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q-Catalan Numbers

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q-analogs of the Catalan numbers $C_n = (1/(n+1))\binom{2n}{n}$ are studied from the viewpoint of Lagrange inversion. The first, due to Carlitz, corresponds to the Andrews–Gessel-Garsia q-Lagrange inversion theory, satisfies a nice recurrence relation and counts inversions of Catalan words. The second, tracing back to Mac Mahon, arise from Krattenthaler's and Gessel and Stanton's q-Lagrange inversion formula, have a nice explicit formula and enumerate the major index. Finally a joint generalization is given which includes also the Polya-Gessel q-Catalan numbers.

1. INTRODUCTION

In this survey on different q-analogs of the Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n} \tag{1.1}$$

we want to stress the importance of using *Lagrange expansions* instead of the customary generating functions. For motivation we briefly repeat the well-known case $q = 1$:

Starting with the recurrence relation

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}, \qquad C_0 = 1, \tag{1.2}$$

which corresponds to a decomposition of Catalan structures, one defines the generating function

$$f(t) = \sum_{n=0}^{\infty} C_n t^n \tag{1.3}$$

Then (1.2) is equivalent with

$$f(t) = 1 + tf(t)^2. \tag{1.4}$$

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Now one usually solves this quadratic equation to obtain the explicit formula (1.1). But we may also use the Lagrange inversion formula, even in two ways:

(i) Either we set $f(t) = 1 + y(t)$ which gives

$$y = t(1+y)^2 \qquad \text{or} \qquad t = y(1+y)^{-2}.$$

Inserting this into (1.3) gives the expansion formula

$$y = \sum_{n=1}^{\infty} C_n \frac{y^n}{(1+y)^{2n}}. \tag{1.5}$$

(ii) Or with $z = tf(t) = \sum_{n=1}^{\infty} C_{n-1} t^n$, (1.4) gives

$$z = t + z^2 \qquad \text{or} \qquad t = z(1-z)$$

and hence another expansion

$$z = \sum_{n=1}^{\infty} C_{n-1} z^n (1-z)^n. \tag{1.6}$$

From both expansions (1.5) and (1.6) one immediately gets the explicit value (1.1) using the Lagrange inversion formula (see, e.g., [6, 23]). Of course the real advantage of (1.5) and (1.6) over the generating function is that they suggest the "right" generalization of Catalan numbers. They will be the starting point of our study of q-Catalan numbers.

Using the q-Lagrange theory of Garsia [8, 9], which unifies the previous works of Andrews [1] and Gessel [10], and where the power $g(z)^n$ is replaced by $g(z)\,g(qz)\cdots g(q^{n-1}z)$, one obtains the q-Catalan numbers invented by Carlitz and Riordan [4] and studied further in detail by Carlitz [3, 5] and other [1, 10, 24, 26]. They satisfy a simple recurrence relation analogous to (1.2) and arise in several q-enumeration problems: They count inversions of Catalan words, and Catalan permutations, area below lattice paths, etc. But no explicit formula like (1.1) is known.

It is shown in Section 3 that the explicit q-Catalan numbers $(1/[n+1])[{2n \atop n}]$ arise instead from a different q-Lagrange formula for expansion (1.5), due to Krattenthaler [16, 17] and Gessel and Stanton [12]. Their combinatorial significance for counting the major index of Catalan words dates back to MacMahon's study of "lattice permutations." In fact we study a more general expansion giving a whole family $c_n(\lambda; q)$ of q-Catalan numbers, which split up into explicit q-Runyon numbers (Sect. 4).

In Section 5 we generalize (1.5) further to obtain 3-variate Catalan numbers $C_n(x; a, b)$ and state their combinatorial meaning and recurrence relations of type (1.2). Finally we show (Sect. 6) that even the Polya–Gessel q-Catalan numbers [22, 10] which count area of polygons are included in the $C_n(x; a, b)$.

Notation. Let $S(n, m)$ be the set of all words $w = w_1 w_2 \cdots w_{n+m}$ consisting of n 0s and m 1s. With $S_+(n, m)$ we denote the subset of $S(n, m)$ consisting of those words, such that no initial segment contains more 1s than 0s. Moreover let $S_-(n, m) = S(n, m) \setminus S_+(n, m)$ and $\mathscr{C}_n = S_+(n, n)$ for short. We identify such words $w \in S(n, m)$ with lattice paths from $(0, 0)$ to $(n+m, n-m)$ in the sense of Feller [7, p. 73], drawing an ascending edge for a 0 and a descending one for a 1. Then a *Catalan word* $w \in \mathscr{C}_n$ corresponds to a lattice path from $(0, 0)$ to $(2n, 0)$, where no edge lies below the x axis.

For our q-analogs we use the three classical statistics: The "down set" $D(w)$ of a word $w = w_1 \cdots w_n$ is defined as

$$D(w) = \{i : w_i > w_{i+1},\ 1 \leqslant i \leqslant n-1\}.$$
$$\operatorname{maj} w = \textstyle\sum \{i : i \in D(w)\}$$
$$\operatorname{inv} w = |\{(i, j) : i < j \text{ and } w_i > w_j\}|$$
$$\operatorname{des} w = |D(w)|.$$

We will also use the standard q-notation

$$[n] = \frac{q^n - 1}{q - 1}$$

$$(x; q)_n = (x)_n = (1-x)(1-qx) \cdots (1-q^{n-1}x) = \sum_{k=0}^{n} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix} (-x)^k$$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q)_n}{(q)_k\,(q)_{n-k}}$$

and the well-known result [2, Chap. 3.4; 6, p. 266; 19, Vol. 2, p. 206]

$$\sum_{w \in S(n,m)} q^{\operatorname{inv} w} = \sum_{w \in S(n,m)} q^{\operatorname{maj} w} = \begin{bmatrix} n+m \\ n \end{bmatrix}.$$

2. Carlitz's q-Catalan Numbers

In analogy to expansion (1.6) we define q-Catalan numbers $C_n = C_n(q)$ as in [1] by means of

$$z = \sum_{n=1}^{\infty} C_{n-1} z^n (1-z)(1-qz) \cdots (1-q^{n-1}z). \tag{2.1}$$

Then we obtain

$$\begin{aligned} z^2 &= \sum_{k \geqslant 1} C_{k-1} z^k (z)_k q^{-k} \cdot q^k z \\ &= \sum_{k \geqslant 1} C_{k-1} z^k (z)_k q^{-k} \sum_{l \geqslant 1} C_{l-1} (q^k z)^l (q^k z)_l \\ &= \sum_{n \geqslant 2} \left(\sum_{k+l=n} C_{k-1} C_{l-1} q^{k(l-1)} \right) z^n (z)_n. \end{aligned}$$

Rewriting (2.1) as

$$z = C_0 z(1-z) + \sum_{n \geqslant 2} C_{n-1} z^n (z)_n$$

gives $C_0 = 1$ and another expansion of z^2. Comparing coefficients leads to

$$C_{n-1} = \sum_{k+l=n-2} C_k C_l q^{(k+1)l}$$

or

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k} q^{(k+1)(n-k)}, \qquad C_0 = 1. \tag{2.2}$$

Writing

$$\tilde{C}_n(q) = q^{\binom{n}{2}} C_n(q^{-1}) \tag{2.3}$$

we obtain the simplest possible q-analog of the classical recurrence relation (1.2) for the Catalan numbers

$$\tilde{C}_{n+1} = \sum_{k=0}^{n} q^k \tilde{C}_k \tilde{C}_{n-k}. \tag{2.4}$$

The first values are

$$\begin{gathered} C_0 = C_1 = 1, \qquad C_2 = 1 + q, \qquad C_3 = 1 + q + 2q^2 + q^3, \\ C_4 = 1 + q + 2q^2 + 3q^3 + 3q^4 + 3q^5 + q^6. \end{gathered}$$

A simple explicit formula like (1.1) is not known. The analog of the second expansion (1.5) leads essentially to the same q-Catalan numbers

$$z = \sum_{n=1}^{\infty} q^{\binom{n-1}{2}} C_n \frac{z^n}{(1+z)(1+qz)\cdots(1+q^{2n-1}z)}. \tag{2.5}$$

Proof. We replace z by qz in (2.5), divide by q, add $C_0=1$ on both sides and divide by $1+z$ to obtain

$$1=\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}C_n z^n}{(-z;q)_{2n+1}}.$$

Now we "square" this equation in the same manner as we have done above with (2.1):

$$\begin{aligned}1&=\sum_{k\geqslant 0} q^{\binom{k}{2}}\frac{C_k z^k}{(-z;q)_{2k+1}}\cdot\sum_{l\geqslant 0} q^{\binom{l}{2}}\frac{C_l(q^{2k+1}z)^l}{(-q^{2k+1}z;q)_{2l+1}}\\&=\sum_{n\geqslant 0} q^{\binom{n}{2}}\frac{z^n}{(-z;q)_{2n+2}}\sum_{k=0}^{n} C_k C_{n-k} q^{(k+1)(n-k)}.\end{aligned}$$

After a multiplication with z we can compare coefficients with (2.5) which gives exactly (2.2). So we see that the numbers C_n defined by (2.5) satisfy the recurrence relation (2.2) and hence coincide with our q-Catalan numbers. ∎

We now turn to the combinatorial meaning of these q-Catalan numbers,

$$C_n=\sum_{w\in\mathscr{C}_n} q^{\operatorname{inv} w}. \tag{2.6}$$

For the proof we decompose a Catalan word $w\in\mathscr{C}_{n+1}$ in the usual way into $w=0w_1 1w_2$ with $w_1\in\mathscr{C}_k$, $w_2\in\mathscr{C}_{n-k}$ for some k with $0\leqslant k\leqslant n$. Then the number of inversions in w is given by

$$\operatorname{inv} w=\operatorname{inv} w_1+\operatorname{inv} w_2+(k+1)(n-k).$$

Hence $\sum\{q^{\operatorname{inv} w}: w\in\mathscr{C}_n\}$ satisfies the same recursion (2.2) as C_n, which completes the proof of (2.6).

Geometrically the inversion number of w means the area of the polygon which lies between the lattice path of w and that of the word $0\cdots 01\cdots 1$ without inversions. Viewing this polygon as the Ferrers graph of a partition (see [2]) gives an interpretation of the Catalan numbers $C_n(q)=\sum_m C_{nm}q^m$ in terms of partitions: C_{nm} is the number of partitions $(\lambda_1, \lambda_2,..., \lambda_r)$ of m with $\lambda_i\leqslant n-i+1$. From this we may infer, e.g., the asymptotic formula for

$$|q|<1: C_n(q)\to\prod_{i=0}^{\infty}(1-q^i)^{-1} \qquad \text{as} \quad n\to\infty, \tag{2.7}$$

i.e., the q-Catalan numbers $C_n(q)$ converge to the partition function. For the $\tilde{C}_n(q)$ the exponent of q counts the area between the paths of w and of $0101\cdots 01$.

Another combinatorial model for Catalan numbers is described in Knuth [15, p. 238f, Exs. 2–5]; he considers a stack where at each of $2n$ steps either one of the numbers $1, 2, \ldots, n$ (in this order) is put onto its top or its top item is taken off. The output of such an operation gives an arrangement or permutation $p = p_1 p_2 \cdots p_n$ of the numbers $1, 2, \ldots, n$, which is characterized by the following property: Each descending block of p is "complete" in the sense that it contains all numbers between the smallest and the largest element of this block that have not appeared before; or more formally: there are no indices $i<j<k$ such that $p_j < p_k < p_i$. Let $\mathscr{C}_n^*$ be the set of these permutations. Coding a "put next number onto" by 0 and "take top number off" by 1, we obtain a bijection between *Catalan permutations* $\mathscr{C}_n^*$ and Catalan words $\mathscr{C}_n$. Now a representation in terms of permutations leads immediately to q-analogs by using the standard q-statistics. In fact, it is easy to see that

$$\tilde{C}_n(q) = \sum_{\sigma \in \mathscr{C}_n^*} q^{\operatorname{inv} \sigma}, \tag{2.8}$$

i.e., the Carlitz q-Catalan numbers count also inversions of Catalan permutations.

At this point a remark on a related number sequence, the *Bell numbers* (see [6, 13]) seems appropriate. Obviously $\mathscr{C}_n^*$ is contained in the set $\mathscr{B}_n$ of those permutations where every basic component consists of only one decreasing block. Such permutations represent partitions by writing elements within each block in decreasing order and ordering the blocks along increasing largest (= first) elements. Then

$$B_n(q) = \sum_{\sigma \in \mathscr{B}_n} q^{\operatorname{inv} \sigma} \tag{2.9}$$

gives a q-analog of the Bell numbers $B_n = |\mathscr{B}_n|$, which fit with Gessel's q-exponential formula [11]: Their generating function is

$$\sum_{n=0}^{\infty} \frac{B_n(q)}{[n]!} t^n = e_q[e_{1/q}(t) - 1] \tag{2.10}$$

and they satisfy the recurrence relation

$$B_{n+1}(q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k+1}{2}} B_{n-k}(q). \tag{2.11}$$

There are a lot of further interesting results on Carlitz's q-Catalan numbers, in particular for their generating function $f(z) = \sum_{n=0}^{\infty} \tilde{C}_n(q) z^n$, which satisfies the relation $f(z) = 1 + zf(z) f(qz)$ and can be written as a continued fraction of Ramanujan (see [1, 3, 4, 5, 10, 24, 26]).

3. THE q-CATALAN NUMBERS $c_n(\lambda; q)$

For expansions like (1.5) there exists another q-analog than (2.5), for which an explicit q-Lagrange formula has been found recently and independently by Krattenthaler [16, 17] and Gessel and Stanton [12] (see also [14] for a unified approach to their theorems).

THEOREM. *If*

$$f(z)=\sum_{n=0}^{\infty} \frac{c_n z^n}{(1-q^{-n}z)\cdots(1-q^{-1}z)(1-az)\cdots(1-q^{n-1}az)}$$
$$=\sum_{n=0}^{\infty} \frac{c_n z^n}{(q^{-1}z;q^{-1})_n\,(az;q)_n},$$

then

$$c_n=\frac{1}{[n]} f'(z)(1-q^{-(n-1)}z)\cdots(1-z)(1-az)\cdots(1-q^{n-1}az)\,|_{z^{n-1}},$$

where $f'(z)$ is the q-derivative $(f(qz)-f(z))/(q-1)z$ of $f(z)$ and $g(z)\,|_{z^n}$ denotes the coefficient of z^n in the formal power series $g(z)$.

Sketch of the proof. Set $g_n(z)=z^n/(q^{-n}z)_n\,(az)_n$. Then the theorem is equivalent to the orthogonality relation

$$g_n'(z)(q^{1-k}z)_k\,(az)_k\,|_{z^{k-1}}=[n]\,\delta_{nk}$$

which is obtained by calculating the q-derivative of g_k/g_n and looking at its residue. ∎

This motivates us to define a new kind of q-Catalan numbers $c_n(\lambda; q)=c_n(\lambda)$ by means of the expansion formula

$$z=\sum_{n=1}^{\infty} \frac{c_n(\lambda;q)\,z^n}{q^{\binom{n}{2}}(-q^{-n}z)_n\,(-q^{\lambda}z)_n}. \tag{3.1}$$

Using the above q-Lagrange formula, we obtain

$$q^{-\binom{n}{2}}c_n(\lambda)=\frac{1}{[n]}\,(-q^{-n+1}z)_n\,(-q^{\lambda}z)_n\,|_{z^{n-1}} \tag{3.2}$$
$$=\frac{1}{[n]}\sum_k \begin{bmatrix} n\\ k\end{bmatrix} q^{\binom{k}{2}}(q^{-n+1}z)^k \sum_l \begin{bmatrix} n\\ l\end{bmatrix} q^{\binom{l}{2}}(q^{\lambda}z)^l\,\Big|_{z^{n-1}}$$
$$=\frac{1}{[n]}\sum_k \begin{bmatrix} n\\ k\end{bmatrix}\begin{bmatrix} n\\ n-k-1\end{bmatrix} q^{\binom{k}{2}-k(n-1)+\lambda(n-1-k)+\binom{n-1-k}{2}}$$

and finally the explicit formula

$$c_n(\lambda)=\frac{1}{[n]}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}n\\k+1\end{bmatrix}q^{k^2+\lambda k}. \tag{3.3}$$

The terms in this sum are q-analogs of the *Runyon numbers* $r_{nk}=(1/n)\binom{n}{k}\binom{n}{k+1}$ from [23, p. 17]. They count the lattice paths w in $\mathscr{C}_n$ with k "valleys" or $k+1$ "peaks," i.e., des $w=k$.

For certain values of λ, (3.2) may be evaluated in a simpler way. In particular, we obtain for $\lambda=1$,

$$\begin{aligned}c_n(1)&=q^{\binom{n}{2}}\frac{1}{[n]}(-zq^{1-n})_{2n}\Big|_{z^{n-1}}\\&=q^{\binom{n}{2}}\frac{1}{[n]}\begin{bmatrix}2n\\n-1\end{bmatrix}q^{(-n+1)(n-1)+\binom{n-1}{2}}\\&=\frac{1}{[n+1]}\begin{bmatrix}2n\\n\end{bmatrix}\end{aligned}\tag{3.4}$$

which is the most obvious q-analog of the Catalan numbers. For $\lambda=0$ a similar calculation gives

$$c_n(0)=\frac{1}{[n+1]}\begin{bmatrix}2n\\n\end{bmatrix}\frac{1+q}{1+q^n}=\frac{[2]}{[n+1]}\begin{bmatrix}2n-1\\n\end{bmatrix}=\frac{[2]}{[2n]}\begin{bmatrix}2n\\n-1\end{bmatrix}.$$

The first values are given by

$$\begin{aligned}&c_0(\lambda)=c_1(\lambda)=1,\qquad c_2(\lambda)=1+q^{1+\lambda},\\&c_3(\lambda)=1+[3]\,q^{1+\lambda}+q^{4+2\lambda},\ldots.\end{aligned}$$

As for Carlitz's q-Catalan numbers, the $c_n(\lambda)$ have a nice combinatorial interpretation:

$$c_n(\lambda)=\sum_{w\in\mathscr{C}_n}q^{\operatorname{maj}w+(\lambda-1)\operatorname{des}w}. \tag{3.5}$$

We will first present a simple proof of the important special case $\lambda=1$ and defer the proof of the general case to the next sections. Using an idea similar to the reflection principle we can state

LEMMA. *There exists a bijection* $\varphi\colon S_-(n,n)\to S(n+1,n-1)$ *which satisfies* maj $\varphi(w)=$ maj $w-1$.

Proof. Given a path $w\in S_-(n,n)$ we determine the first of its "deepest" points. Call it P and let P' be the lattice point on w before P. Now we tip up the decreasing piece $P'P$ (coded by a 1) into an ascending one (coded by a 0), add the remainder (shifted upwards for two units) and call this path $\varphi(w)$. Apparently $\varphi(w) \in S(n+1, n-1)$ and $\operatorname{maj} \varphi(w) = \operatorname{maj} w - 1$.

It is easy to see that this map is a bijection: Starting with $w' \in S(n+1, n-1)$ we find the critical point P' to be now the rightmost of the "deepest" points. ∎

This implies

$$\sum_{w \in S_-(n,n)} q^{\operatorname{maj} w} = \sum_{w' \in S(n+1,n-1)} q^{\operatorname{maj} w' + 1} = q \begin{bmatrix} 2n \\ n+1 \end{bmatrix}.$$

Hence

$$\sum_{w \in \mathscr{C}_n} q^{\operatorname{maj} w} = \begin{bmatrix} 2n \\ n \end{bmatrix} - q \begin{bmatrix} 2n \\ n+1 \end{bmatrix} = \frac{1}{[n+1]} \begin{bmatrix} 2n \\ n \end{bmatrix}$$

which concludes the proof of (3.5) for $\lambda = 1$.

Of course the same works also for $S_-(n, m)$ and $S(n+1, m-1)$ which would give q-ballot numbers. These results are due to MacMahon [19, Vol. 2, p. 214; and 20, p. 1345], who called elements of $S_+(n, m)$ "lattice permutations," but we could not find the above simple combinatorial proof in the literature. Another proof was given by Aissen [27].

4. q-Runyon Numbers and a Theorem of MacMahon

Obviously (3.5) is equivalent with

$$\sum \{q^{\operatorname{maj} w}: w \in \mathscr{C}_n, \operatorname{des} w = k\} = \frac{1}{[n]} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n \\ k+1 \end{bmatrix} q^{k^2+k}. \tag{4.1}$$

As Gessel has pointed out to us, this is actually a special case of the following result on "sublattice permutations" of MacMahon, stated in [20, p. 1429]: Let

$$S(a, b; k) = \{w \in S(a, b): \operatorname{des} w = k\}$$

$$S_\pm(a, b; k) = S(a, b; k) \cap S_\pm(a, b)$$

and $M(a, b; k) = \sum_{w \in S(a,b;k)} q^{\operatorname{maj} w}$ and analogously M_+, M_-. Then for $a \geqslant b$,

$$M_+(a, b; k) = q^{k^2+k} \frac{[a-k+1] \cdots [a-1][b-k+1] \cdots [b]}{[k]!\,[k+1]!} \times ([a-k] + q^{b-k+1}[a-b][k]) \tag{4.2}$$

For $a=b=n$, (4.2) reduces to (4.1). Since MacMahon stated this formula without proof, we take the opportunity to present two proofs of (4.2). An elementary calculation shows that (4.2) may be written in the more suggestive form

$$M_+(a,b;k)=q^{k^2}\left(\begin{bmatrix}a\\k\end{bmatrix}\begin{bmatrix}b\\k\end{bmatrix}-\begin{bmatrix}a-1\\k-1\end{bmatrix}\begin{bmatrix}b+1\\k+1\end{bmatrix}\right). \tag{4.3}$$

Proof. Looking at the last letter of a word $w\in S_+(a,b;k)$ we observe

$$S_+(a,b;k)=S_+(a-1,b;k)\,0+S_+(a,b-1;k)\,1-S_+(a-1,b-1;k)\,10$$
$$+S_+(a-1,b-1;k-1)\,10$$

for $a>b$ and

$$S_+(a,a;k)=S_+(a,a-1;k).$$

This implies the recurrence relation

$$M_+(a,b;k)=M_+(a-1,b;k)+M_+(a,b-1;k)-M_+(a-1,b-1;k)$$
$$+q^{a+b-1}M_+(a-1,b-1;k-1) \qquad \text{for} \quad a>b, \tag{4.4}$$

and

$$M_+(a,a;k)=M_+(a,a-1;k). \tag{4.5}$$

It is now straightforward to verify these relations for (4.3). ∎

We note that relation (4.4) is satisfied by $q^{k^2}\left[\begin{smallmatrix}a-s\\k-s\end{smallmatrix}\right]\left[\begin{smallmatrix}b+s\\k+s\end{smallmatrix}\right]$ for any fixed s and holds also for $M(a,b;k)$ and $M_-(a,b;k)$, but with a different boundary condition instead of (4.5). From this we can conclude the following refinement of (4.3):

$$M(a,b;k)=q^{k^2}\begin{bmatrix}a\\k\end{bmatrix}\begin{bmatrix}b\\k\end{bmatrix} \tag{4.6}$$

$$M_-(a,b;k)=q^{k^2}\begin{bmatrix}a-1\\k-1\end{bmatrix}\begin{bmatrix}b+1\\k+1\end{bmatrix} \qquad \text{for} \quad a\geqslant b. \tag{4.7}$$

We now present a more direct proof of (4.6): Given a word $w\in S(a,b;k)$ we count for each of the k descents of w the number of 0s and 1s to the left of it. Obviously these two sequences $(A_i)_{1\leqslant i\leqslant k}$ and $(B_i)_{1\leqslant i\leqslant k}$, which satisfy

$$0\leqslant A_1<A_2<\cdots<A_k<a \tag{4.8}$$

$$0<B_1<B_2<\cdots<B_k\leqslant b, \tag{4.9}$$

determine the given word $w \in S(a, b; k)$ uniquely, and

$$\operatorname{maj} w = \sum_{i=1}^{k} (A_i + B_i).$$

But now it is well known (see, e.g., [2]) that

$$\sum q^{A_1 + \cdots + A_k} = q^{\binom{k}{2}} \begin{bmatrix} a \\ k \end{bmatrix} \quad \text{and} \quad \sum q^{B_1 + \cdots + B_k} = q^{\binom{k+1}{2}} \begin{bmatrix} b \\ k \end{bmatrix},$$

where the sums are taken over all sequences satisfying (4.8) (resp. (4.9)). This proves (4.6). ▮

A similar but more technical argument is possible for (4.7), like Krattenthaler's [16] related direct proof of (3.5).

5. A Generalization

In this section we give a further proof of (3.5), now using recurrence relations analogous to (1.2), similar to the simple proof of (2.6). This method works in a more general case: We refine the maj-statistic in the following way: For a word $w = w_1 w_2 \cdots w_{n+m} \in S(n, m)$ set

$$\alpha(w) = \sum_{i \in D(w)} |\{ j \leqslant i : w_j = 0 \}|$$

$$\beta(w) = \sum_{i \in D(w)} |\{ j \leqslant i : w_j = 1 \}|,$$

where $D(w)$ is the down set of w. Obviously $\alpha(w) + \beta(w) = \operatorname{maj} w$.

Now we can state our main result:

THEOREM. *The following statements are equivalent*:

$$C_n(x; a, b) = \sum_{w \in \mathscr{C}_n} x^{\operatorname{des} w} a^{\alpha(w)} b^{\beta(w)}; \tag{5.1}$$

$$z = \sum_{n=1}^{\infty} \frac{a^{-\binom{n}{2}} C_n(x; a, b)\, z^n}{(1 + a^{-1}z) \cdots (1 + a^{-n}z)(1 + xbz) \cdots (1 + xb^n z)}; \tag{5.2}$$

$$C_{n+1}(x) = C_n(xa) + x \sum_{k=0}^{n-1} (ab)^{k+1}\, C_k(xa)\, C_{n-k}(x(ab)^{k+1}), \quad C_0 = 1; \tag{5.3}$$

$$C_{n+1}(x) = C_n(xa) + x \sum_{k=1}^{n} (ab)^k\, C_k(x)\, C_{n-k}(xa^{k+1}b^k), \qquad C_0 = 1; \tag{5.4}$$

$$C_{n+1}(x) = C_n(xab) + xb \sum_{k=0}^{n-1} a^{k+1} C_k(xab)\, C_{n-k}(x(ab)^{k+1}), \quad C_0 = 1; \tag{5.5}$$

where we have written $C_n(x)$ instead of $C_n(x; a, b)$ for short.

Proof. (5.1)⇔(5.3). Again we decompose a word $w \in \mathscr{C}_{n+1}$ into $w = 0w_1 1w_2$ with $w_1 \in \mathscr{C}_k$ and $w_2 \in \mathscr{C}_{n-k}$ for some k with $0 \leqslant k \leqslant n$. Then for $k < n$ we have

$$\operatorname{des} w = \operatorname{des} w_1 + 1 + \operatorname{des} w_2$$

$$\alpha(w) = \alpha(w_1) + \operatorname{des} w_1 + (k+1) + \alpha(w_2) + (k+1) \operatorname{des} w_2$$

$$\beta(w) = \beta(w_1) + (k+1) + \beta(w_2) + (k+1) \operatorname{des} w_2;$$

and for $k = n$, $\operatorname{des} w = \operatorname{des} w_1$, $\alpha(w) = \alpha(w_1) + \operatorname{des} w_1$, and $\beta(w) = \beta(w_1)$. This gives immediately the recurrence relation (5.3).

(5.1)⇔(5.4). This is proved in the same way using the decomposition $w = w_1 0 w_2 1$.

(5.2)⇔(5.3). Adding 1 and multiplying by $z/(1+z)$ both sides of (5.2) gives

$$z = \sum_{n=0}^{\infty} \frac{a^{-\binom{n}{2}} C_n(x)\, z^{n+1}}{(1+z)\cdots(1+a^{-n}z)(1+xbz)\cdots(1+xb^n z)}.$$

Now we add a further factor $1 + xb^{n+1}z = 1 + xa^n b^{n+1}(za^{-n})$, replace za^{-n} by the series (5.2) with $xa^n b^{n+1}$ instead of x and obtain

$$z = \sum_{n=0}^{\infty} \frac{a^{-\binom{n}{2}} C_n(x)\, z^{n+1}}{(-z; a^{-1})_{n+1}\, (-xbz; b)_{n+1}}$$

$$\times \left[1 + xa^n b^{n+1} \sum_{j=1}^{\infty} \frac{a^{-\binom{j}{2}} C_j(xa^n b^{n+1})\, z^j a^{-nj}}{(-za^{-n-1}; a^{-1})_j\, (-xb^{n+2}z; b)_j} \right]$$

$$= \sum_{n=0}^{\infty} \frac{a^{-\binom{n}{2}} z^{n+1}}{(-z; a^{-1})_{n+1}\, (-xbz; b)_{n+1}}$$

$$\times \left[C_n(x) + x \sum_{k=0}^{n-1} a^k b^{k+1} C_k(x)\, C_{n-k}(xa^k b^{k+1}) \right].$$

Comparing the coefficients in this series again with (5.2) (after replacing z by az), we obtain the recurrence (5.3).

(5.2)⇔(5.5). This is proved in the same way by adding the factor $1 + xz$ instead of $1 + xb^{n+1}z$.

Remarks. (1) We do not have direct proofs of any other combinations.

(2) Obviously this 3-variate version of the Catalan numbers extends the q-analog of Section 3 by setting $a = b = q$:

$$C_n(q^{\lambda-1}; q, q) = c_n(\lambda; q). \tag{5.6}$$

(5.1) is just (3.5), (5.2) is (3.1), and (5.3)–(5.5) are recurrence relations extending (1.2).

(3) Surprisingly the Carlitz q-Catalan numbers also are covered by them, as we recognize from (5.2) by setting $a=q^{-2}$, $b=q^2$, $x=q^{-1}$, and comparing with (2.5):

$$C_n(q^{-1}; q^{-2}, q^2) = q^{-\binom{n}{2}} C_n(q) \tag{5.7}$$

or

$$\tilde{C}_n(q) = C_n(q; q^2, q^{-2}).$$

But then (5.1) implies that the two statistics "$\binom{n}{2}-\text{inv}$" and "$2\alpha - 2\beta + \text{des}$" have the same distribution on $\mathscr{C}_n$. One can show that they are also equally distributed on $S(n, n)$. We will give a "bijective proof" of this fact in the next section.

(4) It would be interesting to extend the above theorem further by including the inversions in (5.1).

(5) We suppose that most of the things that are known for the statistics (des, maj) can be extended to (des, α, β). E.g., it is easy to generalize (4.6) to

$$\sum_{w \in S(n,m)} s^{\text{des}\, w} p^{\alpha(w)} q^{\beta(w)} = \sum_{k \geq 0} s^k p^{\binom{k}{2}} q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_p \begin{bmatrix} m \\ k \end{bmatrix}_q$$

which is the answer to a general Simon Newcomb problem [2, 6, 19] for two different letters.

6. Polygons and the Polya–Gessel q-Catalan Numbers

Until now we have formulated the combinatorics of our q-Catalan numbers only in terms of 0–1-words and lattice paths. In this last section we describe a further combinatorial model, which is the basis for a q-analog of the Catalan numbers introduced by Polya [22] and Gessel [10].

Following Gessel [10], we consider pairs of lattice paths in the plane, each path starting at the origin, but now consisting of unit horizontal and vertical steps in the positive direction.

Let $\mathscr{P}_{n,j}$ be the set of such path-pairs (π, σ) with the following properties:

(i) both π and σ end at the point $(j, n-j)$,

(ii) π begins with a unit vertical step and σ with a horizontal,

(iii) π and σ do not meet between the origin and their common endpoint.

Obviously the elements of $\mathscr{P}_n = \bigcup_{j=1}^n \mathscr{P}_{n,j}$ are polygons with circumference $2n$. It is well known that $|\mathscr{P}_{n+1}| = C_n$, and $|\mathscr{P}_{n+1,j}| = (1/n)\binom{n}{j}\binom{n}{j-1}$, the Runyon numbers (see [18, 21, 25]). Gessel [10] considered more generally

$$P_n(s; q) = \sum_{j=1}^{n} \sum_{(\pi,\sigma) \in \mathscr{P}_{n,j}} q^{A(\pi,\sigma)} s^j, \tag{6.1}$$

where $A(\pi, \sigma)$ denotes the area enclosed by the polygon (π, σ), and studied their generating function in detail. Polya [22] had considered the case $s = 1$.

We are now going to show that also these q-Catalan numbers can be subsumed under the 3-variate $C_n(x; a, b)$ of Section 5. To this end remember the bijection between words in $S(n, n, k)$ and pairs of sequences

$$\begin{aligned} &0 \leqslant A_1 < A_2 < \cdots < A_k < n \\ &0 < B_1 < B_2 < \cdots < B_k \leqslant n \end{aligned} \tag{6.2}$$

we have constructed in (4.8) and (4.9). With this bijection the subset $S_+(n, n, k)$ corresponds to sequences with $A_i \geqslant B_i$. Now set $\alpha_i = A_i - i + 1$, $\beta_i = B_i - i$ for $i = 1, ..., k$, so that (6.2) is equivalent to

$$\begin{aligned} &0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_k \leqslant n-k \\ &0 \leqslant \beta_1 \leqslant \beta_2 \leqslant \cdots \leqslant \beta_k \leqslant n-k \end{aligned} \tag{6.3}$$

and $\alpha_i > \beta_i$.

We now assign to (6.3) two lattice paths from $(0, 0)$ to $(k+1, n-k)$: The one, which we call π, joins the lattice points (i, α_i) to (i, α_{i+1}) and (i, α_{i+1}) to $(i+1, \alpha_{i+1})$, and the other, σ, joins (i, β_i) to $(i+1, \beta_i)$ and $(i+1, \beta_i)$ to $(i+1, \beta_{i+1})$ for $i = 0, ..., k$, where $\alpha_0 = \beta_0 = 0$ and $\alpha_{k+1} = \beta_{k+1} = n-k$. Then $\alpha_i > \beta_i$ just means the above condition (iii). Obviously,

$$\operatorname{des} w = k$$

$$\alpha(w) = A_1 + \cdots + A_k = \alpha_1 + \cdots + \alpha_k + \binom{k}{2} = A(\pi) - n + \binom{k+1}{2} \tag{6.4}$$

$$\beta(w) = B_1 + \cdots + B_k = \beta_1 + \cdots + \beta_k + \binom{k+1}{2} = A(\sigma) + \binom{k+1}{2},$$

where $A(\pi)$ denotes the area below the path π in the positive quadrant. Thus we have found a bijection between $\mathscr{C}_n$ and $\mathscr{P}_{n+1}$ which translates our three statistics (des, α, β) in a simple way. Hence we can interprete the results of Section 5 in terms of polygons. In particular, (6.4) implies

$A(\pi, \sigma) = A(\pi) - A(\sigma) = \alpha(w) - \beta(w) + n$, so that comparing (6.1) with (5.1) leads to the desired result

$$P_{n+1}(s; q) = sq^n C_n(s; q, q^{-1}). \tag{6.5}$$

Hence our 3-variate $C_n(x; a, b)$ reduce to the Polya–Gessel q-Catalan numbers for $ab = 1$. Looking at (5.7) we observe that for $s = q^{1/2}$ we obtain Carlitz's q-analog again. This is shown also by the following combinatorial argument:

As in [10] we represent a path-pair $(\pi, \sigma) \in \mathscr{P}_{n,j}$ as a sequence of pairs of steps: let v be a vertical step and h the horizontal step. Then we write the pair (π, σ) with $\pi = a_0 a_1 \cdots a_n$, $\sigma = b_0 \cdots b_n$, each a_i and b_i being a v or h, as the sequence of step-pairs $(a_0 b_0) \cdots (a_n b_n)$. In order to find a bijection with $\mathscr{C}_n$, we code a sequence of step-pairs as a word in 0, 1 as follows:

$$(v, h) \to 0\ 0 \qquad (v, v) \to 1\ 0$$
$$(h, v) \to 1\ 1 \qquad (h, h) \to 0\ 1.$$

Omitting one "0" at the beginning at a "1" at the end, we obtain a word w in $\mathscr{C}_n$. It is easy to see that this encoding is bijective and that

$$2A(\pi, \sigma) - 2n + j - 1 = \binom{n}{2} - \operatorname{inv} w$$

which is essentially the area below w, as we observed in Section 2. This implies

$$\tilde{C}_n(q) = \sum_{(\pi,\sigma) \in \mathscr{P}_{n+1}} q^{2A(\pi,\sigma) - 2n + j - 1} = q^{-2n-1} P_{n+1}(q; q^2),$$

the connection between Carlitz's and Polya–Gessel's q-Catalan numbers.

Together with (6.5) this shows

$$\tilde{C}_n(q) = C_n(q; q^2, q^{-2})$$

which gives the promised purely combinatorial proof of (5.7) by combining the above two bijections between $\mathscr{C}_n$ and $\mathscr{P}_{n+1}$.

We should remark that for the Polya–Gessel q-Catalan numbers our Lagrange expansion (5.2) is equivalent to Gessel's results on their generating function if we apply the q-inversion theorem of Garsia [8, Theorem 1.1].

We conclude with the following nice result, which was found by Schwärzler, a student of our institute, and which can be derived from (5.2):

(i) The sum of the areas of the polygons in $\mathscr{P}_{n+1}$ is exactly 4^{n-1}.

(ii) (Conjecture) These polygons may be put together to a square of side 2^{n-1}.

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