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SOME THOUGHTS ON ROOK POLYNOMIALS ON SQUARE CHESSBOARDS

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1. INTRODUCTION

A rook polynomial is a polynomial whose x^k coefficient is the number of ways k rooks can be placed on the squares of an arbitrarily shaped chessboard so that no rooks share the same rows or columns. The k rooks are called *non-taking*. Rook polynomials pattern combinatorial situations, especially those involving restricted permutations. The conventional square board used in the game, chess, is but one configuration.

When all possible board shapes are considered, a wide variety of rook polynomials result. They can be obtained through interesting algebraic, semi-algebraic, or algorithmic operations on the (row,column) matrix designation numbers of the individual squares of the board [1], [3], [7]. Since here we use square boards, our task is simplified because square boards have easily derived, unique rook polynomials.

ROOK POLYNOMIALS OF SQUARE BOARDS

Consider, for example, finding the x^2 coefficient of the rook polynomial of a 4×4 chessboard. The value is the number of ways 2 non-taking rooks can be placed on the board, one of which is shown below.

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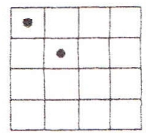

In one row, a rook can be in any of 4 positions and in a second row can be in any of 3 positions. There are $P(4,2)$ ways to place 2 non-taking rooks on any set of two rows. But, there are $C(4,2)$ sets of two rows, so the desired coefficient is $C(4,2)P(4,2) = 72$. Similar reasoning for the general coefficient of x^k in the rook polynomial for an $r \times r$ board yields $C(r,k)P(r,k)$, and a general expression for the rook polynomial becomes $\sum_{k=0}^{r} C(r,k)P(r,k)x^k$. The rook polynomial of our example is

$$1 + 16x + 72x^2 + 96x^3 + 24x^4 \tag{1}$$

THE SQUARE BOARD ROOK POLYNOMIAL TRIANGLE (SBRP)

Note that the numerical coefficients of (1) form the 4th row of Pascal's triangle with each k^{th} entry multiplied by $P(4,k)$. It is interesting to explore some of the extensions to a SBRP triangle, an example of which appears below. The rows start with $r = 0$ and continue through $r = 5$. Positions start at $k = 0$ on the left and continue to the right through $k = r$ for that row. The general term is $C(rk)P(rk) = k![C(r,k)]^2 = \frac{1}{k!}[P(r,k)]^2$ as compared with $C(r,k)$ for Pascal's triangle. An interesting, if minor, feature of the r^{th} row is that the 0th coefficient is always 1, the 1st is r^2, and the r^{th} is $r!$.

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 4 & & 2 & & & \\
 & & 1 & & 9 & & 18 & & 6 & & \\
 & 1 & & 16 & & 72 & & 96 & & 24 & \\
1 & & 25 & & 200 & & 600 & & 600 & & 120
\end{array} \tag{2}$$

ROW GENERATION

Pascal's triangle seems to express its universality by being central to several classes of integer triangles [4], [5]. The SBRP triangle is a member of one such class in that each entry of a next row is a function of the sum two entries immediately above it in the preceding row (as in Pascal's triangle).

Multiplication of the sum of two adjacent r^{th} row entries by the factor, $f_1(r,k)$, yields the immediately below $(r+1)^{th}$ row entry and factor as

$$[C(r,k)P(r,k) + C(r,k+1)P(r,k+1)]f_1(r,k) = C(r+1,k+1)P(r+1,k+1) \tag{3}$$

$$f_1(r,k) = \frac{(r+1)^2}{(k+1)+(r-k)^2} \tag{4}$$

For $r = 4, k = 2$, (3) and (4) check out as $(72+96)\frac{25}{7} = 600$ on triangle (2). We can similarly determine the $(r-1)^{th}$ row entry of (2) from the sum of the two immediately below r^{th} row entries. The $f_2(r,k)$ becomes

$$f_2(r,k) = \frac{(r-k)^2(k+1)}{r^2[(k+1)+(r-k)^2]}. \tag{5}$$

(Although rarely done in Pascal triangles, the previous $(r-1)^{th}$ row entry from the sum of the two immediately below r^{th} row entries is easily found to be $f_3(r,k) = \frac{(r-k)(k+1)}{(r+1)r}$.)

PROPERTIES OF RIGHT-DESCENDING DIAGONALS

Integers of the right-descending diagonals of (2) appear in Sloane and Plouffe's [8] encyclopedia of sequences as absolute value coefficients of Laguerre polynomials. To be consistent with conventional terminology, we will express the sequence terms in index n starting with $n-1$, by first finding the diagonal terms as functions of the row number, s, of the left terminus and the row number, r, of a sequence term. It appears that the first term of a Laguerre sequence is 1 and the second term is always a square. For agreement with our computations and terminology, the Laguerre sequence which has $(s+1)^2$ as its second term is the sequence associated with our parameter s. For example, the diagonal of (2) which reads 1, 9, 72, $\cdots$ has $s = 2$ since $(2+1)^2 = 9$. For the right-descending diagonals, r increases by one for the next term. Recall also that we have to take into account that we start indexing r with 0, not 1. By noting this, we can derive equivalent forms of a general diagonal term as

$$C(r, r-s)P(r, r-s) = [C(r, r-s)]^2(r-s)! = \frac{(r)(r-1)\cdots(r-(s-1))}{(s!)^2}r! \tag{6}$$

The expresssions of (6) have no value for $r < s$, i.e., until $r = s$. However, if r is replaced by $r = n+s-1$, we have the sequence values in the more useful and conventional index n starting with 1 for $n = 1$. With the substitution for r, (6) has the closed form

$$\frac{((n+s-1)!)^2}{(n-1)!(s!)^2} \tag{7}$$

The 0^{th} right-descending diagonal of (2) is 1, 1, 2, 6, 24, $\cdots$, $(n-1)!, \cdots$. Sloane and Plouffe [8] list this sequence as M1675 but use $n!$ as the general term because they start the sequence with the value 1 for $n = 0$. The 1st ($s = 1$) right-descending diagonal of (2) is 1, 4, 18, 96,

$600, \cdots, n \cdot n!, \cdots$ and appears as M3545 in [8]. The 2nd ($s = 2$) right-descending diagonal of (2) is 1, 9, 72, 600, 5400, $\cdots \frac{((n+1)!)^2}{(n-1)!(2!)^2}$. Reference [8] lists this sequence as M4649. At this point, reference [8] includes a closed generating function in the form of a ratio of two polynomials in x for M4649. M4649 is listed as an *exponential* generating function. This and higher s-valued right-descending diagonals are identified as Laguerre polynomial coefficient sequences. However, only M4649 includes a closed generating function. We welcome this chance to contribute missing exponential generating functions in succeeding sections and to develop a general generating function applicable to all higher Laguerre polynomials found in, and suggested by, [8].

GENERATING FUNCTIONS FOR COEFFICIENTS OF LAGUERRE POLYNOMIALS

A quotient of polynomials serves as either an *ordinary* closed generating function or an *exponential* closed generating function depending on how the quotient is expressed. In the ordinary function, the general term is $a_k x^k$ while in the exponential function the same general term is $k!a_k\left(\frac{x^k}{k!}\right) = b_k\left(\frac{x^k}{k!}\right)$. While it is easy to get a_k's by direct division, it is not possible to get b_k's directly this way. (Both Liu [7, pp. 33-34] and Brualdi [2, pp. 237-243] have interesting discussions on this apparent impasse.) A fruitful approach to the inverse problem of finding a closed function from either ordinary or exponential coefficients first involves conversion (if needed) to ordinary coefficients. Finding the closed form generating function is then often a matter of judgment, experience, and exceptionally good luck. Reference [8] devotes an interesting chapter to various aspects of this problem.

The simplest solution attempt is indicated when we have a linear sequence of constant coefficients, and we are reasonably sure that a closed form exists as a ratio of two polynomials. If this is indeed the case, the sequence is index-invariant and a set of successive forward finite differences from the sequence leads to a closed form ratio of polynomials. We use the systematic z-Transform [6] approach which served so well in a similar situation in [4].

As stated earlier, the generating function listed in [8] is an exponential generating function. It seems reasonable to assume that generating functions from higher order right-descending diagonals are also exponential. To avoid difficulties with exponential generating functions, we divide the coefficients of known exponential generating functions by $k!$ for the appropriate index k and treat them as coefficients of ordinary generating functions for which we might find a closed form ratio of polynomials.

To review how the z-Transform approach [4], [6] works and to check our calculations, we examine Sloane and Plouffe's M4649. For completeness, we will then obtain the two generating functions missing from [8] and develop a general expression for the closed generating function for all right-descending diagonals of (2).

When working with z-Transforms [6], it is necessary to include a_0 and any other lowest-indexed initial conditions. Since here $a_0 = 0$, computations are simplified. We start with a triangular table of forward differences of the coefficients based on a finite number of coefficients. The size of the table needed also depends on the order of recursion, which in this case is 4. The first values of M4649 have been divided by the appropriate $n!$ to appear as a_n. The $\Delta^k a_n$'s are k^{th} forward differences where we let $a_n = \Delta^0 a_n$ to complete the table.

n	a_n	$\Delta^1 a_n$	$\Delta^2 a_n$	$\Delta^3 a_n$	$\Delta^4 a_n$
0	0				
		1			
1	1		2.5		
		3.5		1.5	
2	4.5		4		0
		7.5		1.5	
3	12		5.5		0
		13		1.5	
4	25		7		0
		20		1.5	
5	45		8.5		

(8)

In (8), $\Delta^4 a_n = 0, \Delta^3 a_{n+1} - \Delta^3 a_n = 0$. Since $\Delta^2 a_{n+2} - \Delta^2 a_{n+1} = \Delta^3 a_{n+1}$ and $\Delta^2 a_{n+1} - \Delta^2 a_n = \Delta^3 a_n, \Delta^2 a_{n+2} - 2\Delta^2 a_{n+1} + \Delta^2 a_n = 0$. We reduce to $\Delta^1 a_{n+3} - 3\Delta^1 a_{n+2} + 3\Delta^1 a_{n+1} + \Delta^1 a_n = 0$.

Continuing with computing the zero differences leads to the homogeneous difference equation

$$a_{n+4} - 4a_{n+3} + 6a_{n+2} - 4a_{n+1} + a_n = 0 \tag{9}$$

The z-Transform [4], [6] of (9) yields

$$\{z^4 Z(a_n) - z^4 a_0 - z^3 a_1 - 2^2 a_2 - za_3\} - 4\{z^3 Z(a_n) - z^3 a_0 - z^2 a_1 - za_2\} +$$
$$6\{z^2 Z(a_n) - z^2 a_0 - za_1\} - 4\{zZ(a_n) - za_0\} + Z(a_n) = 0 \tag{10}$$

After factoring out $Z(a_n)$, rearranging terms, and substituting for a_0 through a_3 from (8), we have the formula for $Z(a_n)$ and the numerical closed form ordinary generating function in z as

$$Z(a_n) = \frac{z^4 a_0 + z^3(a_1 - 4a_0) + z^2(a_2 - 4a_1 + 6a_0) + z(a_3 - 4a_2 + 6a_1 - 4a_0)}{(z-1)^4}$$
$$= \frac{z^3 + \frac{1}{2}z^2}{(z-1)^4} \tag{11}$$

If the right side of (11) were expanded, the sequence terms of the ordinary generating function would appear as coefficients of powers of $\frac{1}{z}$. By replacing z by $\frac{1}{x}$ in (11), we obtain as (12) the ordinary generating funciton for our a's in x and at the same time the exponential generating function for the terms of the second right-descending diagonal. Recall this verifies M4649 of Sloane and Plouffe [8].

$$\frac{x + \frac{1}{2}x^2}{(1-x)^4} \tag{12}$$

The "4-ness" and relationship to coefficients of the 4th row of Pascal's triangle in (8)-(11), and the fact that the recursion order is $4 = 2s$ for the $s = 2$ diagonal is not a coincidence. All of the right descending diagonals have the same recursion, $2s$. We observe the orderly appearance and arrangement of binomial coefficients from Pascal's triangle and use the inverse z-Transform to get the common generating function in z and x.

$$\frac{z^5 + 2z^4 + \frac{z^3}{3}}{(z-1)^6} \to \frac{x + 2x^2 + \frac{x^3}{3}}{(1-x)^6} \tag{14}$$

By applying the same procedures, we obtain generating functions in z and x for the right-descending diagonals for $s = 4$, 5, and 6.

$$s = 4 : \frac{z^7 + \frac{9z^6}{2} + 3z^5 + \frac{z^4}{4}}{(z-1)^8} \to \frac{x + \frac{9x^2}{2} + 3x^3 + \frac{x^4}{4}}{(1-x)^8} \tag{15}$$

$$s = 5 : \frac{z^9 + 8z^8 + 12z^7 + 4z^6 + \frac{z^5}{5}}{(z-1)^{10}} \to \frac{x + 8x^2 + 12x^3 + 4x^4 + \frac{x^5}{5}}{(1-x)^{10}} \tag{16}$$

$$s = 6 : \frac{z^{11} + \frac{25z^{10}}{2} + \frac{100z^9}{9} + 25x^8 + 5z^7 + \frac{z^6}{6}}{(z-1)^{12}} \to \frac{x + \frac{25x^2}{2} + \frac{100x^3}{3} + 25x^4 + 5x^5 + \frac{x^6}{6}}{(1-x)^{12}} \tag{17}$$

GENERAL GENERATING FUNCTION IN S

Consider the triangle formed by and implied by successive rows of the numerator terms of the right sides of generating functions (14)-(17).

$$\begin{array}{ccccccccccc}
 & & & & & 1x & & & & & \\
 & & & & 1x & & \frac{1}{2}x^2 & & & & \\
 & & & 1x & & 2x^2 & & \frac{1}{3}x^3 & & & \\
 & & 1x & & \frac{9}{2}x^2 & & 3x^3 & & \frac{1}{4}x^4 & & \\
 & 1x & & 8x^2 & & 12x^3 & & 4x^4 & & \frac{1}{5}x^5 & \\
1x & & \frac{25}{2}x^2 & & \frac{100}{3}x^3 & & 25x^4 & & 5x^5 & & \frac{1}{6}x^6
\end{array} \tag{18}$$

A significant fact about (18) is its triangular form whose rows are the successive numerators of the closed (either exponential or ordinary) generating functions like those which we exemplified in (13)-(17). Moreover, should we multiply a rook polynomial by x and divide each coefficient by the factorial of the power of its associated x in the new expression, the result would also be the entries of (18). Another interesting feature of (18) triangle in the development of the extension of (11).

We purposefully chose $s = 2$ as our derivation parameter for demonstrating the z-Transform approach because the derived generating function could then be compared with the only function listed in [8].

The case for $s = 0$ is unique in that the generating function, $-\log(1-x)$, is logarithmic. The terms of its series expansion may be observed as the 0^{th} diagonal of (18), while the adjusted coefficients of a series "exponential" expansion appear as the 0^{th} diagonal of (2).

For $s = 1$, the generating function is $\frac{x}{(1-x)^2}$. The terms of the ordinary series expansion can be seen from the 1st diagonal of (18), while the coefficients of the exponential series expansion appear as the 1st diagonal of (2).

SOME MISSING CLOSED GENERATING FUNCTIONS

The 3rd ($s = 3$) right-descending diagonal of (2) can be extended to yield series coefficients, 1, 16, 200, 2400, 29400 , $\cdots$, $\frac{((n+2)!)^2}{(n-1)!(3!)^2}$, $\cdots$. This sequence is listed in [8] as M5019 with, of course, the Laguerre polynomial connection. The ordinary generating function which shares the same closed form with the exponential generating function of M5019 yields 1, 8, $\frac{100}{3}$, 100, 245, $\frac{1568}{3}$, 1008, 1800, $\cdots$, $\frac{((n+2)!)^2}{(n-1)!(3!)^2 n!}$, $\cdots$. Because $s = 3$, and the recursion order is 6, the difference triangle requires only $a_0 = 0$, and a_1 through a_5 from the above series to find the closed form ordinary generating function. We learned earlier [4] that we can bypass much of the development and substitute the sequence values in

$$z^6 a_0 + z^5(a_1 - 6a_0) + z^4(a_2 - 6a_1 + 15a_0) + z^3(a_3 - 6a_2 + 15a_1 - 20a_0) +$$

$$\frac{z^2(a_4 - 6a_3 + 15a_2 - 20a_1 + 15a_0) + z(a_5 - 6a_4 + 15a_3 - 20a_2 + 15a_1 - 6a_0)}{(z-1)^6} \qquad (13)$$

is that it has a terms of its right-descending diagonals, the terms of series ordinary generating functions. In terms of s, (7) indicates that the general series exponential generating function for right-descending diagonals and corresponding series ordinary generating functions in n, starting with $n = 1$ are, respectively,

$$\sum_{n=1}^{\infty} \frac{((n+s-1)!)^2}{(n-1)!(s!)^2} x^n, \quad \sum_{n=1}^{\infty} \frac{((n+s-1)!)^2}{(n-1)!(s!)^2} x^n \qquad (19)$$

For $s = 3$, for example, the generating function numerator of (14) occupies the $s = 2$ row in (18). This difference of one is s for row occupation in general. However, the denominator term is $(1-x)^4$ or, in general, $(1-x)^{2s}$. Hence, for a given s, the general generating function for either of the parts of (19), depending on exponential or ordinary interpretation, becomes the desired general generating function.

$$\frac{\sum_{k=0}^{s-1} \frac{((s-1)!)^2}{((s-k-1)!)^2(k!)^2(k+1)} x^{(k+1)}}{(1-x)^{2s}} \tag{20}$$

SUMMARY

We have shown elementary, but interesting, comparisons of Square Board Rook Polynomial (SBRP) triangles with Pascal triangles. We have investigated and extended the sequence properties of the right-descending diagonals of SBRP triangles. In so doing, we have added to and generalized known Laguerre sequence listings.

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