

Note

Generalization of a Formula of Touchard for Catalan Numbers

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In this paper a general identity is proved which includes among its special cases a recurrence formula for Catalan numbers given by J. Touchard in 1924.

Touchard [7] proved the formula

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} 2^{n-2k} C(k) = C(n+1), \tag{1}$$

where $C(k) = \binom{2k}{k} / (k+1)$ is the k th Catalan number. Further proofs have been given by Izbicki [3], Riordan [4], and Shapiro [5, 6]. These proofs are combinatorial in nature and the most recent work by Shapiro gives a combinatorial interpretation of the numbers $\binom{n}{2k} 2^{n-2k} C(k)$.

Now there is a somewhat similar formula [1, (3.99)]

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} 2^{n-2k} A(k) = A(n), \tag{2}$$

for the numbers $A(k) = \binom{2k}{k}$, which are the middle numbers in alternate rows of Pascal's triangle. Since also $C(k) = \binom{2k}{k} - \binom{2k}{k-1}$, the difference of the middle number and an adjacent coefficient, it would seem plausible to look for a general algebraic identity that includes both (1) and (2).

The object of the present note is to provide such a generalization in the form

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \binom{2k}{k} 2^{n-2k} R(k) = \binom{2n+2r}{n}, \tag{3}$$

where:

$$R(k) = 1, \quad \text{if } r = 0,$$

$$= \frac{(n+1)(n+2) \cdots (n+r)}{(k+1)(k+2) \cdots (k+r)}, \quad \text{if } r \geq 1.$$

For $r = 0$ this yields (2) and for $r = 1$ it yields (1).

Another way of writing this formula is

$$\sum_{k=0}^{[n/2]} \binom{n}{2k} \binom{2k}{k} \binom{n+r}{n-k} \binom{n}{k}^{-1} 2^{n-2k} = \binom{2n+2r}{n}, \tag{4}$$

or also

$$\sum_{k=0}^{[n/2]} \binom{n+r}{n-k} \binom{n-k}{k} 2^{2n-2k} = 2^n \binom{2n+2r}{n}, \tag{5}$$

which disguises the occurrence of $\binom{2k}{k}$, but makes the formula easier to obtain.

In fact, (5) follows from the equivalent formula

$$\sum_{k=0}^n \binom{n}{k} \binom{k}{j-k} 2^{2k} = 2^j \binom{2n}{j}, \tag{6}$$

upon replacing n by $n+r$, j by n , changing the variable of summation k into $n-k$, and dropping zero terms. Formula (6) then is evident by inversion from the formula [1, (3.64)]

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2k}{j} = (-1)^n \binom{n}{j-n} 2^{2n-j}. \tag{7}$$

Formula (7) is easily proved by equating coefficients of x^j on both sides of the identity $(x^2 + 2x)^n = ((x+1)^2 - 1)^n$.

I wish to note that all of these formulas were found by me prior to 1954 and occur in the unpublished background material for my book [1]. Formula (7) is an old result in the mathematical literature.

Since in general $\binom{2k}{k}$ is not divisible by $(k+1)(k+2) \cdots (k+r)$, a neat combinatorial interpretation of (3) is not available. However, in the form (5), or for the more elegant formula (6), combinatorial derivations undoubtedly exist.

Once formula (5) is established for integers $r \geq 0$, then since both members are polynomials in r of degree n , it follows that (5) is actually true for any real or complex r , and we rephrase the formula as

$$\sum_{k=0}^{[n/2]} \binom{n+x}{n-k} \binom{n-k}{k} 2^{n-2k} = \binom{2x+2n}{n}, \tag{8}$$

where x may be any complex number. Although (5) was not explicitly stated in [1], formula (3.107) there is the case $x = -r$ of (8), so that (8) unifies this with the generalization of the Touchard formula.

Perhaps somewhere in the vast literature on Catalan numbers (see [2] for a bibliography of 431 items), further insight to these formulas may be found.

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