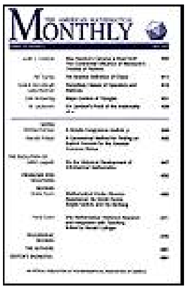

Explicit Formulas for Bernoulli Numbers

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EXPLICIT FORMULAS FOR BERNOULLI NUMBERS

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A recent paper by Higgins [19] offers what is purported to be a new finite double series for the Bernoulli numbers with similar results for the Euler numbers. The paper gives an introductory account of the history of the Bernoulli numbers and quotes from some very old and authoritative sources, as well as recent papers about the numerical computation of the Bernoulli, Euler, and Tangent numbers. However the author seems to have missed other equally valuable papers so that he is constrained to state that "as far as I am aware there has been no explicit evaluation of them apart from this [an integral given by Whittaker and Watson], though values have from time to time been tabulated." The object of the present paper is to set matters straight by presenting a bibliography on explicit formulas for the Bernoulli numbers, and show how one can easily manufacture expressions for these numbers.

Basically, what Higgins found when $a = 0$ in his general formula (2.5) is

$$B_n = \sum_{k=0}^{n} \frac{1}{k+1} \sum_{j=0}^{k} (-1)^j \binom{k}{j} j^n, \qquad n \geqq 0, \tag{1}$$

and the reader will have no difficulty in seeing that the lower limits of summation in both cases may be replaced by $k = 1$ and $j = 1$ so as to agree with the form in which Higgins gives the result, a form valid for $n \geqq 1$. The formula is quite old, and it is difficult to say how to assign priorities, but the interested reader should consult the sources listed here with special attention to the book by Saalschütz [24]. Saalschütz gives [pp. 54–116] a total of 38 explicit formulas for the Bernoulli numbers, usually giving some reference in the older literature together with a proof. The notations used are quite different from recent ones, and the dozens of different notations in use for the numbers of Bernoulli and Stirling, etc., is surely one explanation for the formulas not being widely known. Yet each notation has its own elegance and place.

The book of Saalschütz has been out of print for many decades and has become quite hard to locate, with only a very exceptional library having a copy. The present writer was able to persuade University Microfilms to track down a copy and now a Xerographed version can be gotten from them very easily. The problem was two-fold: question of possible copyright and availability of a copy to photograph. Both problems were solved. The copy from Yale University Library was used. Anyone wishing to work with Bernoulli numbers in any great detail ought to examine the book.

In a recent book review [17] I called attention to formula (1) and deplored the fact that almost no current books on infinite series ever mention or derive (1), and even Knopp in his famous booklet on series asserted that the Bernoulli numbers "cannot be specified by means of a simple formula — except, say, by means of a determinant..." such is the widespread misinformation at hand.

Higgins cites a paper by von Staudt but misses one published five years later [28] in which explicit formulas are given. The formula (1) may be found derived in the

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