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OPERATIONAL RECURRENCES INVOLVING FIBONACCI NUMBERS

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The Fibonacci numbers may be defined by the linear recurrence relation

(1) $$F_{n+1} = F_n + F_{n-1}$$

together with the initial values $F_0 = 0$, $F_1 = 1$.

There are some unorthodox ways of making up sequences which involve Fibonacci numbers, and we should like to mention a few of these. For want of a better name, we shall call the recurrences below 'operational recurrences.'

Instead of taking the next term in a sequence as the sum of the two preceding terms, let us suppose the terms of a sequence are functions of x, and define

(2) $$u_{n+1}(x) = D_x (u_n u_{n-1}),$$

where $D_x = d/dx$. As an example, take $u_0 = 1$, $u_1 = e^x$. Then we find

$$u_2 = D(e^x) = e^x$$

$$u_3 = D(e^{2x}) = 2e^{2x}$$

$$u_4 = D(2e^{3x}) = (2)(3)e^{3x}$$

$$u_5 = (1)(1)(2)(3)(5)e^{5x}$$

and we can easily show by induction that

$$(3) \qquad u_n = (F_1 F_2 F_3 \cdots\cdots F_n)\, e^{F_n x}, \qquad (u_0 = 1,\ u_1 = e^x).$$

Of course, the addition of exponents led to the appearance of the Fibonacci numbers in this case.

Another operation which we may use is differentiation followed by multiplication with x. We define

$$(4) \qquad u_{n+1} = (xD_x)\,(u_n\ u_{n-1}).$$

For an interesting example, let us take

$$u_0 = x^{F_0} = 1, \quad u_1 = x^{F_1} = x\ .$$ Then we

claim that

$$(5) \qquad u_n = x^{F_n} \prod_{k=1}^{n} F_k^{\,F_{n+1-k}}, \qquad n \geq 1.$$

Taking x = 1 we obtain the following table of values as a sample:

n	$u_n(1)$
1	$1^1 = 1$
2	$1^1\,1^1 = 1$
3	$1^2\,1^1\,2^1 = 2$
4	$1^3\,1^2\,2^1\,3^1 = 6$
5	$1^5\,1^3\,2^2\,3^1\,5^1 = 60$
6	$1^8\,1^5\,2^3\,3^2\,5^1\,8^1 = 2880$
7	$1^{13}\,1^8\,2^5\,3^3\,5^2\,8^1\,13^1 = 2,246,400$

For the sake of completeness we give the inductive proof of formula (5). Suppose that

$$u_{n-1} = x^{F_{n-1}} \prod_{k=1}^{n-1} F_k^{F_{n-k}}$$

Then

$$u_n = xD\,(u_{n-1}\,u_{n-2})$$

$$= xD\,\{\, x^{F_{n-1}}\, x^{F_{n-2}} \prod_{k=1}^{n-1} F_k^{F_{n-k}} \prod_{j=1}^{n-2} F_j^{F_{n-1-j}} \}$$

$$= xD\,\{\, x^{F_n} \}\, F_{n-1}^{F_1} \prod_{k=1}^{n-2} F_k^{F_{n-k} + F_{n-1-k}}$$

$$= F_n\, x^{F_n}\, F_{n-1}^{F_1} \prod_{k=1}^{n-2} F_k^{F_{n+1-k}}$$

$$= F_n^{F_1}\, x^{F_n}\, F_{n-1}^{F_2} \prod_{k=1}^{n-2} F_k^{F_{n+1-k}}$$

$$= x^{F_n} \prod_{k=1}^{n} F_k^{F_{n+1-k}}$$

The only 'tricky' part is to recall that $1 = F_1$ and $F_1 = F_2$ so that the factors may be put together at the last step in the desired form.

Suppose that the function $u_n(x)$ has a power series representation of the form

$$(6) \qquad u_n(x) = \sum_{k=0}^{\infty} a_k(n)\, x^k \ .$$

Imposing the operational recurrence (4) we find readily that the coefficients in (6) must obey the convolution recurrence

(7) $$a_k(n) = k \sum_{j=0}^{k} a_j(n-2)\, a_{k-j}(n-1) .$$

Conversely, if (7) holds then $u_n(x)$ satisfies (4).

As a slight variation of (4) let us next define

(8) $$u_{n+1} = x^2 D_x (u_n u_{n-1}) ,$$

and take $u_0 = 1$, $u_1 = x$. Then it is easily shown by induction that

(9) $$u_n = x^{F_{n+2}-1} \prod_{k=4}^{n+2} (F_k - 2)^{F_{n+3-k}} , \quad \text{for } n \geq 2.$$

The reader may find it interesting to derive a formula for the sequence defined by $u_n = u_n(x)$ with

(10) $$u_{n+1} = x^p D_x(u_n u_{n-1}), \quad u_0 = 1, \; u_1 = x, \; p = 3, 4, 5 \ldots$$

As a final example, let us define a sequence by (4) with $u_0 = 1$, $u_1 = e^x$.

Then the first few values of the function sequence are:

$$u_2 = x e^x ,$$

$$u_3 = (2x^2 + x)\, e^{2x}$$

$$u_4 = (6x^4 + 9x^3 + 2x^2)\, e^{3x}$$

$$u_5 = (60x^7 + 192x^6 + 185x^5 + 62x^4 + 6x^3)\, e^{5x}$$

and it is evident that $u_n(x)$ equals $P(x)\, e^{F_n x}$, where $P(x)$ is a polynomial of degree $F_{n+1} - 1$ in x.