SIAM J. MATH. ANAL.
Vol. 19, No. 6, November 1988

018

A COMBINATORIAL INTERPRETATION OF THE INTEGRAL OF THE PRODUCT OF LEGENDRE POLYNOMIALS*

J. GILLIS†, J. JEDWAB‡, AND D. ZEILBERGER§

Abstract. Denote by $P_n(x)$ the Legendre polynomial of degree n and let

$$I_{n_1,\cdots,n_k}=\int_{-1}^{1} P_{n_1}(x)\cdots P_{n_k}(x)\,dx.$$

$I_{n_1,\cdots,n_k}$ is written as a sum involving binomial coefficients and the sum is interpreted via a combinatorial model. This makes possible a combinatorial proof of a number of general theorems concerning $I_{n_1,\cdots,n_k}$, not all of which seem analytically straightforward, including a direct combinatorial derivation of the known formula for $I_{a,b,c}$ and the expression of $I_{a,b,c,d}$ as a simple finite sum. In addition, a number of apparently new combinatorial identities are obtained.

Key words. Legendre polynomials, integrals, digraph

AMS(MOS) subject classifications. primary 33A45; secondary 05A10

1. Introduction. We will be concerned with the Legendre polynomials, defined by

$$P_n(x)=2^{-n}\sum_{\alpha\leqq n/2}(-1)^{\alpha}\binom{n}{\alpha}\binom{2n-2\alpha}{n}x^{n-2\alpha}\qquad(-1\leqq x\leqq 1;\ n=0,1,2,\cdots),$$

which may be written in the equivalent form [4, p. 38]

$$P_n(x)=2^{-n}\sum_{\alpha}\binom{n}{\alpha}^2(x+1)^{\alpha}(x-1)^{n-\alpha}. \tag{1}$$

In (1), as in other combinatorial sums in what follows, we shall omit the limits of summation where these coincide with the natural cut-offs implied by the fact that $\binom{a}{b}=0$ wherever a, b are integers and $b>a>0$ or $a>0>b$.

Let

$$I_{n_1,\cdots,n_k}=\int_{-1}^{1} P_{n_1}(x)\cdots P_{n_k}(x)\,dx, \tag{2}$$

where $n_1,\cdots,n_k$ are nonnegative integers. We will express $I_{n_1,\cdots,n_k}$ as a sum involving binomial coefficients and use a combinatorial interpretation of this sum to derive a number of analytical and combinatorial results.

To simplify notation, write $\underset{\sim}{n}=(n_1,\cdots,n_k)$ and $\underset{\sim}{\alpha}=(\alpha_1,\cdots,\alpha_k)$. It is convenient to write $x=2y-1$ in (1) to obtain

$$P_n(x)=p_n(y)=\sum_{\alpha}\binom{n}{\alpha}^2 y^{\alpha}(y-1)^{n-\alpha}\qquad(0\leqq y\leqq 1),$$

which on substitution in (2) gives

$$\begin{aligned}I_{\underset{\sim}{n}}&=2\sum_{\underset{\sim}{\alpha}}\binom{n_1}{\alpha_1}^2\cdots\binom{n_k}{\alpha_k}^2\int_0^1 y^{\Sigma\alpha_i}(y-1)^{\Sigma n_i-\Sigma\alpha_i}\,dy\\ &=2\sum_{\underset{\sim}{\alpha}}\binom{n_1}{\alpha_1}^2\cdots\binom{n_k}{\alpha_k}^2\left\{\frac{(-1)^{\Sigma\alpha_i}(\Sigma\alpha_i)!\,(\Sigma n_i-\Sigma\alpha_i)!}{(1+\Sigma n_i)!}\right\}\end{aligned} \tag{3}$$

* Received by the editors March 25, 1987; accepted for publication (in revised form) March 1, 1988.

† Department of Applied Mathematics, The Weizmann Institute of Science, Rehovot 76100, Israel.
‡ Emmanuel College, Cambridge, United Kingdom.
§ Department of Mathematics, Drexel University, Philadelphia, Pennsylvania 19104.

$$(4)\qquad =\frac{2}{1+\Sigma n_i}\cdot\sum_{\alpha}(-1)^{\Sigma\alpha_i}\left(\binom{n_1}{\alpha_1}^2\cdots\binom{n_k}{\alpha_k}^2\Big/\binom{\Sigma n_i}{\Sigma\alpha_i}\right).$$

Now consider a set of elements of k different types, ordered by type number i $(i=1,\cdots,k)$ and, within each type number, by a serial number r $(r=1,\cdots,n_i)$. We represent these by points and form a directed graph by connecting them, with one edge going into and one coming out of each of the points. We then color each of the Σn_i edges blue or yellow according to the following *balance condition*:

(*) For each i the number of points of type i at the beginning of blue edges, α_i (say), equals the number at the end of blue edges.

Call each such colored graph a *system*, and let T denote the set of all possible distinct systems. Class a system as *even* or *odd* according to the parity of the total number of blue edges, $\sum_{i=1}^{k}\alpha_i$. Let the difference between the number of even and odd systems, in any subset E of T, be $\Pi_{\underline{n}}(E)$. Clearly,

$$(5)\qquad \begin{aligned}\frac{2}{(1+\Sigma n_i)!}\Pi_{\underline{n}}(T)&=\frac{2}{(1+\Sigma n_i)!}\left\{\sum_{\alpha}(-1)^{\Sigma\alpha_i}\binom{n_1}{\alpha_1}^2\cdots\binom{n_k}{\alpha_k}^2(\Sigma\alpha_i)!(\Sigma n_i-\Sigma\alpha_i)!\right\}\\&=I_{\underline{n}}\end{aligned}$$

by (3).

2. Some elementary considerations. Denote the set of distinct graphs formed by omitting the coloring of each system, in any subset E of T, by E^*. We will refer to an edge beginning at a point of type i and ending at a point of type j $(1\leqq i,j\leqq k)$ as "an $i\to j$ edge," calling it *pure* if $i=j$ and *mixed* if $i\neq j$. Where desired we will indicate the edge color by $i\overset{B}{\to}j$ or $i\overset{Y}{\to}j$.

We recall that any vertex is characterized by a pair of natural numbers (i,j) where i is the number of the type to which it belongs and j $(1\leqq j\leqq n_i)$ is its serial number in that type. If two points P, P' are characterized by (i,j), (i',j'), respectively, then P is said to be of *lower rank* than P' if

$$\begin{array}{ll}\text{either} & i<i'\\ \text{or} & i=i',\quad j<j'.\end{array}$$

Now let P be the set of systems containing at least one pure edge. Given any system in P, select from among the pure edges the one beginning at the point of lowest rank and change its color. This leaves the balance condition (*) satisfied but produces a new system of opposite parity, so that the two systems together give a canceling contribution to $\Pi_{\underline{n}}(T)$. Since this process defines a (1, 1) parity-changing map from P to itself, $\Pi_{\underline{n}}(P)=0$. Writing $T\backslash P=U$, say, this is equivalent to

$$\Pi_{\underline{n}}(T)=\Pi_{\underline{n}}(U);$$

thus, we may disregard P and count only the contribution of systems in U to $\Pi_{\underline{n}}(T)$.

Now consider any graph belonging to U. Take the lowest ranking vertex and call it X^0. Since there is exactly one edge starting at each vertex, there will be a uniquely defined cycle of the form

$$(6)\qquad X^0\to X^1\to X^2\to\cdots\to X^{P-1}\to X^P(=X^0).$$

Since there are no pure edges it follows that $P\geqq 2$ and that each edge $X^i\to X^{i+1}$ is mixed. If this cycle does not cover the entire graph, let Y^0 be the lowest ranking vertex not lying on it. As before, we define a cycle

$$Y^0 \to Y^1 \to \cdots \to Y^{q-1} \to Y^q (= Y^0) \tag{7}$$

and continue until the entire graph has been covered in this way.

Consider any such cycle, e.g., (6). It can be divided up into segments each of which begins and ends with vertices of the same *type* as X^0. For example, if the cycle (6) were

$$(1,1) \to (2,5) \to (3,7) \to (1,6) \to (2,4) \to (7,1) \to (4,3) \to (1,2) \to (2,1) \to (1,1),$$

we should have the segments

$$(1,1) \to (2,5) \to (3,7) \to (1,6),$$

$$(1,6) \to (2,4) \to (7,1) \to (4,3) \to (1,2),$$

$$(1,2) \to (2,1) \to (1,1).$$

We can thus describe the graph structure by a set of segments, which we may order by the ranks of their initial vertices. A segment will be called *odd* or *even* according to the parity of the number of edges which compose it. Now let V be the set of graphs whose structures contain at least one odd segment. Suppose such a graph, G (say), contains the segment

$$Z^0 \to Z^1 \to \cdots \to Z^{r-1} \to Z^r \tag{8}$$

where r is odd, and suppose, moreover, that (8) is the lowest ranking such odd segment in this graph. We change the graph, and its coloring, according to the following rules:

(a) Change the connecting edges of (8) to produce the segment

$$Z^0 \to Z^{r-1} \to Z^{r-2} \to \cdots \to Z^1 \to Z^r. \tag{9}$$

(b) Color these new edges so that the ith edge of (9) $(1 \leqq i \leqq r)$ has the opposite color to that of the $(r+1-i)$th edge of (8).

Call the new system, with its coloring, G'. It is easily verified that G' also satisfies the rule (*). On the other hand, since r is odd, the graphs G, G' will have opposite parities. Moreover the transformation is clearly an involution. We thus have a $(1,1)$ parity changing surjection of V onto itself. It follows that

$$\Pi_{\underline{n}}(V) = 0. \tag{10}$$

Therefore if we write $W = U \setminus V$ we have

$$\Pi_{\underline{n}}(U) = \Pi_{\underline{n}}(W), \tag{11}$$

and therefore it is sufficient to construct the systems belonging to W and calculate their contribution to $\Pi_{\underline{n}}(T)$.

3. Application to the case $k=3$. We now apply these considerations to the particular case $k=3$, writing $\underline{n} = (a, b, c)$. Since the product $P_a(x)P_b(x)P_c(x)$ is a polynomial of parity equal to that of $a+b+c$, its integral will be zero for odd $a+b+c$. We therefore limit the discussion to $a+b+c=2s$, where s is an integer. Moreover it follows from the orthogonality of the polynomials that the integral will vanish unless $s \geqq \max(a, b, c)$. We proceed to study the integral under these assumptions. Let G be any graph of the set W^*, let E be the number of even systems that can be constructed by coloring G, and let Ω be the number of odd systems. By the definition of W, each such graph is made up of segments of one of the following forms:

$$\text{(12)} \qquad \begin{aligned} &\text{(i)} \quad (2\to 3\to)^{l_i}2,\\ &\text{(ii)} \quad 1\to(3\to 2\to)^{m_i}3\to 1,\\ &\text{(iii)} \quad 1\to(2\to 3\to)^{n_i}2\to 1,\end{aligned}$$

where the 1, 2, 3 indicate the types to which the vertices belong and $l_i \geqq 1$, $m_i \geqq 0$, $n_i \geqq 0$. Let the segments in each of these three classes be ordered by the rank of their initial vertex.

For each i, j $(i, j = 1, 2, 3)$ denote by E_{ij} the number of $i\to j$ edges. In any segment of type (i), (ii), or (iii) the number of $i\to j$ edges equals that of $j\to i$ edges and, hence, for the whole graph $E_{ij} = E_{ji}$. Since, by hypothesis, there are no pure edges, it follows that

$$\text{(13)} \qquad \begin{aligned} E_{12} &= E_{21} = s-c,\\ E_{13} &= E_{31} = s-b, \quad \text{and}\\ E_{23} &= E_{32} = s-a.\end{aligned}$$

To simplify the notation we write A, B, C for $s-a$, $s-b$, $s-c$, respectively. It is easily seen that the only possible distributions of colors consistent with (*) must be as shown in the following table:

	$2\to 3$	$3\to 2$	$3\to 1$	$1\to 3$	$1\to 2$	$2\to 1$
Blue	$\alpha+t$	α	$\beta+t$	β	$\gamma+t$	γ
Yellow	$A-\alpha-t$	$A-\alpha$	$B-\beta-t$	$B-\beta$	$C-\gamma-t$	$C-\gamma$

where t, α, β, γ may take any values for which the table entries are all nonnegative integers. Now the distribution of colors among the $2\to 3$ and $3\to 2$ edges is determined when we have chosen $(\alpha+t)$ of the $2\to 3$ edges and $A-\alpha$ from the $3\to 2$ edges, and this can clearly be done in $\binom{2A}{A+t}$ ways. Similar results hold for $3\to 1$ and for $1\to 2$. The total number of blue edges in any such coloring is $2(\alpha+\beta+\gamma)+3t \equiv t \pmod 2$. Hence the difference between the numbers of even and odd systems possible on any such graph is

$$\text{(14)} \qquad \begin{aligned} &\sum_t (-1)^t \binom{2A}{A+t}\binom{2B}{B+t}\binom{2C}{C+t}\\ &\quad = \frac{(2A)!(2B)!(2C)!}{(B+C)!(C+A)!(A+B)!}\sum_t (-1)^t \binom{A+B}{A+t}\binom{B+C}{B+t}\binom{C+A}{C+t}.\end{aligned}$$

But

$$\text{(15)} \qquad \sum_t (-1)^t \binom{A+B}{A+t}\binom{B+C}{B+t}\binom{C+A}{C+t} = \frac{(A+B+C)!}{A!B!C!}.$$

(For an elegant combinatorial proof of this known identity, see [3, p. 65].) Substituting into (17), we obtain

$$\sum_t (-1)^t \binom{2A}{A+t}\binom{2B}{B+t}\binom{2C}{C+t}$$

$$
\begin{aligned}
&= \frac{(2A)!(2B)!(2C)!}{(B+C)!(C+A)!(A+B)!} \cdot \frac{(A+B+C)!}{A!B!C!} \\
(16)\qquad &= \frac{(2s-2a)!(2s-2b)!(2s-2c)!s!}{a!b!c!(s-a)!(s-b)!(s-c)!} \\
&= \binom{2s-2a}{s-a}\binom{2s-2b}{s-b}\binom{2s-2c}{s-c} \cdot \frac{s!(s-a)!(s-b)!(s-c)!}{a!b!c!}.
\end{aligned}
$$

In particular the number is the same for all the graphs of W^*. It remains to determine how many such graphs there are.

Consider (12). Since there are B edges of type $1 \to 3$ in the graph, this will also be the number of segments of type (ii). Similarly there will be C segments of type (iii). Hence the total number of m_i's and n_i's is $B+C=a$. If we write $L=\Sigma l_i$ we see we have to determine the numbers L, $\{m_i\}$, $\{n_i\}$. Since $L+\Sigma m_i+\Sigma n_j$ equals the number of $2 \to 3$ edges, i.e., A, it follows that L, $\{m_i\}$, $\{n_j\}$ are the nonnegative integer solutions of

$$\sum_{i=1}^{a+1} x_i = A$$

and this number is known to be $\binom{A+a}{a} = \binom{s}{a}$.

The number of possibilities with the segments in each of (ii) and (iii) ranked in order is therefore $\binom{s}{a}/(B!C!)$. If the segments of forms (ii), (iii) are connected via vertices of type 1 (and this may be done in $a!$ ways), the graph will be determined except for the numbers l_i and the ranks of the vertices. The vertices not involved in segments of form (i) may be ranked in $a!(b!/L!)(C!/L!)$ ways while it is easily seen that the remaining pairs of $(1, 2)$ points may be connected in cycles and ranked in $(L!)^2$ ways. Hence the total number of possible graphs in W^* is

$$\left\{\binom{s}{a}\Big/(B!C!)\right\} \cdot a! \cdot \{a!b!c!\} = \frac{s!a!b!c!}{A!B!C!}. \tag{17}$$

It follows from (5), (16), and (17) that

$$I_{a,b,c} = \frac{2}{(a+b+c+1)} \cdot \binom{2s-2a}{s-a}\binom{2s-2b}{s-b}\binom{2s-2c}{s-c}\binom{2s}{s}^{-1}. \tag{18}$$

This result was first obtained by Adams [1]. His approach was to evaluate the integral for some low values of the subscripts and, on the basis of this, to guess a general formula, which he then proved by induction. For a succinct history of the problem see Askey [2, pp. 39-40]. In the special case $a=b=c=2\lambda$, (18) becomes

$$\int_{-1}^{1} \{P_{2\lambda}(x)\}^3\, dx = \frac{2\{(3\lambda)!\}^2}{(6\lambda+1)!}\binom{2\lambda}{\lambda}^3. \tag{19}$$

Substituting (18) into (4), we get the binomial identity

$$
\begin{aligned}
&\sum_{\alpha,\beta,\gamma} (-1)^{\alpha+\beta+\gamma}\binom{a}{\alpha}^2\binom{b}{\beta}^2\binom{c}{\gamma}^2\binom{a+b+c}{\alpha+\beta+\gamma}^{-1} \\
(20)\qquad &= \begin{cases} \binom{2s-2a}{s-a}\binom{2s-2b}{s-b}\binom{2s-2c}{s-c}\binom{2s}{s}^{-1} & \text{for } a+b+c=2s, \\ 0 & \text{for } a+b+c \text{ odd.} \end{cases}
\end{aligned}
$$

In the special case $a=b=c=2\lambda$, this becomes

$$\sum_{\alpha,\beta,\gamma} (-1)^{\alpha+\beta+\gamma}\binom{2\lambda}{\alpha}^2\binom{2\lambda}{\beta}^2\binom{2\lambda}{\gamma}^2\binom{6\lambda}{\alpha+\beta+\gamma}^{-1} = \binom{2\lambda}{\lambda}^3\binom{6\lambda}{3\lambda}^{-1}. \tag{21}$$

4. The case $k=4$. Since the product $P_{n_1}, P_{n_2} \cdots P_{n_k}$ is a polynomial, it may be written in the form

$$P_{n_1}(x)P_{n_2}(x)\cdots P_{n_k}(x)=\sum_{\alpha} C_{n_1,n_2,\cdots,n_k,\alpha}P_\alpha(x) \tag{22}$$

where the $C_{n_1,\cdots,n_k}$, α are constants. Now if we apply the well-known relation

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx=\frac{2}{2m+1}\delta_{m,n}, \tag{23}$$

we get

$$\frac{2}{2\alpha+1}C_{n_1,n_2,\cdots,n_k,\alpha}=I_{n_1,n_2,\cdots,n_k,\alpha}. \tag{24}$$

Now let a, b, c, d be nonnegative integers. By (23) and (24)

$$\begin{aligned} P_a(x)P_b(x)&=\sum_{\alpha}\left(\alpha+\frac{1}{2}\right)I_{a,b,\alpha}P_\alpha(x), \quad \text{and}\\ P_c(x)P_d(x)&=\sum_{\beta}\left(\beta+\frac{1}{2}\right)I_{c,d,\beta}P_\beta(x). \end{aligned} \tag{25}$$

Hence,

$$\begin{aligned} I_{a,b,c,d}&=\int_{-1}^{1} P_a(x)P_b(x)P_c(x)P_d(x)\,dx\\ &=\sum_{\alpha,\beta}\left(\alpha+\frac{1}{2}\right)\left(\beta+\frac{1}{2}\right)I_{a,b,\alpha}I_{c,d,\beta}\int_{-1}^{1} P_\alpha(x)P_\beta(x)\,dx\\ &=\sum_{\alpha,\beta}\left(\alpha+\frac{1}{2}\right)\left(\beta+\frac{1}{2}\right)I_{a,b,\alpha}I_{c,d,\beta}\frac{\delta_{\alpha,\beta}}{\alpha+\frac{1}{2}} \quad \text{by (22)}\\ &=\sum_{\alpha}\left(\alpha+\frac{1}{2}\right)I_{a,b,\alpha}I_{c,d,\alpha}. \end{aligned} \tag{26}$$

Since $I_{a,b,c,d}=0$, unless $a+b+c+d$ is even, we may assume in (26) that $a+b\equiv c+d \pmod 2$. Thus we may write

$$\begin{aligned} I_{a,b,c,d}&=\sum_{\gamma}\left(2\gamma+\frac{1}{2}\right)I_{a,b,2\gamma}I_{c,d,2\gamma} \quad \text{if } a+b\equiv c+d\equiv 0 \pmod 2\\ &=\sum_{\gamma}\left(2\gamma+\frac{3}{2}\right)I_{a,b,2\gamma+1}I_{c,d,2\gamma+1} \quad \text{if } a+b\equiv c+d\equiv 1 \pmod 2. \end{aligned} \tag{27}$$

Moreover, since $I_{a,b,c,d}$ is clearly symmetric in the subscripts, we see that

$$\sum_{\alpha}\left(\alpha+\frac{1}{2}\right)I_{a,b,\alpha}I_{c,d,\alpha}=\sum_{\beta}\left(\beta+\frac{1}{2}\right)I_{a,c,\beta}I_{b,d,\beta}. \tag{28}$$

A special case of some interest arises if we take $a=b=c=d$. By (27) we get

$$I_{a,a,a,a}=\sum_{\gamma}\left(2\gamma+\frac{1}{2}\right)I^2_{a,a,2\gamma},$$

i.e.,

$$\int_{-1}^{+1}[P_a(x)]^4\,dx=2\sum_{\gamma}(4\gamma+1)\left\{\frac{[(a+\gamma)!]^2}{(2a+2\gamma+1)}\binom{2\gamma}{\gamma}^2\binom{2a-2\gamma}{a-\gamma}\right\}^2. \tag{29}$$

REFERENCES

[1] J. C. ADAMS, Proc. Roy. Soc., XXVII (1878).
[2] R. ASKEY, *Orthogonal Polynomials and Special Functions*, CBMS-NSF Regional Conference Series in Applied Mathematics 21, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1975.
[3] P. CARTIER AND D. FOATA, *Problèmes combinatoires de commutation et de rearrangement*, Lecture Notes in Math. 85, Springer-Verlag, Berlin, New York, 1969.
[4] H. W. GOULD, *Combinatorial Identities*, 2nd ed., University of West Virginia, Morgantown, WV, 1972.