now a classical one in the theory of Toeplitz matrices; nevertheless, it has undergone many extensions and generalizations in recent years. For example, the book of Böttcher and Silbermann [3] presents a systematic extension of the theory of Toeplitz matrices and operators to operators with matrix-valued symbols, also dealing with Szegö-like limits and other asymptotic results; Tyrtyshnikov [11] extended the Szegö formula to the case of multilevel Toeplitz matrices, that is, $f = f(x_1, \dots, x_p)$ depends on p variables and the matrices $\{T_n(f)\}$, where $n = (n_1, \dots, n_p)$ is now a multiindex, have a p-level Toeplitz structure; moreover, the boundedness assumption on *f* was dropped, and *f* was only assumed to be square integrable over the cube Q^p (throughout, we denote by Q the interval $(-\pi, \pi)$). In the nonhermitian case, an analogous result was proved concerning singular values. In the recent paper [12], Tyrtyshnikov and Zamarashkin proved that the results from [11] still hold when f is only supposed to be integrable over Q^p; finally, such results are further extended in [10] to the case where *f* is simultaneously *p*-variate and matrix-valued, thus embracing as particular cases all the previously known results on the asymptotic spectral distribution of Toeplitz-related matrices.

In this paper, we prove the existence of limits analogous to (1) for multilevel block Hankel matrices generated by the Fourier coefficients of a matrix-valued multivariate measure whose singular part is a finite sum of point masses. For positive natural numbers p, h, k, let μ be a matrix-valued measure, defined on Q^p, with values in $\mathbb{C}^{h\times k}$; throughout, the symbol n always denotes the multiindex $n = (n_1, \dots, n_p)$, and $\pi(n)$ denotes the product $n_1 \cdots n_p$. The p-level block Hankel matrix $H_n(\mu)$ generated by μ has order $h\,\pi(n) \times k\,\pi(n)$ and is defined by

$$H_n(\mu) = \sum_{j_1=1}^{2n_1-1} \cdots \sum_{j_p=1}^{2n_p-1} K_{n_1}^{(j_1)} \otimes \cdots \otimes K_{n_p}^{(j_p)} \otimes a_{j_1,\dots,j_p}(\mu). \tag{2}$$

In the above equality, $K_m^{(l)}$ denotes the matrix of order m whose (i, j) entry equals 1 if $j + i = l + 1$ and equals zero otherwise; the matrices $K_m^{(l)}$, $l = 1, \dots, 2m - 1$, are the natural basis of the linear space of $m \times m$ Hankel matrices, and the tensor notation emphasizes the p-level Hankel structure of $H_n(\mu)$; finally, the innermost blocks $a_{j_1,\dots,j_p}(\mu)$ are $h \times k$ matrices, and they are the Fourier coefficients of μ,

$$a_{j_1,\dots,j_p}(\mu) = \frac{1}{(2\pi)^p}\int_{Q^p} \mathrm{e}^{-\mathrm{i}(j_1x_1+\cdots+j_px_p)}\,\mathrm{d}\mu(x) \in \mathbb{C}^{h\times k},$$

where $x = (x_1, \dots, x_p)$ ranges over Q^p.

The above definitions of $H_n(\mu)$ and $a_{j_1,\dots,j_p}(\mu)$ generalize the more usual definitions of Fourier coefficient $a_{j_1,\dots,j_p}(f)$ of an integrable function f,

$$a_{j_1,\dots,j_p}(f) = \frac{1}{(2\pi)^p}\int_{Q^p} f(x)\,\mathrm{e}^{-\mathrm{i}(j_1x_1+\cdots+j_px_p)}\,\mathrm{d}x, \tag{3}$$

and of the Hankel matrix $H_n(f)$ generated by it. In fact, when μ is univariate, scalar valued (that is, when $h = k = p = 1$) and absolutely continuous, the above defined $H_n(\mu)$ is just the Hankel matrix $H_n(f) \equiv \{a_{i+j-1}\}_{i,j=1}^n$ of order n generated by the function f such that $f(x)\,\mathrm{d}x = \mathrm{d}\mu(x)$.

Spectral properties of Hankel matrices where pioneered by Widom [13], as finite counterparts of certain integral operators. Boundedness and compactness of Hankel operators in ℓ^2 spaces were studied by Nehari and Hartman, see [8]. Moreover, eigenvalues of finite Hankel matrices play an important role in best rational approximation theory, in particular, in the Carathéodory–Fejér and Nehari problems. We refer to the book [9] for a recent account on Hankel matrices and operators. More recent results connecting spectral properties of Hankel matrices to problems in approximation theory can be found in [4,7].

In this paper, we consider unbounded sequences of Hankel matrices. Our results concern the asymptotic spectral distribution of $H_n(\mu)$ as n tends to infinity, and that of $A_n + H_n(\mu)$ where A_n is any sequence of matrices which has some given spectral distribution. We remark that the notation "$\lim_{n\to\infty}$" stands for "$\lim_{\min\{n_i\}\to\infty}$", that is, not only the size of $H_n(\mu)$, but also the size of each inner Hankel level of $H_n(\mu)$ must diverge.

The following theorem characterizes the spectral distribution of block multilevel Hankel matrices, and it states that their spectra are always clustered at zero. It extends a previous result in [5], which covered the particular case where $f(x) = \mathrm{d}\mu(x)/\,\mathrm{d}x$ is univariate, scalar-valued and bounded.

Theorem 1. *If $\mu(x)$ is a matrix-valued measure in Q^p, with values in $\mathbb{C}^{h\times k}$, whose singular part is finitely supported, then for any function F, uniformly continuous and bounded over $\mathbb{R}$,*

$$\lim_{n\to\infty} \frac{1}{(h\wedge k)\,\pi(n)} \sum_{j=1}^{(h\wedge k)\pi(n)} F(\sigma_j(H_n(\mu))) = F(0), \tag{4}$$

where $h\wedge k = \min\{h,k\}$ and $\sigma_j(H_n(\mu))$ denotes the jth singular value of $H_n(\mu)$.

Moreover, if the involved matrices are hermitian, then a similar limit holds for eigenvalues as well.

In other words, the asymptotic spectral measure of the matrices $H_n(\mu)$ is the Heavyside step function with the jump at zero, or, according to the definition of a *cluster* found in [11], the set $\{0\}$ is a cluster for the singular values of $H_n(\mu)$. Moreover, adding a sequence of such Hankel matrices to a sequence with a given spectral distribution does not affect the spectral distribution itself. As a particular case, we obtain a result on spectral distributions of Toeplitz-plus-Hankel matrices, which generalizes the results from [5] concerning the one-level scalar case.

After introducing some definitions and preliminary results in Section 2, we give the proof of Theorem 1 and other related results in Section 3.

2. Notations and preliminary results

If $A \in \mathbb{C}^{h\times k}$, we denote by $\sigma_j(A)$ the jth singular value of A, and if A is hermitian, we denote by $\lambda_j(A)$ the jth eigenvalue of A (both sets are arranged in nondecreasing order, counting multiplicities). Throughout, we endow $\mathbb{C}^{h\times k}$ with the so called *trace norm*,

$$\|A\|_{\mathrm{tr}} = \sum_{j=1}^{h\wedge k} \sigma_j(A),$$

where $h \wedge k = \min\{h, k\}$. For the basic properties of the trace norm, we refer the reader to [1].

We consider functions belonging to the Banach space $L_1(Q^p, \mathbb{C}^{h\times k})$ of all matrix-valued functions which are integrable over Q^p. The L_1-norm is that induced by the trace norm, that is,

$$f \in L_1(Q^p, \mathbb{C}^{h\times k}) \iff \|f\|_{L_1} = \frac{1}{(2\pi)^p} \int_{Q^p} \|f(x)\|_{\mathrm{tr}}\, \mathrm{d}x < +\infty.$$

It is clear that any other matrix norm on $\mathbb{C}^{h\times k}$ would lead to the same space of functions; nevertheless, the choice of the trace norm will turn out to be natural in our setting.

Given $g \in L_1(Q^p, \mathbb{C}^{h\times k})$, the p-level block Toeplitz matrix $T_n(g)$ generated by g has order $h\,\pi(n) \times k\,\pi(n)$, and in analogy with (2), is defined by

$$T_n(g) = \sum_{j_1=-n_1}^{n_1} \cdots \sum_{j_p=-n_p}^{n_p} J_{n_1}^{(j_1)} \otimes \cdots \otimes J_{n_p}^{(j_p)} \otimes a_{j_1,\ldots,j_p}(g), \tag{5}$$

where $J_m^{(l)}$ denotes the matrix of order m whose (i, j) entry equals 1 if $j - i = l$ and equals zero otherwise; as for Hankel matrices, the innermost blocks $a_{j_1,\ldots,j_p}(g)$ are $h \times k$ matrices, namely the Fourier coefficients of g, defined as in (3).

The following result is a particular case of Mirsky's theorem, and will be used in Section 3; a proof and more details can be found in the book of Bhatia [1, pp. 100,101].

Lemma 2. *For any two matrices $A, B \in \mathbb{C}^{h\times k}$, we have*

$$\sum_{j=1}^{h\wedge k} \left|\sigma_j(A) - \sigma_j(B)\right| \leqslant \|A - B\|_{\mathrm{tr}}. \tag{6}$$

If, moreover, A and B are hermitian, then also

$$\sum_{j=1}^{k} \left|\lambda_j(A) - \lambda_j(B)\right| \leqslant \|A - B\|_{\mathrm{tr}}. \tag{7}$$

Finally, let $\mathscr{O}$ denote the set of all sequences $\{K_n\}$ of $h\pi(n) \times k\pi(n)$ matrices such that

$$\lim_{n\to\infty} \frac{1}{\pi(n)} \|K_n\|_{\mathrm{tr}} = 0. \tag{8}$$

The class $\mathscr{O}$ plays an important role in the study of asymptotic spectral properties of sequences of matrices, see e.g. [3, Sections 5.6 and 5.7]. Its relevance is clarified by the fact that the term-by-term sum of a sequence in $\mathscr{O}$ to a sequence of matrices leaves unchanged the asymptotic spectral distribution of the latter:

Theorem 3. *Suppose that $\{K_n\}$ is a sequence in $\mathscr{O}$ and $\{A_n\}$ is a sequence of matrices such that A_n and K_n are $h\,\pi(n) \times k\,\pi(n)$ and such that the limit*

$$\lim_{n\to\infty} \frac{1}{(h \wedge k)\,\pi(n)} \sum_{j=1}^{(h\wedge k)\pi(n)} F(\sigma_j(A_n)) \tag{9}$$

exists for all F uniformly continuous and bounded over $\mathbb{R}$. Then the singular values of $A_n + K_n$ are distributed as those of A_n, that is, the limit

$$\lim_{n\to\infty} \frac{1}{(h \wedge k)\,\pi(n)} \sum_{j=1}^{(h\wedge k)\pi(n)} F(\sigma_j(A_n + K_n)) \tag{10}$$

exists and is equal to the limit (9).

Moreover, if all the involved matrices are hermitian, then the same results hold for eigenvalues as well.

Proof. For any multiindex $n = (n_1, \ldots, n_p)$ we let

$$D_n = \frac{1}{\pi(n)} \left| \sum_{j=1}^{(h\wedge k)\pi(n)} F(\sigma_j(A_n + K_n)) - \sum_{j=1}^{(h\wedge k)\pi(n)} F(\sigma_j(A_n)) \right|.$$

In order to prove that (9) equals (10) it suffices to show that $\lim_{n\to\infty} D_n = 0$. Suppose first that F is smooth and $\|F'\|_\infty = \sup|F'(x)| < +\infty$. Using the mean value theorem and Lemma 2 we obtain

$$D_n \leqslant \frac{\|F'\|_\infty}{\pi(n)} \sum_{j=1}^{(h\wedge k)\pi(n)} |\sigma_j(A_n + K_n) - \sigma_j(A_n)| \leqslant \frac{\|F'\|_\infty}{\pi(n)} \|K_n\|_{\mathrm{tr}}.$$

Taking the limit for $n \to \infty$, the proof is completed. The general case can be managed by a standard approximation argument: If F is uniformly continuous and bounded over $\mathbb{R}$, let $\{F_m\}$ be a sequence of $C^1(\mathbb{R})$ functions such that $\|F - F_m\|_\infty \leqslant \varepsilon$ for all ε and sufficiently large m. Then we have

$$
\begin{aligned}
D_n \leqslant \frac{1}{\pi(n)} \Bigg(& \|F_m'\|_\infty \|K_n\|_{\mathrm{tr}} + \sum_{j=1}^{(h\wedge k)\pi(n)} |F(\sigma_j(K_n)) - F_m(\sigma_j(K_n))| \\
& + \sum_{j=1}^{(h\wedge k)\pi(n)} |F(\sigma_j(A_n+K_n)) - F_m(\sigma_j(A_n+K_n))| \Bigg) \\
\leqslant & \frac{\|F_m'\|_\infty}{\pi(n)} \|K_n\|_{\mathrm{tr}} + 2(h\wedge k)\varepsilon.
\end{aligned}
$$

Consequently, $\limsup_{n\to\infty} D_n \leqslant 2(h \wedge k)\varepsilon$, and since $\varepsilon > 0$ can be chosen arbitralily, we arrive at the assertion.

Finally, if all the matrices are hermitian, in order to prove the statements concerning singular values, it suffices to repeat step by step the above arguments replacing σ_j with λ_j and using (7) in place of (6). □

3. Main results

Consider the following inequality concerning block multilevel Toeplitz matrices:

Lemma 4. *If $g \in L_1(Q^p, \mathbb{C}^{h\times k})$ then, for any multiindex n,*

$$
\frac{1}{\pi(n)} \|T_n(g)\|_{\mathrm{tr}} \leqslant 2\|g\|_{L_1}. \tag{11}
$$

For a proof, see [10, Lemma 3.1], where the estimate is proved when $h = k$; the general case can be easily obtained by adding a suitable number of dummy null rows (if $h < k$) or columns (if $h > k$), which does not affect the norms.

Here we prove an analogous inequality for $H_n(f)$; the idea is that of passing from Hankel to Toeplitz structure by a suitable permutation of rows.

Lemma 5. *If $f \in L_1(Q^p, \mathbb{C}^{h\times k})$ then, for any multiindex n,*

$$
\frac{1}{\pi(n)} \|H_n(f)\|_{\mathrm{tr}} \leqslant 2\|f\|_{L_1}. \tag{12}
$$

Proof. We observe that the matrix $K_m^{(m)}$ is the "reverse" permutation matrix of order m; to point it out, we denote it by R_m. Moreover, let $I_m = J_m^{(0)}$ be the identity matrix of order m. It is immediate to see that

$$
R_m\, K_m^{(h)} = J_m^{(h-m)}, \quad h = 1, 2, \ldots, 2m-1, \tag{13}
$$

that is, R_m turns a Hankel matrix into a Toeplitz. This holds also in the multilevel case; indeed, observing that $R_{\pi(n)} = R_{n_1} \otimes \cdots \otimes R_{n_p}$, we obtain from (2), (5) and (13),

$$
\begin{aligned}
&\left(R_{\pi(n)} \otimes I_h\right) H_n(f)\\
&\quad= \sum_{j_1=1}^{2n_1-1} \cdots \sum_{j_p=1}^{2n_p-1} \left(R_{n_1}\, K_{n_1}^{(j_1)}\right) \otimes \cdots \otimes \left(R_{n_p}\, K_{n_p}^{(j_p)}\right) \otimes a_{j_1,\ldots,j_p}(f)\\
&\quad= \sum_{j_1=1}^{2n_1-1} \cdots \sum_{j_p=1}^{2n_p-1} J_{n_1}^{(j_1-n_1)} \otimes \cdots \otimes J_{n_p}^{(j_p-n_p)} \otimes a_{j_1,\ldots,j_p}(f)\\
&\quad= \sum_{j_1=-n_1}^{n_1} \cdots \sum_{j_p=-n_p}^{n_p} J_{n_1}^{(j_1)} \otimes \cdots \otimes J_{n_p}^{(j_p)} \otimes a_{j_1+n_1,\ldots,j_p+n_p}(f)\\
&\quad= T_n(g),
\end{aligned}
$$

where $g(x_1, \ldots, x_p) = \mathrm{e}^{\mathrm{i}(n_1x_1+\cdots+n_px_p)} f(x_1, \ldots, x_p)$. The trace norm is unitarily invariant (see [1]). Therefore we obtain $\|H_n(f)\|_{\mathrm{tr}} = \|T_n(g)\|_{\mathrm{tr}}$. Since the singular values of $f(x)$ coincide with those of $g(x)$, we have $\|g\|_{L_1} = \|f\|_{L_1}$, and from (11), we obtain (12). □

Theorem 6. *If $f \in L_1(Q^p, \mathbb{C}^{h\times k})$, then the sequence $\{H_n(f)\}$ belongs to $\mathscr{O}$.*

Proof. When f is a trigonometric polynomial,

$$
f(x_1, \ldots, x_p) = \sum_{j_1=-d}^{d} \cdots \sum_{j_p=-d}^{d} a_{j_1,\ldots,j_p} \mathrm{e}^{\mathrm{i}(j_1x_1+\cdots+j_px_p)}, \tag{14}
$$

by removing null rows and columns (which does not affect the trace norm), we obtain

$$
\|H_n(f)\|_{\mathrm{tr}} = \| \sum_{j_1=1}^{d} \cdots \sum_{j_p=1}^{d} K_d^{(j_1)} \otimes \cdots \otimes K_d^{(j_p)} \otimes a_{j_1,\ldots,j_p}(f)\|_{\mathrm{tr}},
$$

and hence $\|H_n(f)\|_{\mathrm{tr}}$ does not depend on n. Therefore, equality (8) is trivially fulfilled with $K_n = H_n(f)$.

When $f \in L_1(Q^p, \mathbb{C}^{h\times k})$, let $\{f_m\}$ be a sequence of functions of the kind (14) (where d may depend on m) converging to f in the L_1-norm. For any given m, from the triangular inequality and Lemma 5 it follows

$$
\begin{aligned}
\frac{1}{\pi(n)}\|H_n(f)\|_{\mathrm{tr}} &\leqslant \frac{1}{\pi(n)}\|H_n(f_m)\|_{\mathrm{tr}} + \frac{1}{\pi(n)}\|H_n(f - f_m)\|_{\mathrm{tr}}\\
&\leqslant \frac{1}{\pi(n)}\|H_n(f_m)\|_{\mathrm{tr}} + 2\|f - f_m\|_{L_1}.
\end{aligned}
$$

Taking the limit as $n \to \infty$ yields

$$
\limsup_{n\to\infty} \frac{1}{\pi(n)}\|H_n(f)\|_{\mathrm{tr}} \leqslant 2\|f - f_m\|_{L_1}
$$

for any fixed m. Finally, taking the limit as $m \to \infty$ we complete the proof. □

As an immediate consequence we have that, if $g \in L_1(Q^p, \mathbb{C}^{h\times k})$, then

$$\lim_{n\to\infty} \frac{1}{\pi(n)} \sum_{j=1}^{k\,\pi(n)} F(\sigma_j(T_n(g) + H_n(f))) = \frac{1}{(2\pi)^p} \int_{Q^p} \sum_{j=1}^{k} F(\sigma_j(g(x)))\,\mathrm{d}x. \tag{15}$$

The proof of (15) is but a particular case of Theorem 3, when A_n equals $T_n(g)$; the fact that the spectra of $T_n(g)$ are distributed as the right-hand side of (15) was proved in [10]. Note that Eq. (15) was shown in [5] in the one-level scalar case under the additional hypothesis that both f and g are bounded.

Proof of Theorem 1. By hypothesis, we have $\mu = \mu_{\mathrm{a}} + \mu_{\mathrm{s}}$, where μ_{a} is absolutely continuous and μ_{s} is finitely supported. If μ is scalar valued and μ_{s} consists of just a unit mass at $\hat{x} = (\hat{x}_1, \dots, \hat{x}_p) \in Q^p$, then, for any multiindices $i = (i_1, \dots, i_p)$ and $j = (j_1, \dots, j_p)$ with $1 \leqslant i_k, j_k \leqslant n_k$ for $k = 1, \dots, p$, the (i, j) entry of $H_n(\mu_{\mathrm{s}})$ is

$$\begin{aligned} a_{i_1+j_1-1,\dots,i_p+j_p-1}(\mu_{\mathrm{s}}) &= \frac{1}{(2\pi)^p} \mathrm{e}^{-\mathrm{i}((i_1+j_1-1)\hat{x}_1+\cdots+(i_p+j_p-1)\hat{x}_p)} \\ &= c \prod_{k=1}^{p} \mathrm{e}^{-\mathrm{i} i_k \hat{x}_k} \prod_{k=1}^{p} \mathrm{e}^{-\mathrm{i} j_k \hat{x}_k} \end{aligned}$$

where $c = \mathrm{e}^{\mathrm{i}(\hat{x}_1+\cdots+\hat{x}_p)}/(2\pi)^p$. Hence the rank of $H_n(\mu_{\mathrm{s}})$ is 1. Analogously, when μ_{s} is supported in l points and $\mu_{\mathrm{s}}(x) \in \mathbb{C}^{h\times k}$, we see that the rank of $H_n(\mu_{\mathrm{s}})$ is not greater than $l(h \wedge k)$. Consequently, if $\mu_{\mathrm{a}} \equiv 0$, equality (4) holds since only a finite number (independently of n) of singular values and eigenvalues are different from zero and F is bounded. In the general case, the assertion follows if we define the sequences $A_n = H_n(\mu_{\mathrm{s}})$ and $K_n = H_n(f)$, where f is the density associated to μ_{a}, and use Theorems 3 and 6 in the decomposition $H_n(\mu) = H_n(\mu_{\mathrm{a}}) + H_n(\mu_{\mathrm{s}})$. $\square$

Acknowledgement

We thank A. Böttcher for offering detailed comments and friendly suggestions on this paper, and for providing us with a draft of Ref. [3].

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