

UNIVERSAL SCHUBERT POLYNOMIALS

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1. Introduction. The aim of this paper is to introduce some polynomials that specialize to all previously known Schubert polynomials: the classical Schubert polynomials of Lascoux and Schützenberger [LS1], [M], the quantum Schubert polynomials of Fomin, Gelfand, and Postnikov [FGP], and the quantum Schubert polynomials for partial flag varieties of Ciocan-Fontanine [CF2]. There are also double versions of these universal Schubert polynomials that generalize the previously known double Schubert polynomials [L], [M], [KM], [CFF]. They describe degeneracy loci of maps of vector bundles, but in a more general setting than the previously known setting of [F2].

These universal Schubert polynomials possess many but not all algebraic properties of their classical specializations. Their extra structure makes them useful for studying their specializations, as it can be easier to find patterns before variables are specialized.

The main geometric setting to which these polynomials apply is the following. We have maps of vector bundles

$$(1) \quad F_1 \rightarrow F_2 \rightarrow \cdots \rightarrow F_n \rightarrow E_n \rightarrow \cdots \rightarrow E_2 \rightarrow E_1$$

on a variety or scheme X , where each F_i and E_i has rank i . We do *not* assume here that the maps $F_i \rightarrow F_{i+1}$ are injective or that the maps $E_{i+1} \rightarrow E_i$ are surjective, as was the case studied in [F2]. For each w in the symmetric group S_{n+1} , there is a degeneracy locus

$$(2) \quad \Omega_w = \{x \in X \mid \text{rank}(F_q(x) \rightarrow E_p(x)) \leq r_w(p, q) \text{ for all } 1 \leq p, q \leq n\},$$

where $r_w(p, q)$ is the number of $i \leq p$ such that $w(i) \leq q$. Such degeneracy loci are described by the double form $\mathfrak{S}_w(c, d)$ of universal Schubert polynomials, evaluated at the Chern classes of all the bundles involved. Unlike the situation studied in [F2], where these Chern classes were determined by their first Chern classes, in the present general setting one must have more general polynomials to describe such loci. There are similar formulas when some of the bundles in (1) are missing.

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All Schubert polynomials are indexed by permutations w in some symmetric group S_{n+1} . We present these universal Schubert polynomials in two forms. The first (in its single form), denoted $\mathfrak{S}_w(c)$, is a polynomial in variables $c_i(k)$, for $1 \leq i \leq k \leq n$. When $c_i(k)$ is specialized to the i th elementary symmetric polynomial $e_i(x_1, \dots, x_k)$ in variables x_1, \dots, x_k , this polynomial becomes the classical Schubert polynomial, denoted $\mathfrak{S}_w(x)$. When $c_i(k)$ is specialized to the i th quantum elementary symmetric polynomial, in variables $x_1, \dots, x_k, q_1, \dots, q_{k-1}$, $\mathfrak{S}_w(c)$ specializes to the quantum Schubert polynomial \mathfrak{S}_w^q of [FGP]. In our geometric setting, $c_i(k)$ is the i th Chern class of a vector bundle of rank k .

We write the second form of the universal Schubert polynomial, denoted $\mathfrak{S}_w(g)$, as a polynomial in variables $g_i[j]$, for $i \geq 1$ and $j \geq 0$ with $i + j \leq n + 1$. We regard $g_i[j]$ as an indeterminate of degree $j + 1$. This polynomial $\mathfrak{S}_w(g)$ is obtained from $\mathfrak{S}_w(c)$ by replacing each $c_i(k)$ by the coefficient of T^i in the determinant of $A + IT$, where A is the k by k matrix with $g_i[j - i]$ in the (i, j) position for $i \leq j$, and with -1 in positions $(i + 1, i)$ below the diagonal, and zero elsewhere. (See Section 4 for another definition of these polynomials.) One recovers the classical Schubert polynomials $\mathfrak{S}_w(x)$ by setting $g_i[0] = x_i$ and $g_i[j] = 0$ for $j \geq 1$, and one recovers the quantum Schubert polynomials \mathfrak{S}_w^q by setting $g_i[0] = x_i$, $g_i[1] = q_i$, and $g_i[j] = 0$ for $j \geq 2$. In Section 4 we see that other specializations give the polynomials defined in [CF2] for Schubert classes in quantum cohomology rings of partial flag varieties. Since the variables $c_i(k)$ and $g_i[j]$ generate the same polynomials ring (i.e., $\mathbb{Z}[c] = \mathbb{Z}[g]$), the two forms of universal polynomials are equivalent.

From the single polynomials, we construct universal double Schubert polynomials $\mathfrak{S}_w(c, d)$ and $\mathfrak{S}_w(g, h)$, which also specialize to the known cases of double Schubert polynomials. In the geometric setting of (1), the variables $c_i(j)$ become the Chern classes $c_i(E_j)$, and the variables $d_i(j)$ become $c_i(F_j)$.

The universal polynomials are constructed in Section 2. The theorems relating them to degeneracy loci are proven in Section 3. Section 4 contains some determinantal formulas for universal Schubert polynomials and some results and questions about their algebra.

In [F2], following the classical approaches of [BGG] and [D], the degeneracy loci formulas were proven—in a universal setting on a flag bundle—by starting with the locus of top codimension, which is realized as the zero of a section of a vector bundle; then the other loci are constructed inductively by a sequence of \mathbb{P}^1 -bundle correspondences. In the present setting, neither of these methods is available. Indeed, the top double classes $\mathfrak{S}_w(c, d)$ do not factor. Our procedure, roughly speaking, is to find a locus in a flag bundle that maps to a given degeneracy locus Ω_w , but where one has injections and surjections of the bundles, so that one can apply the results of [F2]; then this formula is pushed forward to get a formula for Ω_w .

Ionuț Ciocan-Fontanine initiated this project by asking several years ago for a degeneracy locus formula that would apply when the maps $E_{i+1} \rightarrow E_i$ are not surjective. His work in [CF1], [CF2] was a source for this question, and con-

versations with him have been very useful. Algebraically, the universal Schubert polynomials are natural generalizations of the quantum Schubert polynomials of [FGP] (cf. [KM] and [CFF]), and the inspiration of these sources should be clear. I thank Chandler Fulton and Mel Hochster for advice on computer testing related to this study.

In the preprint version of this paper, we conjectured that the universal versions of these degeneracy loci are Cohen-Macaulay. Claudio Procesi pointed out the paper [ADFK], which proved, in characteristic zero, that the reduced structures on these loci are Cohen-Macaulay. Recently, Lakshmibai and Magyar [LM] succeeded in proving that these schemes, with their natural determinantal structures, are reduced and Cohen-Macaulay in all characteristics. This strengthens our results, so that the formulas for the degeneracy loci have the usual meaning in intersection theory, as in [F1, §14].

2. Definitions of universal Schubert polynomials. We give three constructions of the single universal Schubert polynomials $\mathfrak{S}_w(c)$. We consider independent variables $c_i(j)$ for $1 \leq i \leq j \leq n$. It is to be understood that $c_i(j) = 1$ if $i = 0$, and $c_i(j) = 0$ if $i < 0$ or $i > j$. Let w be a permutation in S_{n+1} , and let $l(w)$ denote its length.

The quickest definition of $\mathfrak{S}_w(c)$ is a variation of that used in [FGP] to define quantum Schubert polynomials. In the classical case, such a formula can be found in [LS2, (2.10)]. A classical Schubert polynomial $\mathfrak{S}_w(x)$ can be written uniquely in the form

$$\mathfrak{S}_w(x) = \sum a_{i_1, \dots, i_n} e_{i_1}(x_1) \cdots e_{i_n}(x_1, \dots, x_n),$$

the sum over (i_1, \dots, i_n) with each $i_\alpha \leq \alpha$ and $\sum i_\alpha = l(w)$; here a_{i_1, \dots, i_n} are unique integers (depending on w). Define

$$(3) \quad \mathfrak{S}_w(c) = \sum a_{i_1, \dots, i_n} c_{i_1}(1) \cdots c_{i_n}(n).$$

The preceding definition is based on the following elementary fact (see [LS2, (2.7)] and [FGP, Prop. 3.4]), which we use frequently.

LEMMA 2.1. *Let R be a commutative ring, and let M be the free R -submodule of the polynomial ring $R[c]$ spanned by all monomials $c_{i_1}(1) \cdots c_{i_n}(n)$ with each $i_\alpha \leq \alpha$. Let M' be the free R -submodule of the polynomial ring $R[x]$ spanned by all monomials $x_1^{j_1} \cdots x_n^{j_n}$ with each $j_\alpha \leq n + 1 - \alpha$. Then the map that sends $c_i(j)$ to $e_i(x_1, \dots, x_j)$ determines an isomorphism of M onto M' .*

Universal Schubert polynomials can also be defined by a direct inductive procedure analogous to that for the classical Schubert polynomials. For w_0 the permutation in S_{n+1} of longest length, that is, $w_0(i) = n + 2 - i$ for $1 \leq i \leq n + 1$, set

$$(4) \quad \mathfrak{S}_{w_0}(c) = c_1(1) \cdot c_2(2) \cdots c_n(n).$$

The general universal polynomial is determined by the property that if k is an integer with $w(k) < w(k + 1)$, and $v = ws_k$ is obtained from w by interchanging the values of k and $k + 1$, then

$$(5) \quad \mathfrak{S}_w(c) = \partial_k(\mathfrak{S}_v(c)).$$

Here ∂_k is an additive endomorphism of the free \mathbb{Z} -module M spanned by monomials $c_{i_1}(1) \cdots c_{i_n}(n)$ with each $i_\alpha \leq \alpha$. It takes such a monomial to a sum of signed monomials, each with the same indices except in positions $k - 1$ and k . Write for simplicity $[a, b]$ for the monomial with $c_a(k - 1)$ and $c_b(k)$ in these two positions, with the other positions fixed but arbitrary. Note that $[p, q] = 0$ if p or q is negative, or if $p > k - 1$ or $q > k$. With this notation the formula for ∂_k is

$$(6) \quad \begin{aligned} \partial_k([a, b]) &= \sum_{i \geq 0} [a + i, b - 1 - i] - \sum_{i \geq 1} [b - 1 - i, a + i] \quad \text{if } a \geq b - 1; \\ \partial_k([a, b]) &= \sum_{i \geq 0} [b - 1 + i, a - i] - \sum_{i \geq 1} [a - i, b - 1 + i] \quad \text{if } a \leq b - 2. \end{aligned}$$

For $k = 1$, $\partial_1([1]) = [0]$ and $\partial_1([0]) = 0$. See [LS2, §3] for similar formulas.

To see that this definition is well defined, it suffices to verify that $\partial_k \circ \partial_l = \partial_l \circ \partial_k$ if $|k - l| \geq 2$, and that $\partial_k \circ \partial_{k+1} \circ \partial_k = \partial_{k+1} \circ \partial_k \circ \partial_{k+1}$ for $1 \leq k \leq n - 1$. This can be verified directly from the definition, but it follows easily from the lemma, together with the simple verification that ∂_k is compatible with the standard difference operator $\partial_k^{(x)}$ defined on $R[x] : \partial_k^{(x)}(P) = (P - s_k(P))/(x_k - x_{k+1})$, where $s_k(P)$ is the result of interchanging x_k and x_{k+1} in P (cf. [M, Chapter 2]). It also follows from this argument that the universal quantum Schubert polynomials form a basis for M , but we will see a stronger reason for this in Proposition 2.2.

The third definition defines double versions $\mathfrak{S}_w(c, y)$ of these polynomials, with general first variables $c_i(j)$ as above and special second variables y_1, \dots, y_n . This definition is similar to a construction in [KM] (cf. [CFF]), except that we have included signs with the y variables in order to be consistent with the original notation of Lascoux and Schützenberger. For this, set

$$(7) \quad \mathfrak{S}_{w_0}(c, y) = \prod_{i=1}^n \left(\sum_{j=0}^i c_{i-j}(i) (-y_{n+1-i})^j \right).$$

Now if $k + 1$ appears in the list of values of w to the right of k , and v interchanges the positions of $k + 1$ and k (i.e., $v = s_k w$), then

$$(8) \quad \mathfrak{S}_w(c, y) = -\partial_k^{(y)}(\mathfrak{S}_v(c, y)),$$

where $\partial_k^{(y)}$ is the standard difference operator, acting on the y variables alone. Then

$$(9) \quad \mathfrak{S}_w(c) = \mathfrak{S}_w(c, 0);$$

that is, the single universal Schubert polynomials are obtained from these double polynomials by setting the second set of variables equal to 0.

It is not hard to see that these three definitions agree. That the first and second definitions agree follows from the fact that they both give polynomials in the module M of the lemma which specialize to the classical Schubert polynomials under the isomorphism from M to M' . Next we observe that the double polynomials $\mathfrak{S}_w(c, y)$ specialize to the usual double Schubert polynomials $\mathfrak{S}_w(x, y)$ when each $c_i(j)$ is sent to $e_i(j)$. This follows from the fact that $\mathfrak{S}_w(x, y) = (-1)^{l(w)} \mathfrak{S}_{w^{-1}}(y, x)$; see [M, (6.4)(iii)]. From this it follows that $\mathfrak{S}_w(c, 0)$ specializes to $\mathfrak{S}_w(x) = \mathfrak{S}_w(x, 0)$ when $c_i(j)$ is replaced by $e_i(j)$, and this shows that the third definition agrees with the first two.

There is a natural definition of *universal double Schubert polynomials*, which we denote by $\mathfrak{S}_w(c, d)$, where c stands for the variables $c_i(j)$ and d stands for another set of variables $d_i(j)$. These are defined by the formula

$$(10) \quad \mathfrak{S}_w(c, d) = \sum_{u, v} (-1)^{l(v)} \mathfrak{S}_u(c) \mathfrak{S}_v(d),$$

where the sum is over all u and v in S_{n+1} such that $v^{-1}u = w$ and $l(u) + l(v) = l(w)$. The same argument as in the preceding paragraph, together with [M, (6.3)], shows that $\mathfrak{S}_w(c, d)$ specializes to the polynomials $\mathfrak{S}_w(c, y)$ when each $d_i(j)$ is specialized to the i th elementary symmetric polynomial in y_1, \dots, y_j . In particular, when $c_i(j)$ is also specialized to $e_i(x_1, \dots, x_j)$, then $\mathfrak{S}_w(c, d)$ becomes the classical double Schubert polynomial $\mathfrak{S}_w(x, y)$ of [L] (cf. [M]).

The universal double Schubert polynomials for permutations in S_3 are

$$\begin{aligned} \mathfrak{S}_{321} &= c_1(1)c_2(2) - (c_1(1)c_1(2) - c_2(2))d_1(1) - c_2(2)d_1(2) \\ &\quad + c_1(1)(d_1(1)d_1(2) - d_2(2)) + c_1(2)d_2(2) - d_1(1)d_2(2); \\ \mathfrak{S}_{231} &= c_2(2) - c_1(2)d_1(1) + d_1(1)d_1(2) - d_2(2); \\ \mathfrak{S}_{312} &= c_1(1)c_1(2) - c_2(2) - c_1(1)d_1(2) + d_2(2); \\ \mathfrak{S}_{132} &= c_1(2) - d_1(2); \\ \mathfrak{S}_{213} &= c_1(1) - d_1(1); \\ \mathfrak{S}_{123} &= 1. \end{aligned}$$

One can make specializations of either or both variables. For example, introducing variables $h_i[j]$ analogous to the variables $g_i[j]$, we have polynomials that we denote by $\mathfrak{S}_w(g, h)$. In general we let the position in the alphabet distinguish among these different single and double universal Schubert polynomials, using variables c, d for the first case, variables g, h for the second, and x, y for the classical case. (This seems preferable to introducing different notations for each realization of these polynomials.)

By construction, the universal Schubert polynomials are expressed as a linear combination of monomials $c_{i_1}(1) \cdots c_{i_n}(n)$. We want to say a little more about this expansion. For this we need a modified version of the code of a permutation w . For $w \in S_{n+1}$, we define this code, and denote it $c'(w)$, to be the sequence (i_1, \dots, i_n) , where i_k is defined by the formula

$$(11) \quad i_k = \text{Card}\{j \leq k \mid w(j) > w(k+1)\}.$$

This number i_k is the number of boxes in the k th row of a modified diagram $D'(w)$ of w , which is constructed as follows. Form an n by n square of boxes arranged as in a matrix, and, for $2 \leq i \leq n+1$, remove all the boxes from the row directly above the position $(i, w(i))$ that are strictly to the left of this position, and also all the boxes in the column directly to the left of and strictly above that position. So $D'(w)$ consists of boxes (i, j) such that $w(i+1) \leq j$ and $w^{-1}(j+1) \leq i$. It is easily seen that if $c'(w) = (i_1, \dots, i_n)$, then the Lehmer code of the permutation $w_0 w w_0$ (cf. [M, p. 9]) is $(i_n, \dots, i_1, 0)$. In particular, this code $c'(w)$ determines w , and the sum of the integers in $c'(w)$ is the length of w —properties that are easily proved directly.

PROPOSITION 2.2. *Let $c'(w) = (i_1, \dots, i_n)$. Then*

$$\mathfrak{S}_w(c) = c_{i_1}(1) \cdots c_{i_n}(n) + \sum m_{j_1, \dots, j_n} c_{j_1}(1) \cdots c_{j_n}(n),$$

where the sum is over (j_1, \dots, j_n) that are strictly smaller than (i_1, \dots, i_n) in the lexicographic ordering.

Proof. This follows from the second construction of the universal Schubert polynomials. The assertion is trivial when $w = w_0$, so we may assume it for all v of length greater than the length of a given w . Let k be the smallest integer such that $w(k) < w(k+1)$, and let $v = w s_k$. Since $\mathfrak{S}_w(c) = \partial_k(\mathfrak{S}_v(c))$, it suffices to show that the assertion of the proposition for v implies the assertion for w . When $k = 1$, this is completely straightforward, since ∂_k is so simple in this case. For $k > 1$, the code $I = c'(w)$ has $i_j = j$ for $1 \leq j \leq k-1$. The code $H = c'(v)$ is the same as I except in positions $k-1$ and k , where it is i_k and k , respectively. Write c_J for $c_{j_1}(1) \cdots c_{j_n}(n)$. It suffices to check that $\partial_k(c_H) = c_I \pm$ smaller terms, and that, if $J < H$, then $\partial_k(c_J)$ consists entirely of terms that are smaller than c_I . The verification of these facts is straightforward from (6). □

Note that this proposition implies (and is equivalent to) the corresponding assertion for the expression of the classical Schubert polynomials in terms of elementary symmetric monomials, or the quantum Schubert polynomials in terms of quantum elementary symmetric polynomials. It shows effectively why the universal Schubert polynomials form a basis for the module M that appears in Lemma 2.1.

Another property, which follows immediately from the definitions, is the *stability* of these universal polynomials: if i is the canonical embedding of S_{n+1} in S_{n+2} , then $\mathfrak{S}_{i(w)}(c, d) = \mathfrak{S}_w(c, d)$. Thus single and double universal polynomials are defined for w in $S_\infty = \bigcup S_n$.

Finally, we have the expected duality property, which is also an immediate consequence of the definitions,

$$(12) \quad \mathfrak{S}_w(d, c) = (-1)^{l(w)} \mathfrak{S}_{w^{-1}}(c, d).$$

3. Formulas for degeneracy loci. In this section we explain how the universal double Schubert polynomials describe degeneracy loci of appropriate maps of vector bundles. We assume that we are given the situation of vector bundles and maps as described in (1) of the introduction, on an algebraic scheme X over a field. (This is only to simplify the exposition; the procedures of [F1, §20] show how to modify the arguments for schemes of finite type over an arbitrary regular base scheme.) Set $c_i(j) = c_i(E_j)$, the i th Chern class of E_j , and set $d_i(j) = c_i(F_j)$. We claim that $\mathfrak{S}_w(c, d)$ is the formula for the locus Ω_w defined in (2).

As usual, of course, this assertion must be interpreted correctly, depending on assumptions about how general the maps between the vector bundles are. When X is an algebraic scheme of pure dimension k , there is a class Ω_w in the Chow group $A_{k-l(w)}(\Omega_w)$, whose image in $A_{k-l(w)}(X)$ is $\mathfrak{S}_w(c, d) \cap [X]$. When Ω_w has the expected dimension $k - l(w)$, then Ω_w is a positive cycle whose support is Ω_w . If, in addition, X is Cohen-Macaulay, then $\Omega_w = [\Omega_w]$, where Ω_w has its natural subscheme structure defined locally by vanishing of determinants. More generally, $\Omega_w = [\Omega_w]$ whenever $\text{depth}(\Omega_w, X) = l(w)$. Without any depth conditions, $[\Omega_w]$ can have larger multiplicities than Ω_w . The construction of the class Ω_w is compatible with proper pushforward and pullback by flat or local complete intersection morphisms. In fact, the class Ω_w can be constructed to live in the bivariant group $A^{l(w)}(\Omega_w \rightarrow X)$; for complex varieties, similar constructions produce classes in the relative cohomology groups $H^{2l(w)}(X, X - \Omega_w)$. This whole package is what is meant by the phrase “ $\mathfrak{S}_w(c, d)$ gives a formula for Ω_w .” For details about these interpretations, see [F1, §14 and §17] and [FP, App. A].

The construction of Ω_w is carried out as follows. Let

$$H = \bigoplus_{i=1}^{n-1} \text{Hom}(F_i, F_{i+1}) \oplus \text{Hom}(F_n, E_n) \oplus \bigoplus_{i=1}^{n-1} \text{Hom}(E_{i+1}, E_i),$$

regarded as a bundle over X . On H there are universal or tautological maps between the pullbacks of the bundles, and hence there are universal loci $\tilde{\Omega}_w \subset H$. If X is reduced, it follows from the theorem of Lakshmibai and Magyar [LM] that these schemes $\tilde{\Omega}_w$ are reduced, of codimension $l(w)$. Moreover, if X is Cohen-Macaulay, their theorem implies that $\tilde{\Omega}_w$ is Cohen-Macaulay. Note that on any open set U of X on which the bundles are trivial, H is a product $U \times H_0$, where H_0 is the corresponding bundle constructed from vector spaces, and $\tilde{\Omega}_w$ is the product of U and the corresponding universal locus in H_0 ; it is these universal local loci that are studied in [LM].

The given maps on X determine a section $s : X \rightarrow H$, and $\Omega_w = s^{-1}(\tilde{\Omega}_w)$. We define Ω_w to be the refined pullback of the class of $\tilde{\Omega}_w$, in the sense of [F1], that is,

$$\Omega_w = s^![\tilde{\Omega}_w].$$

To prove that $\mathfrak{S}_w(c, d)$ is the formula for the locus Ω_w , it suffices to prove that

$$(13) \quad [\tilde{\Omega}_w] = \mathfrak{S}_w(c, d) \cap [H]$$

in $A_{\dim(H)-l(w)}(H)$. As in [F1, §14] and [FP, App. A], the fact that the universal local loci are Cohen-Macaulay is what makes this construction have all the stated properties. In the proofs that follow, we therefore replace X by H , so we may assume the maps are locally universal in this sense.

When the maps $F_i \rightarrow F_{i+1}$ are injective and the maps $E_{i+1} \rightarrow E_i$ are surjective, the formula is exactly that proved in [F2].

Now consider the situation where we do not assume that the maps from each F_i to F_{i+1} are injective, but we do assume that the maps $E_{i+1} \rightarrow E_i$ are surjective.

PROPOSITION 3.1. *The formula for Ω_w is $\mathfrak{S}_w(x, d)$, where $x_i = c_1(\text{Ker}(E_i \rightarrow E_{i-1}))$ and $d_i(j) = c_i(F_j)$.*

Proof. We may assume that the map from F_n to E_n factors into an inclusion $F_n \rightarrow V$ followed by a surjection $V \rightarrow E_n$, where V is a vector bundle of rank $n + 1$. Indeed, as in [F2], one first considers the factorization $F_n \rightarrow F_n \oplus E_n \rightarrow E_n$ given by the graph of $F_n \rightarrow E_n$, and then pulls back to flag bundles to fill in between F_n and $V = F_n \oplus E_n$ and between V and E_n ; this replaces $n + 1$ by $2n$.

Let $\rho : F \rightarrow X$ be the bundle of complete flags in V , with universal subbundles U_i , so we have

$$U_1 \subset \dots \subset U_n \subset \rho^*(V) \rightarrow \rho^*(E_n) \rightarrow \dots \rightarrow \rho^*(E_1).$$

From these bundles and maps, by the case just considered, for each w in S_{n+1} , we have a locus $\tilde{\Omega}_w$ on F , and we know that

$$[\tilde{\Omega}_w] = \mathfrak{S}_w(x, y) \cap [F],$$

where $y_i = c_1(U_i/U_{i-1})$.

Let $Z_n \subset F$ be the locus given by the vanishing of the canonical map from $\rho^*(F_n)$ to V/U_n . This is given by the vanishing of a section of the bundle $\rho^*(F_n)^\vee \otimes V/U_n$, so

$$[Z_n] = c_n(\rho^*(F_n)^\vee \otimes V/U_n) \cap [F].$$

Notice that on Z_n the map from $\rho^*(F_n)$ to V factors through U_n . On Z_n we have the locus Z_{n-1} given by the vanishing of the map from $\rho^*(F_{n-1})$ to U_n/U_{n-1} , so Z_{n-1} is represented on Z_n by a top Chern class. On Z_{n-1} the map from $\rho^*(F_{n-1})$ to U_n factors through U_{n-1} . Continuing in this way, we get a sequence of loci $Z_1 \subset \dots \subset Z_n \subset F$. Let $Z = Z_1$. Then

$$[Z] = \prod_{i=1}^n (c_i(\rho^*(F_i)^\vee \otimes U_{i+1}/U_i)) \cap [F].$$

It follows from the fact that the given maps are locally universal that $\tilde{\Omega}_w$ meets Z properly and that ρ maps $\tilde{\Omega}_w \cap Z$ birationally onto Ω_w . (See the remark following the proof.) It therefore suffices to show that

$$(14) \quad \rho_* \left(\prod_{i=1}^n (c_i(\rho^*(F_i)^\vee \otimes U_{i+1}/U_i)) \cdot \mathfrak{S}_w(x, y) \right) = \mathfrak{S}_w(x, d),$$

where ρ_* is the pushforward from $A^{m+l(w)}(F)$ to $A^{l(w)}(X)$, with $m = n(n + 1)/2$. Let Y_k be the flag bundle of subbundles of V of all ranks from k through n . So ρ factors into a composite of projective bundle projections

$$F = Y_1 \rightarrow Y_2 \rightarrow \dots \rightarrow Y_n = \mathbb{P}^*(V) \rightarrow Y_{n+1} = X.$$

On Y_k we have universal bundles $U_k \subset \dots \subset U_n \subset V$. (Here and elsewhere in the proof we use the same notation for bundles as for their pullbacks by canonical projections.) We know that we can write

$$\mathfrak{S}_w(x, y) = \sum a_I(x) c_{i_1}(U_1) c_{i_2}(U_2) \dots c_{i_n}(U_n),$$

where the sum is over $I = (i_1, \dots, i_n)$ with $i_\alpha \leq \alpha$, and the $a_I(x)$ are polynomials in the x variables. (Note for this that $c_j(U_i) = e_j(y_1, \dots, y_i)$.) It suffices to prove that for each such I , and for each $2 \leq k \leq n + 1$, the pushforward of

$$\prod_{i=1}^n c_i(F_i^\vee \otimes U_{i+1}/U_i) \cdot c_{i_1}(U_1) \dots c_{i_n}(U_n)$$

from $F = Y_1$ to Y_k is equal to

$$\prod_{i=k}^n c_i(F_i^\vee \otimes U_{i+1}/U_i) \cdot c_{i_1}(F_1) \cdots c_{i_{k-1}}(F_{k-1}) \cdot c_{i_k}(U_k) \cdots c_{i_n}(U_n).$$

To prove this, using the projection formula, it suffices by induction to show that the pushforward from Y_k to Y_{k+1} of $c_k(F_k^\vee \otimes U_{k+1}/U_k) \cdot c_i(U_k)$ is equal to $c_i(F_k)$. But this is a special case of the following elementary Gysin formula for projective bundles.

LEMMA 3.2. *Let G be a vector bundle of rank $k + 1$ on a scheme Y , and let $\mathbb{P} = \mathbb{P}^*(G)$ be the projective bundle of hyperplanes in G , with projection $p : \mathbb{P} \rightarrow Y$ and tautological sequence*

$$0 \rightarrow H \rightarrow p^*(G) \rightarrow \mathcal{O}(1) \rightarrow 0$$

of bundles on \mathbb{P} . Let K be a vector bundle of rank k on Y . Then, for $0 \leq i \leq k$,

$$p_*(c_k(p^*(K)^\vee \otimes \mathcal{O}(1)) \cdot c_i(H)) = c_i(K).$$

Proof. Let $\zeta = c_1(\mathcal{O}(1))$. We use the basic fact that for any integer r , $p_*(\zeta^{r+k}) = (-1)^r s_r(G)$, where $s_r(G)$ denotes the r th Segre (inverse Chern) class of the bundle G . We use the formula for the top Chern class of a tensor product with a line bundle

$$c_k(p^*(K)^\vee \otimes \mathcal{O}(1)) = \sum_{a=0}^k (-1)^a c_a(p^*K) \zeta^{k-a},$$

and the Whitney formula $c_i(H) = \sum_{b=0}^i (-1)^b c_{i-b}(p^*G) \zeta^b$. This gives

$$p_*(c_k(p^*(K)^\vee \otimes \mathcal{O}(1)) \cdot c_i(H)) = \sum_{a=0}^k (-1)^a c_a(K) \left(\sum_{b=0}^i (-1)^{b+b-a} c_{i-b}(G) s_{b-a}(G) \right).$$

The inner sum vanishes unless $a = i$, when it gives $(-1)^i$, so the right side is the required $c_i(K)$.

This completes the proof of Lemma 3.2, which constructs the required class in the case where the right maps are surjective but the left maps are not necessarily injective. The dual case, where the left maps are injective and the right maps are arbitrary, can be handled by a dual construction, or simply by taking the duals of all the maps, and applying the case just considered to the situation

$$E_1^\vee \rightarrow \cdots \rightarrow E_n^\vee \rightarrow F_n^\vee \rightarrow \cdots \rightarrow F_1^\vee.$$

The locus Ω_w for the original maps is the locus Ω_{w-1} for this dual sequence. So we define the class Ω_w for the original maps to be the class Ω_{w-1} for this dual. The duality property (12) guarantees that this class has the right image in the Chow group of X .

Remark 3.3. In the preceding proof, we used the fact that the scheme $\tilde{\Omega}_w \cap Z$ is reduced when X is reduced. Although this can be seen directly, this argument can be avoided. Otherwise, the argument would prove that $\mathfrak{S}_w(x, d)$ represents some multiple of the class of the reduced subscheme of Ω_w . This would imply that the polynomial $\mathfrak{S}_w(x, d)$ is a power of some other polynomial, but we know that this is not the case even after specializing to $\mathfrak{S}_w(x, 0)$.

PROPOSITION 3.4. *Suppose we are given, on a scheme X , vector bundles and maps*

$$F_1 \rightarrow \cdots \rightarrow F_n \rightarrow E_n \rightarrow \cdots \rightarrow E_1,$$

the ranks of E_i and F_i being i . Then for each w in S_{n+1} the formula for Ω_w is the universal double Schubert polynomial $\mathfrak{S}_w(c, d)$, where $c_i(j) = c_i(E_j)$ and $d_i(j) = c_i(F_j)$.

Proof. We proceed exactly as in Proposition 3.1, first reducing to the case where one has an intermediate bundle V . One then has the formula for the corresponding locus $\tilde{\Omega}_w$ on the flag bundle F , this time applied to the case where the second maps are arbitrary. This class maps to $\mathfrak{S}_w(c, y) \cap [F]$ in the Chow group of F , where $c_i(j) = c_i(E_j)$ and $y_i = c_1(U_i/U_{i-1})$. We apply the same construction as before, pushing down the product of this class and the class $[Z]$. This time we must prove the formula

$$\rho_* \left(\prod_{i=1}^n (c_i(\rho^*(F_i)^\vee \otimes U_{i+1}/U_i)) \cdot \mathfrak{S}_w(c, y) \right) = \mathfrak{S}_w(c, d).$$

The only difference is that the polynomials $a_i(x)$ that appeared in the preceding proof are replaced by polynomials in the classes $c_i(E_j)$ for this proof.

Remark 3.5. As the proof shows, the Schubert polynomials $\mathfrak{S}_w(c, y)$ represent the loci Ω_w for the situation of the theorem, but under the conditions that the maps $F_i \rightarrow F_{i+1}$ are injective. Here $c_i(j) = c_i(E_j)$ and $y_i = c_1(F_i/F_{i-1})$. One has a dual interpretation for $\mathfrak{S}_w(x, d)$.

There is a corresponding and more general theorem for maps of bundles of arbitrary increasing and decreasing ranks. Suppose we have vector bundles and maps

$$F_1 \rightarrow \cdots \rightarrow F_s \rightarrow E_r \rightarrow \cdots \rightarrow E_1,$$

with $\text{rank}(E_i) = a_i, \text{rank}(F_i) = b_i$, such that

$$a_1 < a_1 < \dots < a_r \quad \text{and} \quad b_1 < b_1 < \dots < b_s.$$

Let w be a permutation whose diagram $D'(w)$ is contained in $A \times B$, with $A = \{a_1, \dots, a_r\}$ and $B = \{b_1, \dots, b_s\}$. There is a degeneracy locus $\Omega_w \subset X$ defined by the conditions that the rank of the map from F_j to E_i is at most $r_w(a_i, b_j)$ for all i and j .

LEMMA 3.6. *If $D'(w) \subset A \times B$, then no $c_i(k)$ occurs in $\mathfrak{S}_w(c, d)$ with $k \notin A$, and no $d_j(l)$ occurs in $\mathfrak{S}_w(c, d)$ with $l \notin B$.*

Proof. This follows from two claims, valid for any permutation w with code $c'(w) = (i_1, \dots, i_n)$ and an integer k such that $i_k = 0$. The first claim is that all terms $c_{j_1}(1) \cdots c_{j_n}(n)$ occurring in $\mathfrak{S}_w(c)$ have $j_k = 0$; this is proved as in the proof of Proposition 2.2. The second claim is that if i is an integer with $w^{-1}(i + 1) < w^{-1}(i)$, and $v = s_i w$, and $c'(w) = (j_1, \dots, j_n)$, then $j_k = 0$; this follows immediately from the definition. It then follows that in the sum (10) expressing $\mathfrak{S}_w(c, d)$, all the $\mathfrak{S}_u(c)$ have no term $c_i(k)$ with $k \notin A$. The assertion for the $d_j(l)$ then follows from (12). □

Let $\mathfrak{S}_w(c(E_\bullet), c(F_\bullet))$ denote the result of specializing $c_i(a_p)$ to $c_i(E_p)$ and $d_i(b_q)$ to $c_i(F_q)$ in $\mathfrak{S}_w(c, d)$.

THEOREM 3.7. *The formula for Ω_w is $\mathfrak{S}_w(c(E_\bullet), c(F_\bullet))$.*

Proof. Take $n = \max(r, s)$, and add trivial bundles to the given bundles so that one has two sets of bundles of all ranks between 1 and n . Insert maps

$$F_k \rightarrow F_k \oplus \mathbf{1} \rightarrow \dots \rightarrow F_k \oplus \mathbf{1}^{b_{k+1}-b_k-1} \rightarrow F_{k+1},$$

where the maps to successive additions of trivial factors are the evident inclusions, and the map from the last bundle to F_{k+1} is the given map on the factor F_k and the zero map on the trivial factors. The dual construction is made on the other side. Then one is in a position where Proposition 3.4 applies. The hypotheses on w guarantee that all of the rank conditions on the added bundles follow from the rank conditions on the given bundles, so that the locus Ω_w is the same whether defined for the given bundles or for all the bundles. Proposition 3.4 gives the required formula.

Remark 3.8. If w is a permutation satisfying the weaker conditions that the descents of w are contained in A and the descents of w^{-1} are contained in B , then the same construction as in the proof of Theorem 3.7 produces a class Ω_w in $A_{\dim(X)-l(w)}(\Omega_w)$. Its image in $A_{\dim(X)-l(w)}(X)$ is the polynomial obtained from $\mathfrak{S}_w(c, d)$ by specializing $c_i(k)$ to $c_i(E_p)$ if $a_p \leq k < a_{p+1}$, with $c_i(k)$ sent to zero if $k < a_1$, and to $c_i(E_r)$ if $k \geq a_r$; similarly $d_i(l)$ is sent to $c_i(F_q)$ if $b_q \leq l < b_{q+1}$.

Remark 3.9. The same class is obtained if arbitrary bundles are used in place of trivial bundles in the construction in the proof of the theorem. However, this is no longer true for the more general classes constructed in the preceding remark.

Remark 3.10. Even in the situation of Proposition 3.4, this theorem applies to more general loci Ω_w , where w is not in S_{n+1} but $D'(w) \subset [n] \times [n]$.

Remark 3.11. The degeneracy loci Ω_w are defined by many rank conditions, but some of them are superfluous. In fact, the rank conditions that are needed on $F_j \rightarrow E_i$ are precisely those described by the boxes (a_i, b_j) in the diagram we denoted by $D'(w)$ in Section 2. In the case when the first maps were injective and the second maps surjective, we saw in [F2] that the essential rank conditions were given by those in a subset of the diagram $D(w)$, which we called the essential set of w . In fact, this essential set is exactly the intersection of $D(w)$ with $D'(w)$. For general maps as considered here, the entire set $D'(w)$ is needed. The proof of this statement is the same as in [F2, Prop. 4.2]. Note that $D'(w)$ consists of $l(w)$ boxes, which is exactly the expected codimension of Ω_w .

4. Determinantal formulas. We first prove a determinantal formula for certain of the polynomials $\mathfrak{S}_w(c, y)$, which is used to show that the universal Schubert polynomials specialize to the quantum polynomials for partial flag varieties of [CF2]. For this, fix an integer $l \geq 2$, and fix a set

$$N = \{n_1 < n_2 < \dots < n_{l-1} < n_l\}$$

of l positive integers. Set $n_0 = 0$, and set $k_i = n_i - n_{i-1}$ for $1 \leq i \leq l$. Let

$$S^{(N)} = \{w \in S_{n_l} \mid w(i) < w(i+1) \text{ if } i \notin N\}$$

be the permutations in S_{n_l} with descents in N . Let $w = w_0^{(N)}$ be the element of longest length in $S^{(N)}$; that is,

$$w(n_{p-1} + i) = n_l - n_p + i \quad \text{for } 1 \leq i \leq k_p, 1 \leq p \leq l.$$

For nonnegative integers a, b , a positive integer k , and an arbitrary integer m , set

$$(15) \quad f_m(k, a, b) = \sum_{p=0}^m (-1)^p c_{m-p}(a) h_p(y_{b+1}, \dots, y_{b+k}),$$

where $h_p(z_1, \dots, z_k)$ denotes the p th complete symmetric polynomial in z_1, \dots, z_k . Equivalently,

$$\sum_m f_m(k, a, b) t^m = \sum_{i \geq 0} c_i(a) t^i \Big/ \prod_{j=b+1}^{b+k} (1 + y_j t).$$

For $k > 0, a \geq 0, b \geq 0$, set

$$(16) \quad D(k, a, b) = \det(f_{a+j-i}(k, a+k-i, b))_{1 \leq i, j \leq k}.$$

PROPOSITION 4.1. For $w = w_0^{(N)}$,

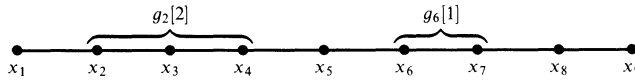
$$\mathfrak{S}_w(c, y) = \prod_{i=1}^{l-1} D(k_{i+1}, n_i, n_l - n_{i+1}).$$

Proof. Note that both sides of the identity to be proved are in the R -module M described in Lemma 2.1, where $R = \mathbb{Z}[y_1, \dots, y_{n_l-1}]$. By that lemma, it therefore suffices to prove the formula after specializing each $c_i(j)$ to $e_i(x_1, \dots, x_j)$. The permutation w is dominant, so, by [M, (6.14)], $\mathfrak{S}_w(x, y) = \prod (x_p - y_q)$, the product over all (p, q) that appear in the diagram $D(w)$ of w . That is,

$$\mathfrak{S}_w(x, y) = \prod_{i=1}^{l-1} \prod_{p=1}^{n_i} \prod_{q=n_{i-1}+1}^{n_i-n_{i+1}} (x_p - y_q).$$

We want to show that the i th term in the product of the proposition specializes to the i th term of this product. Equivalently, we must show that $D(k, n, 0)$ specializes to $\prod_{p=1}^n \prod_{q=1}^k (x_p - y_q)$ for any positive n and k . But this also follows from [M, (6.14)] (cf. [F2, (9.6)]).

We next want to see what happens to the Schubert polynomials $\mathfrak{S}_w(c)$ when we carry out a specialization $c \mapsto g$. For this, it is useful to have a more graphic description of this specialization. Write x_1, \dots, x_{n+1} as the vertices of the Dynkin diagram for (A_n) , and regard $g_1[1], \dots, g_n[1]$ as the edges, with $g_i[1]$ connecting x_i to x_{i+1} . Now regard $g_i[j]$ as the path starting at vertex x_i , moving j steps to the right, and ending at vertex x_{i+j} ; in particular, this identifies $g_i[0]$ with x_i . With this interpretation, $c_i(k)$ is the sum of all products of disjoint paths that cover exactly i vertices, all in $\{x_1, \dots, x_k\}$.



This figure illustrates, for example, that the monomial $x_1 g_2[2] x_5 g_6[1] x_9$ appears in the expansion of $c_8(9)$.

This definition is equivalent to the inductive definition

$$(17) \quad c_i(k) = \sum_{j=0}^i g_{k-j}[j] c_{i-j-1}(k-j-1).$$

This inductive definition is seen to be equivalent to the matrix definition given in Section 1 by expanding the determinant along the right column.

We consider the determinant $D(k, a, b)$ and polynomial $f_m(k, a, b)$ defined before Proposition 4.1, but with each $c_i(j)$ replaced by the corresponding sums of products of $g_s[t]$'s. We need the following fact.

LEMMA 4.2. *Suppose a, b , and k are given, and assume that $g_i[j] = 0$ whenever $a < i + j < a + k$ and $j > 0$. Then*

$$D(k, a, b) = \det(f_{a+j-i}(k, a, b))_{1 \leq i, j \leq k}.$$

Proof. The hypotheses imply that $f_i(k, p, b) = f_i(k, p - 1, b) + x_p f_{i-1}(k, p - 1, b)$ for $a < p < a + k$. It is then a matter of elementary row reduction to show that the rows of the matrix for $D(k, a, b)$ can be replaced by the rows of the determinant on the right, by successively subtracting linear combinations of lower rows. □

We next show that the polynomials that give Giambelli formulas in quantum cohomology for Schubert varieties in partial flag manifolds [CF2] can also be realized as specializations of the universal Schubert polynomials. With N a set of l positive integers as above, let $F^{(N)}$ be the flag variety of flags $V_1 \subset \dots \subset V_{l-1} \subset V$, with $V = V_l$ a fixed n_l -dimensional vector space, and $\dim(V_i) = n_i$. For computations in quantum cohomology of $F^{(N)}$, one introduces variables q_1, \dots, q_{l-1} , where q_i has degree $n_{i+1} - n_{i-1}$. For a permutation w in $S^{(N)}$, we let $\mathfrak{S}_w^{(N)}$ be the result of specializing in $\mathfrak{S}_w(g)$ the variables $g_i[0]$ to x_i , the variables

$$(18) \quad g_{n_{i+1}+1}[k_i + k_{i+1} - 1] \text{ to } (-1)^{k_{i+1}+1} q_i$$

for $1 \leq i \leq l - 1$, and all other variables $g_i[j]$ are set equal to zero.

Another procedure is used in [CF2], which amounts to the following. First do the substitutions $c_i(j) \mapsto c_i(n_k)$ for $j \in [n_k, n_{k+1})$, and then do the preceding substitutions from the $c_i(n_k)$ to polynomials in x 's and q 's.

PROPOSITION 4.3. *For w in $S^{(N)}$, these two procedures give the same polynomial $\mathfrak{S}_w^{(N)}$ in $\mathbb{Z}[x_1, \dots, x_n, q_1, \dots, q_{l-1}]$.*

Proof. When $w = w_0^{(N)}$, this follows from Proposition 4.1 and Lemma 4.2; note that the vanishing of many $g_i[j]$ guarantee that the lemma applies to each determinant $D(k_{i+1}, n_i, n_l - n_{i+1})$. The general case then follows from this case, since every other Schubert polynomial, for w in $S^{(N)}$, can be obtained from the polynomial for $w_0^{(N)}$ by a sequence of operations $\partial_k^{(y)}$ acting on the y variables alone, and specializing in the c variables commutes with these operations. □

Other determinantal formulas for universal Schubert polynomials can be deduced from corresponding formulas for classical Schubert polynomials by the same procedure as in Proposition 4.1. For example, this is the case whenever all

the appearances of $c_i(j)$ in entries of the matrix never have the same j appearing in two different rows, or if the same j never appears in two different columns. Indeed, the determinant of such a matrix is in the module denoted M in Section 2. In fact, as in the proof of Lemma 4.2, the same is true if the matrix can be transformed by an appropriate sequence of elementary row (resp., column) operations into a matrix of this form.

For example, suppose w is a Grassmannian permutation with descent at r . The shape of w is the partition $\lambda = (w(r) - r, \dots, w(2) - 2, w(1) - 1)$. Let $\mu = \tilde{\lambda}$, which is the shape of w^{-1} . Let ϕ be the flag of w^{-1} ; this is an increasing sequence of positive integers, constructed from the Lehmer code (c_1, \dots, c_n) of w^{-1} by arranging the integers of the form $\max\{j \geq i \mid c_j \geq c_i\}$, taken over those i with $c_i \neq 0$, in weakly increasing order (see [M, p. 14]). Let $f_m(k, a) = f_m(k, a, 0)$, where $f_m(k, a, b)$ is defined in (15). Then we have the following proposition.

PROPOSITION 4.4. *If w is a Grassmannian permutation, then*

$$\mathfrak{S}_w(c, y) = \det(f_{\mu_i+j-i}(\phi_i, r + j - 1)),$$

with μ and ϕ as defined in the preceding paragraph.

Proof. The second indices $r + j - 1$ occurring in the matrix guarantee that the determinant is in the $\mathbb{Z}[y]$ -module M . It therefore suffices to prove the corresponding formula in the classical case, that is, to prove that

$$\mathfrak{S}_w(x, y) = \det(e_{\mu_i+j-i}(\phi_i, r + j - 1)),$$

where $e_m(u, v)$ denotes the coefficient of t^m in $\prod_{i=1}^v (1 + x_i t) / \prod_{j=1}^u (1 + y_j t)$. This formula is known (see [M, (6.15), (3.8)]; cf. [F2, (9.18)]), except that in these references the determinant is of the matrix $(e_{\mu_i+j-i}(\phi_i, r))$. But these two determinants are seen to be equal by doing elementary column operations, adding multiples of left columns to those on the right. \square

Remark 4.5. When the universal Schubert polynomial $\mathfrak{S}_w(c, y)$ is specialized to the quantum variables, one recovers the formula of Kirillov [K] for Grassmannian permutations.

Other formulas can be proved in the same way; they give formulas for some Schubert polynomials $\mathfrak{S}_w(c)$ as determinants of matrices with entries $c_i(j)$, where j is constant in columns, but with different values in different columns.

However, some experimentation indicates that there may be more determinantal formulas if one looks for such matrices with j being constant in rows, with different values in different rows. In fact, many of these polynomials have the following special form. For arbitrary sequences a_1, \dots, a_n and b_1, \dots, b_n of non-negative integers, form the $n \times n$ matrix $C(a, b)$ whose (i, j) entry is $c_{a_i+j-i}(b_i)$, except that, if $a_i = 0$, then the (i, j) entry of $C(a, b)$ is the Kronecker index δ_{ij} . Note that $C(a, b)$ has the numbers $c_{a_i}(b_i)$ appearing down the diagonal. Define

$D_{a_1 \dots a_n}(b_1 \dots b_n)$ to be the determinant of this matrix $C(a, b)$. The formulas we are looking for have the form

$$(19) \quad \mathfrak{S}_w(c) = D_{i_{\sigma(1)} \dots i_{\sigma(n)}}(\sigma(1) \dots \sigma(n)).$$

Here w is a permutation in S_{n+1} , $I = (i_1, \dots, i_n)$ is the code $c'(w)$ defined in Section 2, and σ is some permutation in S_n . Note that the product of the diagonal terms in such a determinant is the leading term of $\mathfrak{S}_w(c)$ found in Proposition 2.2. Here are some examples, for permutations in S_5 :

$$w = 5\ 1\ 4\ 2\ 3 \quad I = (1, 1, 2, 2) \quad \mathfrak{S}_w(c) = D_{2211}(4\ 3\ 2\ 1)$$

$$w = 3\ 5\ 1\ 2\ 4 \quad I = (0, 2, 2, 1) \quad \mathfrak{S}_w(c) = D_{1022}(4\ 1\ 3\ 2)$$

$$w = 3\ 2\ 5\ 1\ 4 \quad I = (1, 0, 3, 1) \quad \mathfrak{S}_w(c) = D_{1013}(4\ 2\ 1\ 3).$$

Note that the last permutation is not vexillary. In fact, all but 8 of the 120 permutations in S_5 have such determinantal expressions, but these do not include all of the 103 vexillary permutations. For example, $w = 1\ 5\ 3\ 2\ 4$ is vexillary, but has no such determinantal expression. It would be interesting to characterize those permutations that have such a determinantal expression or to give a rule for a permutation σ (which is often not unique) for those that do. As before, such a formula can be verified by computing its classical specialization, but the formulas seem easier to detect for the universal polynomials.

One can write a product of two universal Schubert polynomials as a linear combination of universal Schubert polynomials, with coefficients in the ring $\mathbb{Z}[g]$ generated by all $g_i[j]$, $j \geq 0$:

$$(20) \quad \mathfrak{S}_u(g) \cdot \mathfrak{S}_v(g) = \sum a_w(g) \mathfrak{S}_w(g).$$

Even when specialized all the way to the classical case, formulas for the coefficients are only known in special cases (cf. [S]). Many of these special cases have been extended to the quantum setting (see [FGP], [CF2]). In these settings, the coefficients that appear with monomials in the x and q variables are all positive, for geometric reasons. This is no longer the case for these universal polynomials. Indeed, given the signs with which the q variables appear in the quantum partial flag specializations, one knows that certain of these coefficients must be negative.

There is one special case where one does have a positive expansion, which is the case of multiplying two single terms $c_i(k) \cdot c_j(k)$. Note that $c_i(k) = \mathfrak{S}_w(c)$, where w is the Grassmannian permutation in S_{k+1} with descent at k and $w(k+1) = k+1-i$.

To state this result, let i, j , and k be integers, with k positive and $0 \leq i, j \leq k$. Let $\mathcal{A}(i, j, k)$ be the set of Grassmannian permutations w in S_{k+1} with descent at $k-1$, such that $w(k) \leq k+1 - \max(i, j)$ and $w(k) + w(k+1) = 2k+3 - (i+j)$.

PROPOSITION 4.6. For $0 \leq i, j \leq k$,

$$c_i(k) \cdot c_j(k) = \sum_{w \in \mathcal{A}(i+1, j+1, k+1)} \mathfrak{S}_w(c) + \sum_{w \in \mathcal{A}(i, j, k)} g_k[1] \mathfrak{S}_w(c) + \sum_{p=1}^{k-1} g_{k-p}[p+1] A_p,$$

where $A_p = \sum \mathfrak{S}_u(c)$, the sum over u of the form $u = wt_{k-p, k}$, where w varies over those permutations in $\mathcal{A}(i, j, k)$ for which $w(k-p) > w(k)$, and $u = wt_{k-p, k}$ is the result of interchanging the values of $k-p$ and k .

Proof. The proof is by explicit calculation, using the following facts. First, for $w \in \mathcal{A}(i, j, k)$, with $a = w(k)$ and $b = w(k+1)$,

$$(21) \quad \mathfrak{S}_w(c) = c_{k-a}(k-1)c_{k+1-b}(k) - c_{k-b}(k-1)c_{k+1-a}(k).$$

This is a special case of Proposition 4.4. It then follows from (6) that for $u = wt_{k-p, k}$ as in the sum for A_p ,

$$(22) \quad \mathfrak{S}_u(c) = c_{k-a-p}(k-1-p)c_{k+1-b}(k) - c_{k-b-p}(k-1-p)c_{k+1-a}(k).$$

With these formulas, one can expand each $\mathfrak{S}_w(c)$ and $\mathfrak{S}_u(c)$ that occurs in the statement of the proposition. On the other hand, from the definitions we have the formula

$$(23) \quad c_s(k+1) = c_s(k) + \sum_{r=0}^{s-1} g_{k+1-r}[r]c_{s-r-1}(k-r).$$

One then substitutes (23) for each occurrence of any $c_s(k+1)$ in the expansion of the first sum in the formula of the proposition and verifies that one has an identity.

Remark 4.7. The first sum in the formula can be written in the form

$$\sum_{l \geq 0} c_{i-l}(k+l)c_{j+l}(k) - \sum_{l \geq 1} c_{i-l}(k)c_{j+l}(k+1).$$

When specialized to the classical case, the other sums vanish, and the proposition becomes a standard formula for multiplying elementary symmetric polynomials.

Remark 4.8. The preceding proposition can be used inductively—in the explicit form given in the proof—to write an arbitrary polynomial in variables $c_i(k)$ as a linear combination of such monomials so that no product $c_i(k)c_j(k)$ occurs with positive i and j . Such a polynomial can be written as a linear combination of universal Schubert polynomials, using Proposition 2.2.

One can construct a *universal ring* R_n , which is a natural place for calculations, and which specializes to the classical and quantum cohomology rings of flag and partial flag varieties. This is an algebra over the polynomial ring $\mathbb{Z}[g]$ in variables $g_i[j]$ for $j > 0$ and $i + j \leq n + 1$. The ring can be defined as

$$(24) \quad R_n = \mathbb{Z}[g][x_1, \dots, x_{n+1}] / (c_1(n+1), \dots, c_{n+1}(n+1)),$$

where each $c_i(n+1)$ is the polynomial in x and g variables defined in Section 2 or (17). The Schubert polynomials $\mathfrak{S}_w(g)$, as w varies over S_{n+1} , form a basis for this algebra over $\mathbb{Z}[g]$.

This ring has an obvious inner product $\langle \cdot, \cdot \rangle$, with values in $\mathbb{Z}[g]$, which is obtained by multiplying and then picking off the coefficient of $x_1^n x_2^{n-1} \dots x_n$ or the coefficient of $\mathfrak{S}_{w_0}(g)$. Computations in low degrees lead one to conjecture the orthogonality

$$(25) \quad \langle \mathfrak{S}_u(g), \omega(\mathfrak{S}_{v_{w_0}}) \rangle = \delta_{uv},$$

where ω is the involution defined by $\omega(g_i[j]) = (-1)^{j+1} g_{n+2-i-j}[j]$. Such an orthogonality was difficult to prove in the quantum case (see [FGP], [KM]) and promises to be even more difficult for this generalization. Similar calculations lead one to believe¹ that

$$(26) \quad \mathfrak{S}_w(c, c) = 0 \quad \text{for } w \neq 1.$$

In a more recent work [BF], formulas for the general degeneracy loci described in [ADFK] and [LM] are found by different methods. When specialized to the loci considered here, they give different formulas for the universal double Schubert polynomials.

It should be interesting to look for analogues of these universal polynomials for the other classical groups.

REFERENCES

[ADFK] S. ABEASIS, A. DEL FRA, AND H. KRAFT, *The geometry of representations of A_m* , Math. Ann. **256** (1981), 401–418.
 [BGG] I. BERNSTEIN, I. GEL'FAND, AND S. GEL'FAND, *Schubert cells, and the cohomology of the spaces G/P* , Russian Math. Surveys **28** (1973), 3–26.
 [BF] A. BUCH AND W. FULTON, *Chern class formulas for quiver varieties*, to appear in Invent. Math.
 [CF1] I. CIOCAN-FONTANINE, *Quantum cohomology of flag varieties*, Internat. Math. Res. Notices **1995**, 263–277.
 [CF2] ———, *On quantum cohomology rings of partial flag varieties*, preprint.
 [CFF] I. CIOCAN-FONTANINE AND W. FULTON, *Quantum double Schubert polynomials*, Institut Mittag-Leffler report no. 6, 1996/97, Appendix J in [FP].

¹ A. N. Kirillov reports that he can prove (25) and (26).

- [D] M. DEMAZURE, *Désingularisation des variétés de Schubert généralisées*, Ann. Sci. École Norm. Sup. (4) **7** (1974), 53–88.
- [FGP] S. FOMIN, S. GEL'FAND, AND A. POSTNIKOV, *Quantum Schubert polynomials*, J. Amer. Math. Soc. **10** (1997), 565–596.
- [F1] W. FULTON, *Intersection Theory*, 2d ed., Ergeb. Math. Grenzgeb. (3) **2**, Springer-Verlag, Berlin, 1998.
- [F2] ———, *Flags, Schubert polynomials, degeneracy loci, and determinantal formulas*, Duke Math. J. **65** (1992), 381–420.
- [FP] W. FULTON AND P. PRAGACZ, *Schubert Varieties and Degeneracy Loci*, Lecture Notes in Math. **1689**, Springer-Verlag, Berlin, 1998.
- [K] A. N. KIRILLOV, *Quantum Schubert polynomials and quantum Schur functions*, preprint.
- [KM] A. N. KIRILLOV AND T. MAENO, *Quantum double Schubert polynomials, quantum Schubert polynomials and Vafa-Intriligator formula*, preprint.
- [LM] V. LAKSHMIBAI AND P. MAGYAR, *Degeneracy schemes, quiver schemes, and Schubert varieties*, Internat. Math. Res. Notices **1998**, 627–640.
- [L] A. LASCoux, *Classes de Chern des variétés de drapeaux*, C. R. Acad. Sci. Paris Sér. I Math. **295** (1982), 393–398.
- [LS1] A. LASCoux AND M.-P. SCHÜTZENBERGER, *Polynômes de Schubert*, C. R. Acad. Sci. Paris Sér. I Math **294** (1982), 447–450.
- [LS2] ———, *Fonctorialité des polynômes de Schubert*, Contemp. Math. **88** (1989), 585–598.
- [M] I. G. MACDONALD, *Notes on Schubert polynomials*, Monographies du LaCIM **6**, Université du Québec, Montréal, 1991.
- [S] F. SOTTILE, *Pieri's formula for flag manifolds and Schubert polynomials*, Ann. Inst. Fourier (Grenoble) **46** (1996), 89–110.

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