unpublished note

Hankel and Toeplitz Determinants

Steven Finch

March 17, 2014

The most famous Hankel matrix is the Hilbert matrix

$$H_n = \left(\frac{1}{i+j-1}\right)_{1\leq i,j\leq n}$$

which has determinant equal to a ratio of Barnes G-function values:

$$\det(H_n) = \frac{\prod_{k=1}^{n-1}(k!)^4}{\prod_{\ell=1}^{2n-1}\ell!} = \frac{G(n+1)^4}{G(2n+1)} \to 0$$

as $n \to \infty$. More precisely [1],

$$\frac{\det(H_n)}{4^{-n^2}(2\pi)^n n^{-1/4}} \to 2^{1/12}e^{1/4}A^{-3} = 0.6450024485...$$

where A denotes the Glaisher-Kinkelin constant [2]. Such Hankel determinants are important in random matrix theory and applications [3], but we shall forsake all this, giving instead only a few examples [4, 5, 6]. Another interesting fact is that $\det(H_n)$ is always the reciprocal of a positive integer [7].

The Hankel determinant of Euler numbers [8] is, in absolute value,

$$\begin{aligned} |E_{i+j}|_{0\leq i,j\leq n-1} &= \prod_{k=1}^{n-1}(k!)^2 = G(n+1)^2 \\ &\sim \frac{e^{\frac{1}{6}}}{A^2}e^{-\frac{3}{2}n^2}(2\pi)^n n^{n^2-\frac{1}{6}} \end{aligned}$$

as $n \to \infty$. The simplicity of this result contrasts with the following. The Hankel determinant of Bernoulli numbers [9] is, in absolute value,

$$\begin{aligned} |B_{i+j}|_{0\leq i,j\leq n-1} &= \prod_{k=1}^{n-1}\frac{(k!)^6}{(2k)!(2k+1)!} \\ &= \frac{2^{\frac{1}{12}}e^{\frac{1}{4}}}{A^3}4^{-n^2}(2\pi)^n\frac{G(n+1)^4}{G(n+1/2)G(n+3/2)} \\ &\sim \frac{2^{\frac{1}{12}}e^{\frac{5}{12}}}{A^5}4^{-n^2}e^{-\frac{3}{2}n^2}(2\pi)^{2n}n^{n^2-\frac{5}{12}} \end{aligned}$$

as $n \to \infty$. We mention three formulas of Krattenthaler [10]:

$$\left|\frac{B_{2i+2j+2}}{(2i+2j+2)!}\right|_{0\le i,j\le n-1} = 4^{-n^2}\prod_{k=1}^{2n-1}(2k+1)^{-2n+k},$$

$$\left|\frac{B_{2i+2j+4}}{(2i+2j+4)!}\right|_{0\le i,j\le n-1} = 4^{-n^2-n}9^{-n}\prod_{k=1}^{2n-1}(2k+3)^{-2n+k},$$

$$\left|\frac{B_{2i+2j+6}}{(2i+2j+6)!}\right|_{0\le i,j\le n-1} = (n+1)(2n+3)4^{-n^2-2n}\prod_{k=1}^{2n+1}(2k+1)^{-2n-2+k}$$

which are always reciprocals of integers (unlike $|E_{i+j}|$ and $|B_{i+j}|$). The asymptotics of these three sequences remain open.

More difficult are determinants of Riemann zeta function values:

$$a_n^{(0)} = |\zeta(i+j)|_{1\le i,j\le n}, \qquad a_n^{(1)} = |\zeta(i+j+1)|_{1\le i,j\le n}$$

which evidently satisfy

$$a_n^{(0)} \sim C\cdot\left(\frac{2n+1}{e^{3/2}}\right)^{-(n+1/2)^2}, \qquad a_n^{(1)} \sim \frac{e^{9/8}}{\sqrt{6}}C\cdot\left(\frac{2n}{e^{3/2}}\right)^{-n^2+3/4}$$

thanks to numerical experiments by Zagier [11]. No closed-form expression for the constant $C = 0.351466738331...$ is known.

A famous Toeplitz matrix, called the alternating Hilbert matrix in [12], is

$$\tilde{H}_n = \left(\frac{1}{i-j}\right)_{1\le i,j\le n}$$

where we understand the diagonal elements to be 0. Schur [13] proved long ago that the maximum eigenvalue (in modulus) of both H_n and $\tilde{H}_n$ is less than π and approaches π as $n \to \infty$. The determinant is, of course, the product of all eigenvalues. When n is odd, $\det(\hat{H}_n) = 0$. When n is even, a closed-form expression for $\det(\hat{H}_n)$ seems to be unavailable, despite the existence of a combinatorial approach [14]. Note that the "symbol" associated with $\hat{H}_n$ is

$$\sum_{r=1} \frac{e^{ir\theta}}{-r} + \sum_{r=1} \frac{e^{-ir\theta}}{r} = i(\theta-\pi)$$

for $0 < \theta < 2\pi$, hence a theorem due to Grenander & Szegő [15] gives

$$\lim_{\substack{n\to\infty \\ n \text{ even}}} \frac{1}{n}\ln\left(\det(\hat{H}_n)\right) = \frac{1}{2\pi}\int_0^{2\pi} \ln\left[i(\theta-\pi)\right]d\theta = -1+\ln(\pi) = 0.1447298858....$$

A refined estimate shown subsequently in [15], potentially governing the value of

$$\lim_{n\to\infty} \det(\hat{H}_n)\cdot\left(\frac{\pi}{e}\right)^n,$$

has conditions that must be verified.

Consider finally another Toeplitz matrix

$$K_n = \left(\frac{1}{1+|i-j|}\right)_{1\le i,j\le n}$$

for which little is known. The "symbol" here is

$$\sum_{r=0}^{\infty}\frac{e^{ir\theta}}{1+r}+\sum_{r=1}^{\infty}\frac{e^{-ir\theta}}{1+r} = -1-e^{i\theta}\ln\left(1-e^{-i\theta}\right)-e^{-i\theta}\ln\left(1-e^{i\theta}\right)$$

for $0<\theta<2\pi$, hence

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\ln\left(\det(K_n)\right) &= \frac{1}{2\pi}\int_0^{2\pi}\ln\left[-1-e^{i\theta}\ln\left(1-e^{-i\theta}\right)-e^{-i\theta}\ln\left(1-e^{i\theta}\right)\right]d\theta\\ &= -0.3100863233....\end{aligned}$$

An exact formula for this constant is desired; might, at least, the integral be simplified in some way?

0.1. Combinatorial Approach.

Assume that n is even. Let S denote the set of all $(n/2)$-tuples of ordered pairs:

$$(p_k, q_k)_{k=1}^{n/2}$$

of positive integers $p_k < q_k$ satisfying

$$\bigcup_{k=1}^{n/2}\{p_k, q_k\} = \{1,2,\ldots,n\}$$

and $p_1 < p_2 < \ldots < p_{n/2}$. Note that the qs need not be in ascending order. Let us verify a formula in [14]:

$$\det(\hat{H}_n) = \sum_{(p_k,q_k)_{k=1}^{n/2}\in S}\ \prod_{k=1}^{n/2}\frac{1}{(q_k-p_k)^2}$$

for $n = 4$. Three such 2-tuples exist:

$$p_1 = 1 < p_2 = 2 < q_1 = 3 < q_2 = 4,$$

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yielding

$$\frac{1}{(3-1)^2(4-2)^2} + \frac{1}{(4-1)^2(3-2)^2} + \frac{1}{(2-1)^2(4-3)^2} = \frac{169}{144} = \det(\hat{H}_4).$$

The case $\det(\hat{H}_2) = 1$ is trivial; the case $\det(\hat{H}_6) = 6723649/4665600$ will require some effort. We wonder if a simple method for computing the size of S, as a function of n, can be found. An analogous approach for $\det(K_n)$ would also be good to see.

0.2. Acknowledgements. I am grateful to Olivier Lévêque for evaluating $\det(K_n)$ asymptotics and Hartmut Monien for helpful correspondence.

References

[1] Wikipedia contributors, Hilbert matrix, Wikipedia, The Free Encyclopedia, 10 March 2014, http://en.wikipedia.org/wiki/Hilbert_matrix.

[2] S. R. Finch, Glaisher-Kinkelin constant, Mathematical Constants, Cambridge Univ. Press, 2003, pp. 135–145.

[3] E. L. Basor, Y. Chen and H. Widom, Determinants of Hankel matrices, J. Funct. Anal. 179 (2001) 214–234; MR1807258 (2001m:47043).

[4] A. Junod, Hankel determinants and orthogonal polynomials, Expo. Math. 21 (2003) 63–74; MR1955218 (2004a:15009).

[5] C. Krattenthaler, Advanced determinant calculus, Sémin. Lothar. Combin. 42 (1999) B42q; arXiv:math/9902004; MR1701596 (2002i:05013).

[6] C. Krattenthaler, Advanced determinant calculus: a complement, Linear Algebra Appl. 411 (2005) 68–166; arXiv:math/0503507; MR2178686 (2006g:05022).

[7] N. J. A. Sloane, On-Line Encyclopedia of Integer Sequences, A005249 and A067689.

[8] S. R. Finch, Catalan's constant, Mathematical Constants, Cambridge Univ. Press, 2003, pp. 53–59.

[9] S. R. Finch, Apéry's constant, Mathematical Constants, Cambridge Univ. Press, 2003, pp. 40–53.

[10] S. Fukuhara and Y. Yang, Period polynomials and explicit formulas for Hecke operators on $\Gamma_0(2)$, Math. Proc. Cambridge Philos. Soc. 146 (2009) 321–350; arXiv:math/0608372; MR2475970 (2009m:11062).

[11] H. Monien, Hankel determinants of Dirichlet series, arXiv:0901.1883.

[12] N. Kurokawa and H. Ochiai, Spectra of alternating Hilbert operators, Spectral Analysis in Geometry and Number Theory, Proc. 2007 Nagoya conf., ed. M. Kotani, H. Naito and T. Tate, Amer. Math. Soc., 2009, pp. 89–101; arXiv:0709.2675; MR1500140 (2010c:11113).

[13] I. Schur, Bemerkungen zur Theorie der beschränkten Bilinearformen mit unendlich vielen Veränderlichen, J. Reine Angew. Math. 140 (1911) 1–28.

[14] E. Preissmann and O. Lévêque, On generalized weighted Hilbert matrices, Pacific J. Math. 265 (2013) 199–219; arXiv:1303.1085; MR3095119.

[15] U. Grenander and G. Szegő, Toeplitz Forms and Their Applications, Chelsea, 1984, pp. 62–80; MR0890515 (88b:42031).