
Laguerre functions and representations of $su(1, 1)$

Dedicated to Tom Koornwinder on the occasion of his 60th birthday.

by Wolter Groenevelt

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ABSTRACT

Spectral analysis of a certain doubly infinite Jacobi operator leads to orthogonality relations for confluent hypergeometric functions, which are called Laguerre functions. This doubly infinite Jacobi operator corresponds to the action of a parabolic element of the Lie algebra $su(1, 1)$. The Clebsch-Gordan coefficients for the tensor product representation of a positive and a negative discrete series representation of $su(1, 1)$ are determined for the parabolic bases. They turn out to be multiples of Jacobi functions. From the interpretation of Laguerre polynomials and functions as overlap coefficients, we obtain a product formula for the Laguerre polynomials, given by an integral over Laguerre functions, Jacobi functions and continuous dual Hahn polynomials.

1. INTRODUCTION

Many special functions of hypergeometric type have an interpretation in representation theory of Lie groups and Lie algebras, see for example Koornwinder's paper [11] and the book by Vilenkin and Klimyk [17]. In this paper we consider the three-dimensional Lie algebra $su(1, 1)$, generated by H , B and C . Elements of $su(1, 1)$ are either elliptic, parabolic or hyperbolic elements, which correspond to the three conjugacy classes of the Lie group $SU(1, 1)$. The self-adjoint element $X_a = -aH + B - C$, $a \in \mathbb{R}$, is an elliptic element for $|a| > 1$, a parabolic element for $|a| = 1$, and a hyperbolic element for $|a| < 1$. In [10] Koelink and Van der Jeugt consider the action of X_a in tensor products of positive discrete series representations. This leads to convolution identities for several hypergeometric orthogonal polynomials. The action of X_a in the tensor product of a positive and a negative discrete series representation is considered

in [6] for the elliptic case, and in [7] for the hyperbolic case. In this paper we investigate the remaining, parabolic, case. In [5] the quantum version of X_a is studied. The Lie algebra $su(1, 1)$ is replaced by the quantized universal enveloping algebra $U_q(su(1, 1))$ and X_a is replaced by a twisted primitive element. It turns out that in $U_q(su(1, 1))$ the three cases are all the same.

There are four classes of irreducible unitary representations of $su(1, 1)$, the positive and negative discrete series, the principal unitary series and the complementary series. The tensor product of a positive and a negative discrete series representation decomposes into a direct integral over the principal unitary series. Discrete terms can occur, and these terms correspond to one complementary series, or a finite number of discrete series. The Clebsch-Gordan coefficients for the standard bases are multiples of continuous dual Hahn polynomials.

We consider the element $X = -H + B - C$, which is a parabolic element. In the discrete series X acts on the standard (elliptic) basis as a Jacobi operator, which corresponds to the three-term recurrence relation for Laguerre polynomials. In the principal unitary series and the complementary series X acts on the standard basis as a doubly infinity Jacobi operator, which corresponds to the recurrence relation for Laguerre functions. So the Laguerre polynomials and functions appear as overlap coefficients between the (generalized) eigenvectors of X and the standard basis vectors. Using the differential equation for the Laguerre polynomials, we realize the generators H , B and C in the discrete series as differential operators. In these realizations the action of the Casimir operator can be identified with the hypergeometric differential equation, which leads to Jacobi functions as Clebsch-Gordan coefficients for parabolic basis vectors. This leads to an identity for the overlap coefficients, which is a product formula for the Laguerre polynomials.

This paper is organized as follows. In §2 we consider a certain doubly infinity Jacobi operator, which corresponds to the action of X in the principal unitary series. Spectral analysis leads to orthogonality relations for Laguerre functions. This section is based on [14] by Masson and Repka.

In §3 we turn to representations of the Lie algebra $su(1, 1)$. We introduce the orthogonal polynomials and functions that we need in §3.1, and we give some of their properties. In §3.2 we introduce the Lie algebra $su(1, 1)$ and give the irreducible unitary representations. In §3.3 we diagonalize the element X in the various representations, and we give generalized eigenvectors. In §3.4 the generators H , B and C are realized as differential operators. Then the Casimir operator in the tensor product can be identified with the hypergeometric differential operator, and this leads to Jacobi functions as Clebsch-Gordan coefficients. As a result we obtain a product formula for Laguerre polynomials, which involves Jacobi functions, Laguerre functions and continuous dual Hahn polynomials.

Notations. If $d\mu(x)$ is a positive measure, we use the notation $d\mu^{\frac{1}{2}}(x)$ for the positive measure with the property

$$d(\mu^{\frac{1}{2}} \times \mu^{\frac{1}{2}})(x, x) = d\mu(x).$$

The hypergeometric series is defined by

$${}_pF_q \left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} ; z \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n z^n}{(b_1)_n \dots (b_q)_n n!},$$

where $(a)_n$ denotes the Pochhammer symbol, defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)(a+2) \dots (a+n-1), \quad n \in \mathbb{Z}_{\geq 0}.$$

For the confluent hypergeometric function we use the notation

$${}_1F_1(a; b; z) = {}_1F_1 \left(\begin{matrix} a \\ b \end{matrix} ; z \right),$$

and the second solution of the confluent hypergeometric differential equation is defined by

$$(1.1) \quad U(a; b; z) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)} {}_1F_1(a; b; z) + \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} {}_1F_1(a-b+1; 2-b; z),$$

see [16, (1.3.1)]. This is a many-valued functions of z , and we take as its principal branch that which lies in the complex plane cut along the negative real axis $(-\infty, 0]$.

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2. LAGUERRE FUNCTIONS

In this section we determine the spectral measure of a certain doubly infinite Jacobi operator. This operator is obtained from the action of the self-adjoint element X of the Lie algebra $su(1, 1)$ in the principal unitary series representation, see §3.3. The eigenfunctions which are needed to describe the spectral measure, are called Laguerre functions. See [14] or [9] for doubly infinite Jacobi operators. The calculation of the eigenfunctions and the Wronskian is obtained from [14], but we repeat the calculations here briefly.

The doubly infinite Jacobi operator $L : \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z})$ is defined by

$$(2.1) \quad Le_k = a_k e_{k+1} + b_k e_k + a_{k-1} e_{k-1},$$

where $\{e_k\}_{k \in \mathbb{Z}}$ is the standard orthonormal basis of $\ell^2(\mathbb{Z})$ and

$$\begin{aligned} a_k &= a_k(\rho, \varepsilon) = \left| k + \varepsilon + \frac{1}{2} + i\rho \right|, \\ b_k &= b_k(\rho, \varepsilon) = 2(k + \varepsilon), \end{aligned}$$

where $\rho \geq 0$, $\varepsilon \in [0, 1)$ and $(\rho, \varepsilon) \neq (0, \frac{1}{2})$.

Remark 2.1. There exists a symmetry for the parameters of L . Let us denote $L = L(\rho, \varepsilon)$. The unitary operator $U : e_k \mapsto (-1)^k e_{-k}$ intertwines $L(\rho, \varepsilon)$ with

$-L(-\rho, -\varepsilon)$. So if $f(z; \varepsilon, \rho, k)$ is a solution to the eigenvalue equation $Lf = zf$, then $(-1)^k f(-z; -\varepsilon, -\rho, -k)$ is another solution to the same eigenvalue equation.

Observe that L is an unbounded, symmetric operator. The domain D of L is the dense subspace of finite linear combinations of the basis vectors e_k . We define for a function $f = \sum_{k=-\infty}^{\infty} f_k e_k \in \ell^2(\mathbb{Z})$

$$L^*f = \sum_{k=-\infty}^{\infty} (a_k f_{k+1} + b_k f_k + a_{k-1} f_{k-1}) e_k.$$

on its domain

$$D^* = \{f \in \ell^2(\mathbb{Z}) \mid L^*f \in \ell^2(\mathbb{Z})\},$$

then (L^*, D^*) is the adjoint of (L, D) . Note that $L^*|_D = L$.

Solutions to $Lv = -zv$ can be given in terms of confluent hypergeometric functions (see [16]).

Proposition 2.2. *The following functions are solutions to $Lv = -zv$:*

$$s_k(z; \rho, \varepsilon) = (-1)^k \frac{|\Gamma(k + \varepsilon + \frac{1}{2} + i\rho)|}{\Gamma(k + \varepsilon + \frac{1}{2} - i\rho)} {}_1F_1(k + \varepsilon + \frac{1}{2} + i\rho; 1 + 2i\rho; z),$$

$$t_k(z; \rho, \varepsilon) = \frac{|\Gamma(\frac{1}{2} - k - \varepsilon - i\rho)|}{\Gamma(\frac{1}{2} - k - \varepsilon + i\rho)} {}_1F_1(\frac{1}{2} - k - \varepsilon - i\rho; 1 - 2i\rho; -z),$$

$$u_k(z; \rho, \varepsilon) = (-1)^k |\Gamma(k + \varepsilon + \frac{1}{2} + i\rho)| U(k + \varepsilon + \frac{1}{2} + i\rho; 1 + 2i\rho; z), \quad z \notin (-\infty, 0]$$

$$v_k(z; \rho, \varepsilon) = |\Gamma(\frac{1}{2} - k - \varepsilon - i\rho)| U(\frac{1}{2} - k - \varepsilon - i\rho; 1 - 2i\rho; -z), \quad z \notin [0, \infty).$$

Proof. The first solution s_k follows from [16, (2.2.1)]

$$(b - a) {}_1F_1(a - 1; b; z) + (2a - b + z) {}_1F_1(a; b; z) - a {}_1F_1(a + 1; b; z) = 0.$$

The second solution t_k follows from the first using the symmetry relation for the parameters, cf. Remark 2.1. In the same way we find from [16, (2.2.8)]

$$U(a - 1; b; z) - (2a - b + z)U(a; b; z) + a(a - b + 1)U(a + 1; b; z) = 0,$$

that u_k and v_k are solutions to $Lv = -zv$. \square

The solution space to $Lv = -zv$ is two-dimensional, since for a fixed $n \in \mathbb{Z}$, v is completely determined by the initial values v_{n-1} and v_n . So the eigenfunctions given in Proposition 2.2 can be expanded in terms of each other.

Proposition 2.3 *We have the connection formulas*

$$u_k(z) = A(z)s_k(z) + B(z)t_k(z), \quad z \notin (-\infty, 0]$$

$$v_k(z) = C(z)s_k(z) + D(z)t_k(z), \quad z \notin [0, \infty),$$

where

$$\begin{aligned}
A(z) &= \Gamma(-2i\rho), \\
B(z) &= z^{-2i\rho} e^z \Gamma(2i\rho) \frac{\sin \pi(\varepsilon + \frac{1}{2} + i\rho)}{|\sin \pi(\varepsilon + \frac{1}{2} - i\rho)|}, \\
C(z) &= (-z)^{2i\rho} e^{-z} \Gamma(-2i\rho) \frac{\sin \pi(\varepsilon + \frac{1}{2} + i\rho)}{|\sin \pi(\varepsilon + \frac{1}{2} - i\rho)|}, \\
D(z) &= \Gamma(2i\rho).
\end{aligned}$$

Or equivalently, for $z \in \mathbb{C} \setminus \mathbb{R}$

$$\begin{aligned}
s_k(z) &= E(z)u_k(z) + F(z)v_k(z), \\
t_k(z) &= G(z)u_k(z) + H(z)v_k(z),
\end{aligned}$$

where

$$\begin{aligned}
E(z) &= \frac{1}{\pi} \sin \pi(\varepsilon + \frac{1}{2} - i\rho) \Gamma(1 + 2i\rho) e^{i\pi\xi(\varepsilon + \frac{1}{2} + i\rho)}, \\
F(z) &= -\frac{1}{\pi} |\sin \pi(\varepsilon + \frac{1}{2} + i\rho)| \Gamma(1 + 2i\rho) e^z z^{-2i\rho} e^{i\pi\xi(\varepsilon + \frac{1}{2} + i\rho)}, \\
G(z) &= \frac{1}{\pi} |\sin \pi(\varepsilon + \frac{1}{2} + i\rho)| \Gamma(1 - 2i\rho) e^{-z} (-z)^{2i\rho} e^{i\pi\xi(\varepsilon + \frac{1}{2} + i\rho)}, \\
H(z) &= -\frac{1}{\pi} \sin \pi(\varepsilon + \frac{1}{2} - i\rho) \Gamma(1 - 2i\rho) e^{i\pi\xi(\varepsilon + \frac{1}{2} + i\rho)},
\end{aligned}$$

where $\xi = \operatorname{sgn} \Im(z)$.

Proof. The first connection formula follows from (1.1), the reflection formula for the Γ -function, and Kummer's transformation: ${}_1F_1(a; b; z) = e^z {}_1F_1(b - a; b; -z)$. The second connection formula can be derived from the first using the symmetry for the parameters, see Remark 2.1.

Proof. The other two connection formulas follow from [16, (1.9.1),(1.4.10)] or they can be derived from the first two.

Definition 2.4. For two functions $f(z) = \sum_{k=-\infty}^{\infty} f_k(z)e_k$ and $g(z) = \sum_{k=-\infty}^{\infty} g_k(z)e_k$, the Wronskian is defined by

$$[f(z), g(z)]_k = a_k(f_k(z)g_{k+1}(z) - f_{k+1}(z)g_k(z)).$$

If $f(z)$ and $g(z)$ are solutions to the eigenvalue equation $Lv = -zv$, the Wronskian $[f(z), g(z)]_k$ is independent of k , so the Wronskian can be found by taking the limit $k \rightarrow \pm\infty$. Moreover, $f(z)$ and $g(z)$ are linearly independent solutions if and only if $[f(z), g(z)] \neq 0$.

Lemma 2.5. For $0 < |\arg(z)| < \pi$ and $k \rightarrow \infty$

$$\begin{aligned}
u_k(z) &= (-1)^k \sqrt{\pi} e^{\frac{1}{2}z - \sqrt{4(k+\varepsilon)z}} z^{-\frac{1}{4} - i\rho} k^{-\frac{1}{4}} \left(1 + O(k^{-\frac{1}{2}})\right), \\
v_{-k}(z) &= \sqrt{\pi} e^{-\frac{1}{2}z - \sqrt{-4(k-\varepsilon)z}} (-z)^{-\frac{1}{4} + i\rho} k^{-\frac{1}{4}} \left(1 + O(k^{-\frac{1}{2}})\right), \\
s_k(z) &= \frac{(-1)^k}{2\sqrt{\pi}} e^{\frac{1}{2}z + \sqrt{4(k+\varepsilon)z}} z^{-\frac{1}{4} - i\rho} \Gamma(1 + 2i\rho) k^{-\frac{1}{4}} \left(1 + O(k^{-\frac{1}{2}})\right), \\
t_{-k}(z) &= \frac{1}{2\sqrt{\pi}} e^{-\frac{1}{2}z + \sqrt{-4(k-\varepsilon)z}} (-z)^{-\frac{1}{4} + i\rho} \Gamma(1 - 2i\rho) k^{-\frac{1}{4}} \left(1 + O(k^{-\frac{1}{2}})\right).
\end{aligned}$$

Proof. This follows from the asymptotic behaviour for $|y| \rightarrow \infty$ of the modified Bessel functions

$$I_\nu(y) = \frac{e^y}{\sqrt{2\pi y}} \left(1 + O\left(\frac{1}{|y|}\right)\right), \quad K_\nu(y) = \sqrt{\frac{\pi}{2y}} e^{-y} \left(1 + O\left(\frac{1}{|y|}\right)\right), \quad |\arg y| < \frac{\pi}{2},$$

and the asymptotic expansions for the confluent hypergeometric functions in terms of modified Bessel functions [16, (4.6.42), (4.6.43)]. \square

For $z \in \mathbb{C} \setminus \mathbb{R}$ we introduce the spaces

$$\begin{aligned}
S_z^+ &= \left\{ f(z) = \sum_{k=-\infty}^{\infty} f_k(z) e_k \mid L^* f(z) = -zf(z) \text{ and } \sum_{k=0}^{\infty} |f_k(z)|^2 < \infty \right\}, \\
S_z^- &= \left\{ f(z) = \sum_{k=-\infty}^{\infty} f_k(z) e_k \mid L^* f(z) = -zf(z) \text{ and } \sum_{k=-\infty}^{-1} |f_k(z)|^2 < \infty \right\}.
\end{aligned}$$

Then the deficiency space for L is $S_z^+ \cap S_z^-$. Note that $\dim S_z^\pm \leq 2$, and in case $\dim S_z^\pm = 2$, we have $S_z^+ = S_z^-$, since the solution space to $Lf = -zf$ is two dimensional.

Next we put $\phi_z = z^{i\rho} u(z)$, $z \notin (-\infty, 0]$, and $\Phi_z = (-z)^{-i\rho} v(z)$, $z \notin [0, \infty)$. From the transformation $U(a; b; z) = z^{1-b} U(a-b+1; 2-b; z)$ it follows that

$$\begin{aligned}
(2.2) \quad u_l(\bar{z}) &= \bar{z}^{-2i\rho} \overline{u_l(z)}, \quad z \notin (-\infty, 0], \\
v_k(\bar{z}) &= (-\bar{z})^{2i\rho} \overline{v_k(z)}, \quad z \notin [0, \infty).
\end{aligned}$$

So we have $\overline{(\phi_z)_k} = (\phi_{\bar{z}})_k$ and $\overline{(\Phi_z)_k} = (\Phi_{\bar{z}})_k$, and in particular $(\phi_x)_k \in \mathbb{R}$ for $x > 0$ and $(\Phi_x)_k \in \mathbb{R}$ for $x < 0$. Note that $(\phi_x)_k$ and $(\Phi_x)_k$ are even in ρ .

We calculate the Wronskian $[\phi_z, \Phi_z]$. From Lemma 2.5 and $a_k = k + O(1)$, for $k \rightarrow \infty$, we find for $z \notin (-\infty, 0]$

$$[s(z), u(z)] = \lim_{k \rightarrow \infty} \frac{1}{2} e^z z^{-\frac{1}{2} - 2i\rho} \Gamma(1 + 2i\rho) k^{\frac{1}{2}} \left(e^{\sqrt{4(k+\varepsilon+1)z} - \sqrt{4(k+\varepsilon)z}} - e^{\sqrt{4(k+\varepsilon)z} - \sqrt{4(k+\varepsilon+1)z}} \right).$$

And since

$$\left(e^{\sqrt{4(k+\varepsilon+1)z} - \sqrt{4(k+\varepsilon)z}} - e^{\sqrt{4(k+\varepsilon)z} - \sqrt{4(k+\varepsilon+1)z}} \right) = 2\sqrt{\frac{z}{k}} \left(1 + O(k^{-\frac{1}{2}})\right), \quad k \rightarrow \infty,$$

we obtain

$$[s(z), u(z)] = e^z z^{-2i\rho} \Gamma(2i\rho + 1), \quad z \notin (-\infty, 0].$$

Then we find from the connection formulas of Proposition 2.3

$$[s(z), u(z)] = F(z)[v(z), u(z)], \quad z \in \mathbb{C} \setminus \mathbb{R},$$

and this gives

$$(2.3) \quad [\phi_z, \Phi_z] = z^{i\rho}(-z)^{-i\rho}[u(z), v(z)] = \frac{-\pi e^{-i\pi(\varepsilon + \frac{1}{2})}}{|\sin \pi(\varepsilon + \frac{1}{2} + i\rho)|}, \quad 0 < |\arg(z)| < \pi.$$

So we find that ϕ_z and Φ_z are linearly independent.

Proposition 2.6. *For $0 < |\arg(z)| < \pi$, we have $S_z^+ = \text{span}\{\phi_z\}$, and $S_z^- = \text{span}\{\Phi_z\}$, and L is essentially self-adjoint.*

Proof. From Lemma 2.5 we see that $\phi_z = z^{i\rho}u(z) \in S_z^+$ and $\Phi_z = (-z)^{-i\rho}v(z) \in S_z^-$, for $0 < |\arg(z)| < \pi$. Masson and Repka prove in [14, Thm.2.1] that the deficiency indices of L are obtained by adding the deficiency indices of the two Jacobi operators J^\pm obtained by restricting L to $\ell^2(\mathbb{Z}_{\geq 0})$ (setting $a_{-1} = 0$) and to $\ell^2(-\mathbb{N})$ (setting $a_0 = 0$). Since $\sum_{k=0}^\infty 1/a_k$ and $\sum_{k=-\infty}^{-1} 1/a_k$ are divergent, [3, Ch.VII, Thm.1.3] proves that J^\pm have deficiency indices $(0, 0)$, and hence so has L . So $\dim S_z^\pm = 1$, and the proposition follows. \square

We use the Stieltjes-Perron inversion formula, see [4, §XII.4], to calculate the spectral measure;

$$E_{f,g}((a, b)) = \lim_{\eta \downarrow 0} \lim_{\delta \downarrow 0} \frac{1}{2\pi i} \int_{a+\eta}^{b-\eta} \langle G(x+i\delta)f, g \rangle - \langle G(x-i\delta)f, g \rangle dx.$$

In this case the resolvent $G(z)$ can be calculated explicitly by

$$(2.4) \quad \langle G(z)f, g \rangle = \frac{1}{[\phi_z, \Phi_z]} \sum_{k \leq l} (\Phi_z)_k (\phi_z)_l (f_l \bar{g}_k + f_k \bar{g}_l) (1 - \frac{1}{2} \delta_{lk}).$$

Proposition 2.7 *The spectral measure for the operator $-L$ defined by (2.1), is described by the following integral, for $B \subseteq \mathbb{R}$ a Borel set,*

$$\begin{aligned} \langle E(B)f, g \rangle &= \frac{1}{\pi^2} \int_{B \cap (-\infty, 0)} e^x |\sin \pi(\varepsilon + \frac{1}{2} + i\rho)|^2 \langle f, v(x) \rangle \langle v(x), g \rangle dx \\ &\quad + \frac{1}{\pi^2} \int_{B \cap (0, \infty)} e^{-x} |\sin \pi(\varepsilon + \frac{1}{2} + i\rho)|^2 \langle f, u(x) \rangle \langle u(x), g \rangle dx, \end{aligned}$$

where $f, g \in \ell^2(\mathbb{Z})$ and with the notation of Proposition 2.2

$$u(x) = \sum_{k=-\infty}^{\infty} u_k(x) e_k, \quad v(x) = \sum_{k=-\infty}^{\infty} v_k(x) e_k.$$

Moreover, 0 is not contained in the point spectrum of L .

Proof. We define

$$(2.5) \quad \Delta(x) = \lim_{\delta \downarrow 0} \frac{(\Phi_{x+i\delta})_k(\phi_{x+i\delta})_l}{[\phi_{x+i\delta}, \Phi_{x+i\delta}]} - \frac{(\Phi_{x-i\delta})_k(\phi_{x-i\delta})_l}{[\phi_{x-i\delta}, \Phi_{x-i\delta}]},$$

then we have, using (2.3) and $\xi = \operatorname{sgn}\Im(z)$,

$$\begin{aligned} \Delta(x) = & - \lim_{\delta \downarrow 0} \frac{1}{\pi} |\sin \pi(\varepsilon + \frac{1}{2} + i\rho)| \left(e^{i\pi(\varepsilon+\frac{1}{2})} (\Phi_{x+i\delta})_k(\phi_{x+i\delta})_l \right. \\ & \left. - e^{-i\pi(\varepsilon+\frac{1}{2})} (\Phi_{x-i\delta})_k(\phi_{x-i\delta})_l \right). \end{aligned}$$

For $x < 0$ we have $\lim_{\delta \downarrow 0} (\Phi_{x+i\delta})_k = \lim_{\delta \downarrow 0} (\Phi_{x-i\delta})_k = (\Phi_x)_k$, and from the connection formulas we find

$$\begin{aligned} & \lim_{\delta \downarrow 0} e^{i\pi(\varepsilon+\frac{1}{2})}(\phi_{x+i\delta})_l - e^{-i\pi(\varepsilon+\frac{1}{2})}(\phi_{x-i\delta})_l \\ &= \lim_{\delta \downarrow 0} \left(e^{i\pi(\varepsilon+\frac{1}{2})} (x+i\delta)^{i\rho} A(x+i\delta) s_l(x+i\delta) - e^{-i\pi(\varepsilon+\frac{1}{2})} (x-i\delta)^{i\rho} A(x-i\delta) s_l(x-i\delta) \right) \\ &+ \lim_{\delta \downarrow 0} \left(e^{i\pi(\varepsilon+\frac{1}{2})} (x+i\delta)^{i\rho} B(x+i\delta) t_l(x+i\delta) - e^{-i\pi(\varepsilon+\frac{1}{2})} (x-i\delta)^{i\rho} B(x-i\delta) t_l(x-i\delta) \right) \\ &= 2i \sin \pi(\varepsilon + \frac{1}{2} + i\rho) (-x)^{i\rho} \Gamma(-2i\rho) s_l(x) + 2ie^x (-x)^{-i\rho} \Gamma(2i\rho) |\sin \pi(\varepsilon + \frac{1}{2} + i\rho)| t_l(x) \\ &= 2ie^x (-x)^{-i\rho} |\sin \pi(\varepsilon + \frac{1}{2} + i\rho)| v_l(x). \end{aligned}$$

Here we used

$$\lim_{\delta \downarrow 0} (-y \pm i\delta)^a = e^{\pm i\pi a} y^a, \quad y > 0.$$

For $x > 0$ we use the symmetry for the parameters, cf. Remark 2.1. So we find

$$(2.6) \quad \Delta(x) = \begin{cases} -\frac{2i}{\pi} e^x (-x)^{-2i\rho} |\sin \pi(\varepsilon + \frac{1}{2} + i\rho)|^2 v_k(x) v_l(x), & x < 0, \\ -\frac{2i}{\pi} e^{-x} x^{2i\rho} |\sin \pi(\varepsilon + \frac{1}{2} + i\rho)|^2 u_k(x) u_l(x), & x > 0. \end{cases}$$

Both expressions are clearly symmetric in k and l , so the sum in (2.4) can be antisymmetrized using (2.2). Now, if 0 is not contained in the point spectrum of L , the result follows from the Stieltjes-Perron inversion formula and (2.2).

To show that 0 is not an element of the point spectrum of L , we show that $\ker L = \{0\}$. First we calculate the Wronskian $[s(0), t(0)]$, using Definition 2.4 with $k = 0$. A straightforward calculation gives

$$[s(0), t(0)] = 2i\rho \frac{\sin \pi(\varepsilon + \frac{1}{2} - i\rho)}{|\sin \pi(\varepsilon + \frac{1}{2} + i\rho)|},$$

hence $s(0)$ and $t(0)$ are linearly independent. So if $f \in \ker L$, $f \neq 0$, then f is a linear combination of $s(0)$ and $t(0)$. But $s(0), t(0) \notin \ell^2(\mathbb{Z})$, since $|s_k(0)| = 1$ and $|t_k(0)| = 1$, and therefore $\ker L = \{0\}$.

Remark 2.8. The result of Proposition 2.7 remains valid if $i\rho$ is replaced by $\lambda + \frac{1}{2}$ where $\lambda \in (-\frac{1}{2}, -\varepsilon)$ and $\varepsilon \in [0, \frac{1}{2})$, or $\lambda \in (-\frac{1}{2}, \varepsilon - 1)$ and $\varepsilon \in (\frac{1}{2}, 1)$. In this

case the operator L is obtained from the action of the self-adjoint element X in the complementary series representation of $su(1, 1)$, see §3.3.

Let us define the Laguerre functions $\psi_n(x; \rho, \varepsilon)$, $n \in \mathbb{Z}$, by

$$\psi_n(x; \rho, \varepsilon) = \begin{cases} v_n(x; \rho, \varepsilon), & x < 0, \\ u_n(x; \rho, \varepsilon), & x > 0, \end{cases}$$

and we define the weight function $w(x; \rho, \varepsilon)$ by

$$w(x; \rho, \varepsilon) = \frac{1}{\pi^2} \left| \sin \pi \left(\varepsilon + \frac{1}{2} + i\rho \right) \right|^2 e^{-|x|}.$$

From Proposition 2.7 we find the following.

Theorem 2.9. *For $\rho \geq 0$, $\varepsilon \in [0, 1)$ and $(\rho, \varepsilon) \neq (0, \frac{1}{2})$, the Laguerre functions $\psi_n(x; \rho, \varepsilon)$ form an orthonormal basis of $L^2(\mathbb{R}, w(x; \rho, \varepsilon)dx)$.*

Proof. The orthonormality of the Laguerre functions follows from Proposition 2.7 by replacing f and g by standard orthonormal basis vectors e_n and e_m , $n, m \in \mathbb{Z}$, and using $B = \mathbb{R}$. Completeness of the Laguerre functions follows from the uniqueness of the spectral measure.

Remark 2.10. It would be nice to have a definition the Laguerre functions for $x = 0$. In order to find the “natural” definition in $x = 0$ we calculate $\Delta(0)$, where $\Delta(x)$ is defined by (2.5). We stress that the Laguerre functions are defined almost everywhere on \mathbb{R} , so the calculation of the Laguerre functions in $x = 0$ is only a formal calculation.

Using the connection coefficients from Proposition 2.3 we find

$$\begin{aligned} \Delta(0) &= -\lim_{\delta \downarrow 0} \frac{1}{\pi} \left| \sin \pi \left(\varepsilon + \frac{1}{2} + i\rho \right) \right| \left(e^{i\pi(\varepsilon+\frac{1}{2})} (\Phi_{i\delta})_k(\phi_{i\delta})_l - e^{-i\pi(\varepsilon+\frac{1}{2})} (\Phi_{-i\delta})_k(\phi_{-i\delta})_l \right) \\ &= -\lim_{\delta \downarrow 0} \frac{1}{\pi} \left| \sin \pi \left(\varepsilon + \frac{1}{2} + i\rho \right) \right| \left(e^{i\pi(\varepsilon+\frac{1}{2}+i\rho)} [C(i\delta)s_k(i\delta) + D(i\delta)t_k(i\delta)] [A(i\delta)s_l(i\delta) + B(i\delta)t_l(i\delta)] \right. \\ &\quad \left. - e^{-i\pi(\varepsilon+\frac{1}{2}+i\rho)} [C(-i\delta)s_k(-i\delta) + D(-i\delta)t_k(-i\delta)] [A(-i\delta)s_l(-i\delta) + B(-i\delta)t_l(-i\delta)] \right). \end{aligned}$$

To compute this limit, we use

$$\begin{aligned} \lim_{\delta \downarrow 0} s_k(i\delta) &= \lim_{\delta \downarrow 0} s_k(-i\delta) = s_k(0), \\ \lim_{\delta \downarrow 0} t_k(i\delta) &= \lim_{\delta \downarrow 0} t_k(-i\delta) = t_k(0), \\ \lim_{\delta \downarrow 0} A(i\delta)D(i\delta) &= \lim_{\delta \downarrow 0} A(-i\delta)D(-i\delta) = |\Gamma(2i\rho)|^2, \\ \lim_{\delta \downarrow 0} B(i\delta)C(i\delta) &= e^{-i\pi 2i\rho} |\Gamma(2i\rho)|^2 \frac{\sin \pi(\varepsilon + \frac{1}{2} + i\rho)}{\sin \pi(\varepsilon + \frac{1}{2} - i\rho)}, \\ \lim_{\delta \downarrow 0} B(-i\delta)C(-i\delta) &= e^{i\pi 2i\rho} |\Gamma(2i\rho)|^2 \frac{\sin \pi(\varepsilon + \frac{1}{2} + i\rho)}{\sin \pi(\varepsilon + \frac{1}{2} - i\rho)}, \end{aligned}$$

then we find

$$\Delta(0) = -\frac{2i}{\pi} \sin \pi \left(\varepsilon + \frac{1}{2} + i\rho \right) \left| \sin \pi \left(\varepsilon + \frac{1}{2} + i\rho \right) \right| |\Gamma(2i\rho)|^2 (t_k(0) \overline{s_l(0)} + s_k(0) t_l(0)).$$

From Euler's reflection formula for the Γ -function we obtain

$$t_k(0) = \frac{\sin \pi \left(\varepsilon + \frac{1}{2} - i\rho \right)}{\left| \sin \pi \left(\varepsilon + \frac{1}{2} + i\rho \right) \right|} \overline{s_k(0)},$$

and this gives

$$\begin{aligned} \Delta(0) &= -\frac{2i}{\pi} \left| \sin \pi \left(\varepsilon + \frac{1}{2} + i\rho \right) \right| |\Gamma(2i\rho)|^2 (t_k(0) \overline{t_l(0)} + \overline{t_k(0)} t_l(0)) \\ &= -\frac{2i}{\pi} \left| \sin \pi \left(\varepsilon + \frac{1}{2} + i\rho \right) \right| |\Gamma(2i\rho)|^2 \begin{pmatrix} \overline{t_k(0)} \\ t_k(0) \end{pmatrix}^* \begin{pmatrix} \overline{t_l(0)} \\ t_l(0) \end{pmatrix}. \end{aligned}$$

Comparing this result with (2.6), we see that for $x = 0$ the Laguerre function can be defined formally by

$$\psi_n(0; \rho, \varepsilon) = |\Gamma(2i\rho)| \begin{pmatrix} \overline{t_k(0)} \\ t_k(0) \end{pmatrix}.$$

3. CLEBSCH-GORDAN COEFFICIENTS FOR PARABOLIC BASIS VECTORS OF $su(1, 1)$

3.1. Orthogonal polynomials and functions

The Wilson polynomials, see Wilson [18] or [1, §3.8], are polynomials on top of the Askey-scheme of hypergeometric polynomials, see Koekoek and Swarttouw [8]. The continuous dual Hahn polynomials are a three-parameter subclass of the Wilson polynomials, and are defined by

$$(3.1) \quad s_n(y; a, b, c) = (a+b)_n (a+c)_n {}_3F_2 \left(\begin{matrix} -n, a+ix, a-ix \\ a+b, a+c \end{matrix}; 1 \right), \quad x^2 = y.$$

For real parameters a, b, c , with $a+b, a+c, b+c$ positive, the continuous dual Hahn polynomials are orthogonal with respect to a positive measure, supported on a subset of \mathbb{R} . The orthonormal continuous dual Hahn polynomials are defined by

$$S_n(y; a, b, c) = \frac{(-1)^n s_n(y; a, b, c)}{\sqrt{n! (a+b)_n (a+c)_n (b+c)_n}}$$

By Kummer's transformation, see e.g. [1, Cor. 3.3.5], the polynomials s_n and S_n are symmetric in a, b and c . Without loss of generality we assume that a is the smallest of the real parameters a, b and c . Let $d\mu(\cdot; a, b, c)$ be the measure defined by

$$\int_{\mathbb{R}} f(y) d\mu(y; a, b, c) = \frac{1}{2\pi} \int_0^{\infty} \left| \frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)}{\Gamma(2ix)} \right|^2 \frac{f(x^2)}{\Gamma(a+b)\Gamma(a+c)\Gamma(b+c)} dx + \frac{\Gamma(b-a)\Gamma(c-a)}{\Gamma(-2a)\Gamma(b+c)} \sum_{k=0}^K (-1)^k \frac{(2a)_k (a+1)_k (a+b)_k (a+c)_k}{(a)_k (a-b+1)_k (a-c+1)_k k!} f(-(a+k)^2),$$

where K is the largest non-negative integer such that $a + K < 0$. In particular, the measure $d\mu(\cdot; a, b, c)$ is absolutely continuous if $a \geq 0$. The measure is positive under the conditions $a + b > 0$, $a + c > 0$ and $b + c > 0$. Then the polynomials $S_n(y; a, b, c)$ are orthonormal with respect to the measure $d\mu(y; a, b, c)$.

The Laguerre polynomials are defined by

$$(3.2) \quad L_n^{(\alpha)}(x) = \frac{(\alpha + 1)_n}{n!} {}_1F_1(-n; \alpha + 1; x).$$

The orthonormal Laguerre polynomials

$$l_n^{(\alpha)}(x) = \sqrt{\frac{n!}{(\alpha + 1)_n}} L_n^{(\alpha)}(x).$$

are orthonormal on $[0, \infty)$ with respect to the weight function

$$w^{(\alpha)}(x) = \frac{x^\alpha e^{-x}}{\Gamma(\alpha + 1)}.$$

They satisfy the three-term recurrence relation

$$x l_n^{(\alpha)}(x) = -\sqrt{(n+1)(\alpha+n+1)} l_{n+1}^{(\alpha)}(x) + (2n+\alpha+1) l_n^{(\alpha)}(x) - \sqrt{n(n+\alpha)} l_{n-1}^{(\alpha)}(x),$$

and the differential equation

$$x y''(x) + (\alpha + 1 - x) y'(x) + n y(x) = 0, \quad y(x) = l_n^{(\alpha)}(x).$$

The Jacobi functions, see [12], are defined by

$$(3.3) \quad \varphi_\lambda^{(\alpha, \beta)}(x) = {}_2F_1\left(\frac{1}{2}(\alpha + \beta + 1 - i\lambda), \frac{1}{2}(\alpha + \beta + 1 + i\lambda); -x\right).$$

Here we use the unique analytic continuation to $\mathbb{C} \setminus [1, \infty)$ of the hypergeometric function. The Jacobi functions are eigenfunctions of the hypergeometric differential operator

$$-x(1+x) \frac{d^2}{dx^2} - [\alpha + 1 + (\alpha + \beta + 2)x] \frac{d}{dx}$$

for eigenvalue $\frac{1}{4}[(\alpha + \beta + 1)^2 + \lambda^2]$. Spectral analysis of the hypergeometric differential operator leads to a unitary integral transform called the Jacobi-function transform. The Jacobi-function transform is given by

$$(3.4) \quad \begin{cases} (Ff)(\lambda) = \int_0^\infty f(x) \varphi_\lambda^{(\alpha, \beta)}(x) \Delta_{\alpha, \beta}(x) dx \\ f(x) = \frac{1}{2\pi} \int (Ff)(\lambda) \varphi_\lambda^{(\alpha, \beta)}(x) d\nu(\lambda) \end{cases}$$

where $\alpha > -1$, $\beta \in \mathbb{R}$, $\Delta_{\alpha, \beta}(x) = 2^{2\alpha+2\beta+1} x^\alpha (1+x)^\beta$, and $d\nu(\lambda)$ is the measure given by

$$\begin{aligned} \frac{1}{2\pi} \int g(\lambda) d\nu(\lambda) &= \frac{1}{2\pi} \int_0^\infty g(\lambda) |c_{\alpha, \beta}(\lambda)|^{-2} d\lambda - i \sum_{\lambda \in D} g(\lambda) \operatorname{Res}_{\mu=\lambda} (c_{\alpha, \beta}(\mu) c_{\alpha, \beta}(-\mu))^{-1}, \\ c_{\alpha, \beta}(\lambda) &= \frac{2^{\alpha+\beta+1-i\lambda} \Gamma(\alpha+1) \Gamma(i\lambda)}{\Gamma(\frac{1}{2}(\alpha+\beta+1+i\lambda)) \Gamma(\frac{1}{2}(\alpha-\beta+1+i\lambda))}, \\ D &= \left\{ i(|\beta| - \alpha - 1 - 2j) \mid j \in \mathbb{Z}_{\geq 0}, |\beta| - \alpha - 1 - 2j > 0 \right\}. \end{aligned}$$

Observe that the measure $d\nu(\lambda)$ is absolutely continuous if $|\beta| \leq \alpha + 1$.

3.2. The Lie algebra $su(1, 1)$

The Lie algebra $su(1, 1)$ is a three dimensional Lie algebra, generated by H , B and C satisfying the commutation relations

$$(3.5) \quad [H, B] = 2B, \quad [H, C] = -2C, \quad [B, C] = H.$$

There is a $*$ -structure defined by $H^* = H$ and $B^* = -C$. The Casimir operator Ω is a central element of $U(su(1, 1))$, and Ω is given by

$$(3.6) \quad \Omega = -\frac{1}{4}(H^2 + 2H + 4CB).$$

There are four classes of irreducible unitary representations of $su(1, 1)$, see [17, §6.4]:

The positive discrete series representations π_k^+ are representations labelled by $k > 0$. The representation space is $\ell^2(\mathbb{Z}_{\geq 0})$ with orthonormal basis $\{e_n\}_{n \in \mathbb{Z}_{\geq 0}}$. The action is given by

$$(3.7) \quad \begin{aligned} \pi_k^+(H) e_n &= 2(k+n) e_n, \\ \pi_k^+(B) e_n &= \sqrt{(n+1)(2k+n)} e_{n+1}, \\ \pi_k^+(C) e_n &= -\sqrt{n(2k+n-1)} e_{n-1}, \\ \pi_k^+(\Omega) e_n &= k(1-k) e_n. \end{aligned}$$

The negative discrete series representations π_k^- are labelled by $k > 0$. The representation space is $\ell^2(\mathbb{Z}_{\geq 0})$ with orthonormal basis $\{e_n\}_{n \in \mathbb{Z}_{\geq 0}}$. The action is given by

$$(3.8) \quad \begin{aligned} \pi_k^-(H) e_n &= -2(k+n) e_n, \\ \pi_k^-(B) e_n &= -\sqrt{n(2k+n-1)} e_{n-1}, \\ \pi_k^-(C) e_n &= \sqrt{(n+1)(2k+n)} e_{n+1}, \\ \pi_k^-(\Omega) e_n &= k(1-k) e_n. \end{aligned}$$

The principal series representations $\pi^{\rho,\varepsilon}$ are labelled by $\varepsilon \in [0, 1)$ and $\rho \geq 0$, where $(\rho, \varepsilon) \neq (0, \frac{1}{2})$. The representation space is $\ell^2(\mathbb{Z})$ with orthonormal basis $\{e_n\}_{n \in \mathbb{Z}}$. The action is given by

$$(3.9) \quad \begin{aligned} \pi^{\rho,\varepsilon}(H) e_n &= 2(\varepsilon + n) e_n, \\ \pi^{\rho,\varepsilon}(B) e_n &= |(n + \varepsilon + \frac{1}{2} + i\rho)| e_{n+1}, \\ \pi^{\rho,\varepsilon}(C) e_n &= -|(n + \varepsilon - \frac{1}{2} + i\rho)| e_{n-1}, \\ \pi^{\rho,\varepsilon}(\Omega) e_n &= (\rho^2 + \frac{1}{4}) e_n. \end{aligned}$$

For $(\rho, \varepsilon) = (0, \frac{1}{2})$ the representation $\pi^{0, \frac{1}{2}}$ splits into a direct sum of a positive and a negative discrete series representation: $\pi^{0, \frac{1}{2}} = \pi_+^{0, \frac{1}{2}} \oplus \pi_-^{0, \frac{1}{2}}$. The representation space splits into two invariant subspaces: $\{e_n \mid n < \frac{1}{2}\} \oplus \{e_n \mid n \geq 0\}$.

The complementary series representations $\pi^{\lambda,\varepsilon}$ are labelled by ε and λ , where $\varepsilon \in [0, \frac{1}{2})$ and $\lambda \in (-\frac{1}{2}, -\varepsilon)$ or $\varepsilon \in (\frac{1}{2}, 1)$ and $\lambda \in (-\frac{1}{2}, \varepsilon - 1)$. The representation space is $\ell^2(\mathbb{Z})$ with orthonormal basis $\{e_n\}_{n \in \mathbb{Z}}$. The action is given by

$$(3.10) \quad \begin{aligned} \pi^{\lambda,\varepsilon}(H) e_n &= 2(\varepsilon + n) e_n, \\ \pi^{\lambda,\varepsilon}(B) e_n &= \sqrt{(n + \varepsilon + 1 + \lambda)(n + \varepsilon - \lambda)} e_{n+1}, \\ \pi^{\lambda,\varepsilon}(C) e_n &= -\sqrt{(n + \varepsilon + \lambda)(n + \varepsilon - \lambda - 1)} e_{n-1}, \\ \pi^{\lambda,\varepsilon}(\Omega) e_n &= -\lambda(1 + \lambda) e_n. \end{aligned}$$

Note that formally for $\lambda = -\frac{1}{2} + i\rho$ the actions in the principal series and in the complementary series are the same.

We remark that the operators (3.7)-(3.10) are unbounded, with domain the set of finite linear combinations of the basis vectors. The representations are *-representations in the sense of Schmüdgen [15, Ch.8].

The decomposition of the tensor product of a positive and a negative discrete series representation of $su(1, 1)$ is determined in [6, Thm.2.2], see also [17, §8.7.7] for the group $SU(1, 1)$.

Theorem 3.1. *For $k_1 \leq k_2$ the decomposition of the tensor product of positive and negative discrete series representations of $su(1, 1)$ is*

$$\begin{aligned} \pi_{k_1}^+ \otimes \pi_{k_2}^- &\cong \int_0^{\oplus \infty} \pi^{\rho,\varepsilon} d\rho, & k_1 - k_2 \geq -\frac{1}{2}, k_1 + k_2 \geq \frac{1}{2}, \\ \pi_{k_1}^+ \otimes \pi_{k_2}^- &\cong \int_0^{\oplus \infty} \pi^{\rho,\varepsilon} d\rho \oplus \pi^{\lambda,\varepsilon}, & k_1 + k_2 < \frac{1}{2}, \\ \pi_{k_1}^+ \otimes \pi_{k_2}^- &\cong \int_0^{\oplus \infty} \pi^{\rho,\varepsilon} d\rho \oplus \bigoplus_{\substack{j \in \mathbb{Z}_{\geq 0} \\ k_2 - k_1 - \frac{1}{2} - j > 0}} \pi_{k_2 - k_1 - j}^-, & k_1 - k_2 < -\frac{1}{2}, \end{aligned}$$

where $\varepsilon = k_1 - k_2 + L$, L is the unique integer such that $\varepsilon \in [0, 1)$, and $\lambda = -k_1 - k_2$. Further, under the identification above,

$$(3.11) \quad e_{n_1} \otimes e_{n_2} = (-1)^{n_2} \int_{\mathbb{R}} S_n(y; n_1 - n_2) e_{n_1 - n_2 - L} d\mu^{\frac{1}{2}}(y; n_1 - n_2),$$

where $n = \min\{n_1, n_2\}$, $S_n(y; p)$ is an orthonormal continuous dual Hahn polynomial,

$$S_n(y; p) = \begin{cases} S_n(y; k_1 - k_2 + \frac{1}{2}, k_1 + k_2 - \frac{1}{2}, k_2 - k_1 - p + \frac{1}{2}), & p \leq 0, \\ S_n(y; k_2 - k_1 + \frac{1}{2}, k_1 + k_2 - \frac{1}{2}, k_1 - k_2 + p + \frac{1}{2}), & p \geq 0, \end{cases}$$

and

$$d\mu(y; p) = \begin{cases} d\mu(y; k_1 - k_2 + \frac{1}{2}, k_1 + k_2 - \frac{1}{2}, k_2 - k_1 - p + \frac{1}{2}), & p \leq 0, \\ d\mu(y; k_2 - k_1 + \frac{1}{2}, k_1 + k_2 - \frac{1}{2}, k_1 - k_2 + p + \frac{1}{2}), & p \geq 0. \end{cases}$$

The inversion of (3.11) can be given explicitly, e.g. for an element

$$f \otimes e_{r-L} = \int_0^\infty f(x) e_{r-L} dx \in L^2(0, \infty) \otimes \ell^2(\mathbb{Z}) \cong \int_0^\infty \ell^2(\mathbb{Z}) dx$$

in the representation space of the direct integral representation, we have

$$(3.12) \quad f \otimes e_{r-L} = \begin{cases} \sum_{p=0}^\infty (-1)^{p-r} \left[\int_{\mathbb{R}} S_p(y; r) f(y) d\mu^{\frac{1}{2}}(y; r) \right] e_p \otimes e_{p-r}, & r \leq 0, \\ \sum_{p=0}^\infty (-1)^p \left[\int_{\mathbb{R}} S_p(y; r) f(y) d\mu^{\frac{1}{2}}(y; r) \right] e_{p+r} \otimes e_p, & r \geq 0. \end{cases}$$

For the discrete components in Theorem 3.1 we can replace f by a Dirac delta function at the appropriate points of the discrete mass of $d\mu(\cdot; r)$. In the following subsections we assume that discrete terms do not occur in the tensor product decomposition. From the calculations it is clear how to extend the results to the general case.

3.3. Parabolic basisvectors

We consider the self-adjoint element

$$X = -H + B - C \in su(1, 1),$$

which is a parabolic element. We determine the spectral decomposition of X in the various representations. We also give (generalized) eigenvectors of X . This is done in the same way as in [10], using (doubly infinite) Jacobi operators. First we consider X in the discrete series. The action of X can be identified with the three-term recurrence relation for the Laguerre polynomials.

Proposition 3.2. *The operators Θ^\pm defined by*

$$\Theta^\pm : \ell^2(\mathbb{Z}_{\geq 0}) \rightarrow L^2([0, \infty), w^{(2k-1)}(x)dx)$$

$$e_n \mapsto l_n^{(2k-1)}(\cdot),$$

are unitary and intertwine $\pi_k^\pm(X)$ with $M_{\mp x}$.

Here M denotes the multiplication operator: $M_f g(x) = f(x)g(x)$.

In terms of generalized eigenvectors, Proposition 3.2 states that

$$v^\pm(x) = \sum_{n=0}^{\infty} l_n^{(2k-1)}(x)e_n, \quad x \in [0, \infty),$$

is a generalized eigenvector of $\pi_k^\pm(X)$ for eigenvalue $\mp x$. These eigenvectors can be considered as parabolic basis vectors for $su(1, 1)$.

Next we consider X in the principal unitary series. We find that $\pi^{\rho, \varepsilon}(X)$ extends to a doubly infinite Jacobi operator which corresponds to the recurrence relation for the Laguerre functions. The spectral analysis of $\pi^{\rho, \varepsilon}(X)$ is carried out in §2.

Proposition 3.3. *The operator $\Theta^{\rho, \varepsilon}$ defined by*

$$\Theta^{\rho, \varepsilon} : \ell^2(\mathbb{Z}) \rightarrow L^2(\mathbb{R}, w(x; \rho, \varepsilon)dx)$$

$$e_n \mapsto (-1)^n \psi_n(\cdot; \rho, \varepsilon)$$

is unitary and intertwines $\pi^{\rho, \varepsilon}(X)$ with M_x .

So, for $x \in \mathbb{R} \setminus \{0\}$,

$$v^{\rho, \varepsilon}(x) = \sum_{n=-\infty}^{\infty} (-1)^n \psi_n(x; \rho, \varepsilon)e_n,$$

is a generalized eigenvector of $\pi^{\rho, \varepsilon}(X)$ for eigenvalue x . We exclude the point $x = 0$ because the Laguerre functions are not defined at that point.

Next we consider the action of X in the tensor product. Recall that $\Delta(Y) = 1 \otimes Y + Y \otimes 1$ for $Y \in su(1, 1)$. Then we find from Proposition 3.2 the following result.

Proposition 3.4. *The operator Υ defined by*

$$\Upsilon : \ell^2(\mathbb{Z}_{\geq 0}) \otimes \ell^2(\mathbb{Z}_{\geq 0}) \rightarrow L^2([0, \infty) \times [0, \infty), w^{(2k_1-1)}(x_1)w^{(2k_2-1)}(x_2)dx_1 dx_2)$$

$$e_{n_1} \otimes e_{n_2} \mapsto l_{n_1}^{(2k_1-1)}(x_1)l_{n_2}^{(2k_2-1)}(x_2),$$

is unitary and intertwines $\pi_{k_1}^+ \otimes \pi_{k_2}^-(\Delta(X))$ with $M_{x_2 - x_1}$.

So

$$v^+(x_1) \otimes v^-(x_2) = \sum_{n_1, n_2=0}^{\infty} l_{n_1}^{(2k_1-1)}(x_1)l_{n_2}^{(2k_2-1)}(x_2) e_{n_1} \otimes e_{n_2}$$

is a generalized eigenvector of $\pi_{k_1}^+ \otimes \pi_{k_2}^-(\Delta(X))$ for eigenvalue $x_2 - x_1$.

3.4. Clebsch-Gordan coefficients

We want to determine the Clebsch-Gordan coefficients between the bases of uncoupled eigenvectors $v^+(x_1) \otimes v^-(x_2)$ and coupled eigenvectors $\int^{\oplus} v^{\rho, \varepsilon}(x_2 - x_1) d\rho$ of the operator $\pi_{k_1}^+ \otimes \pi_{k_2}^-(\Delta(X))$ for eigenvalue $x_2 - x_1$. So we want to find the Clebsch-Gordan coefficient g , a function of ρ depending on x_1, x_2 (and k_1, k_2), satisfying

$$(3.13) \quad v^+(x_1) \otimes v^-(x_2) = \int_0^\infty g(\rho) v^{\rho, \varepsilon}(x_2 - x_1) d\rho.$$

Note that this is an expression for generalized eigenvectors of the self-adjoint operator $\pi_{k_1}^+ \otimes \pi_{k_2}^-(\Delta(X)) = \int^{\oplus} \pi^{\rho, \varepsilon}(X) d\rho$. Because the self-adjoint operators $\pi_{k_1}^+(X), \pi_{k_2}^-(X), \pi^{\rho, \varepsilon}(X)$ have simple spectrum, cf. Propositions 3.2 and 3.3, we see that g is uniquely determined by (3.13) (almost everywhere). The function g can be determined from the above expression by taking inner products with $e_{n_1} \otimes e_{n_2} = \int S_n e_{n_1 - n_2 - L} d\mu^{\frac{1}{2}}$. This comes down to finding the function g , for which the operator \mathcal{T}_g defined by

$$\begin{aligned} \mathcal{T}_g : L^2(0, \infty) \otimes \ell^2(\mathbb{Z}) &\cong \oplus \int_0^\infty \ell^2(\mathbb{Z}) dx \rightarrow \oplus \int_0^\infty L^2(\mathbb{R}, w(t; \rho, \varepsilon) dt) d\rho, \\ f \otimes e_n &\mapsto (-1)^n \int_0^\infty f(\rho) g(\rho) \psi_n(t; \rho, \varepsilon) d\rho, \end{aligned}$$

is the same as \mathcal{T} . Note that it follows from (3.13) that the Clebsch-Gordan coefficient g does not depend on n_1 and n_2 .

From Theorem 3.1 and (3.9) we know that the Clebsch-Gordan coefficient g must be an eigenfunction of the Casimir operator Ω in the tensor product for eigenvalue $\rho^2 + \frac{1}{4}$. So first we determine the actions of the generators H, B and C on parabolic basis vectors.

We start with a very simple lemma, which is based on the fact that $sl(2, \mathbb{C})$ is semi-simple, so $[sl(2, \mathbb{C}), sl(2, \mathbb{C})] = sl(2, \mathbb{C})$.

Lemma 3.5.

$$B = \frac{1}{4}[H, X] + \frac{1}{2}X + \frac{1}{2}H, \quad C = \frac{1}{4}[H, X] - \frac{1}{2}X - \frac{1}{2}H.$$

Proof. From the definition of X and the commutation relations (3.5) we find $[H, X] = 2B + 2C$. This proves the lemma. \square

This lemma shows that to find the action of the generators H, B and C , it is enough to find the action of H , since the action of X is known.

Proposition 3.6. *In the positive discrete series, the generators H, B, C have a realization as densely defined differential operators acting on $L^2([0, \infty), w^{(2k-1)}(x) dx)$:*

$$\begin{aligned}\pi_k^+(H) &= -2x \frac{d^2}{dx^2} - 2(2k-x) \frac{d}{dx} + 2k, \\ \pi_k^+(B) &= -x \frac{d^2}{dx^2} - 2(k-x) \frac{d}{dx} + (2k-x), \\ \pi_k^+(C) &= x \frac{d^2}{dx^2} + 2k \frac{d}{dx}.\end{aligned}$$

In the negative discrete series H, B, C have a realization as densely defined differential operators acting on $L^2([0, \infty), w^{(2k-1)}(x)dx)$:

$$\begin{aligned}\pi_k^-(H) &= 2x \frac{d^2}{dx^2} + 2(2k-x) \frac{d}{dx} - 2k, \\ \pi_k^-(B) &= x \frac{d^2}{dx^2} + 2k \frac{d}{dx}, \\ \pi_k^-(C) &= -x \frac{d^2}{dx^2} - 2(k-x) \frac{d}{dx} + (2k-x).\end{aligned}$$

Proof. We show that Θ^+ intertwines the actions of H, B and C given by (3.7), with the differential operators given in the proposition.

From the differential equation for the Laguerre polynomials, we find for the action of H

$$\begin{aligned}\Theta^+ \pi_k^+(H) e_n &= (2n+2k) l_n^{(2k-1)}(x) \\ &= -2x \frac{d^2}{dx^2} l_n^{(2k-1)}(x) - 2(2k-x) \frac{d}{dx} l_n^{(2k-1)}(x) + 2k l_n^{(2k-1)}(x) \\ &= \left(-2x \frac{d^2}{dx^2} - 2(2k-x) \frac{d}{dx} + 2k \right) \Theta^+ e_n.\end{aligned}$$

So we have realized $\pi_k^+(H)$ as a differential operator. By Proposition 3.2 $\pi_k^+(X)$ is realized as the multiplication operator M_{-x} . A direct calculation shows that

$$\Theta^+ \pi_k^+([H, X]) e_n = \left(4x \frac{d}{dx} + 2(2k-x) \right) \Theta^+ e_n.$$

Then Lemma 3.5 proves the proposition for the positive discrete series. Note that the differential operators are defined on the space consisting of polynomials which is a dense subspace of $L^2([0, \infty), w^{(2k-1)}(x)dx)$. We find the action in the negative discrete series in the same way, or we use the Lie-algebra isomorphism ϑ , given by

$$\vartheta(H) = -H, \quad \vartheta(B) = C, \quad \vartheta(C) = B.$$

Then $\pi_k^+(\vartheta(Y)) = \pi_k^-(Y)$ for $Y \in su(1, 1)$.

A straightforward calculation shows that these operators indeed satisfy the $su(1, 1)$ commutation relations. \square

It is also possible to find the actions of H, B and C on the eigenvectors $v^{\rho, \varepsilon}(x)$. This is done using the differential equation for the Laguerre functions, which follows from the confluent hypergeometric differential equation. We do not need these actions here.

Next we want to calculate $\pi_{k_1}^+ \otimes \pi_{k_2}^- (\Delta(\Omega))$ for these realizations. From (3.6) we obtain

$$(3.14) \quad \Delta(\Omega) = 1 \otimes \Omega + \Omega \otimes 1 - \frac{1}{2} H \otimes H - (C \otimes B + B \otimes C).$$

Proposition 3.6 shows that $\pi_{k_1}^+ \otimes \pi_{k_2}^- (\Delta(\Omega))$ is a differential operator acting on a dense subspace of $L^2([0, \infty) \times [0, \infty), w^{(2k_1-1)}(x_1)w^{(2k_2-1)}(x_2)dx_1dx_2)$. Let $p(x_1)$ and $q(x_2)$ be polynomials in x_1 , respectively x_2 , then we find after a long calculation in which many terms cancel,

$$(3.15) \quad \begin{aligned} & \pi_{k_1}^+ \otimes \pi_{k_2}^- (\Delta(\Omega))p(x_1)q(x_2) = \\ & -x_1x_2 \left(p''(x_1)q(x_2) + 2p'(x_1)q'(x_2) + p(x_1)q''(x_2) \right) \\ & + (2x_1x_2 - 2k_1x_2 - 2k_2x_1) \left(p'(x_1)q(x_2) + p(x_1)q'(x_2) \right) \\ & + (2k_1x_2 + 2k_2x_1 - 2k_1k_2 - x_1x_2 \\ & + k_1(1 - k_1) + k_2(1 - k_2))p(x_1)q(x_2), \end{aligned}$$

where $p'(x_1) = \frac{d}{dx_1}p(x_1)$ and $q'(x_2) = \frac{d}{dx_2}q(x_2)$. For $t > 0$ let H_t be the space consisting of polynomials of the form $p(x)q(x+t)$, and for $t < 0$ let H_t be the space consisting of polynomials of the form $p(x-t)q(x)$.

Proposition 3.7. *For fixed $t \in \mathbb{R} \setminus \{0\}$ we have*

$$\begin{aligned} & M_{e^x}^{-1} \circ \pi_{k_1}^+ \otimes \pi_{k_2}^- (\Delta(\Omega))|_{H_t} \circ M_{e^x} \\ & = \begin{cases} -x(x-t) \frac{d^2}{dx^2} - (2k_1x + 2k_2(x-t)) \frac{d}{dx} + ((k_1(1-k_1) \\ \quad + k_2(1-k_2) - 2k_1k_2), & t < 0, \\ -x(x+t) \frac{d^2}{dx^2} - (2k_1(x+t) + 2k_2x) \frac{d}{dx} \\ \quad + ((k_1(1-k_1) + k_2(1-k_2) - 2k_1k_2), & t > 0. \end{cases} \end{aligned}$$

Proof. First we assume $t > 0$. Put $x = x_1$, $t = x_2 - x_1$ and $e^x\varphi(x; t) = p(x)q(x+t)$ in (3.15), then

$$\begin{aligned} & \pi_{k_1}^+ \otimes \pi_{k_2}^- (\Delta(\Omega))e^x\varphi(x; t) = \\ & e^x \left[-x(x+t) \frac{d^2\varphi}{dx^2} - (2k_1(x+t) + 2k_2x) \frac{d\varphi}{dx} \right. \\ & \quad \left. + ((k_1(1-k_1) + k_2(1-k_2) - 2k_1k_2)\varphi \right]. \end{aligned}$$

This proves the proposition for $t > 0$. The case $t < 0$ is proved similarly. \square

For $t > 0$ the differential operator in Proposition 3.7 has the Jacobi functions $\varphi_{2\rho}^{(2k_1-1, 2k_2-1)}(x/t)$ as eigenfunctions for eigenvalue $\rho^2 + \frac{1}{4}$, and acts on the space $L^2(\mathbb{R}, \Delta_{2k_1-1, 2k_2-1}(x/t)dx/t)$, since $w^{(2k_1-1)}(x+t)w^{(2k_2-1)}(x) =$

$C e^{-2x} \Delta_{2k_1-1, 2k_2-1}(x/t)$ where C is a factor independent of x . A similar observation can be made for $t < 0$. So the spectral analysis of the self-adjoint operator $\Delta(\Omega)_t = \pi_{k_1}^+ \otimes \pi_{k_2}^-(\Delta(\Omega))|_{H_t}$ leads to the Jacobi function transform, and we can identify the spectrum of $\Delta(\Omega)_t$ with the support of the measure $d\nu(2\rho)$. We find that the support is exactly the same as the support of the orthogonality measure for the continuous dual Hahn polynomials given in Theorem 3.1. In particular the spectrum of $\Delta(\Omega)_t$ is simple, so the Clebsch-Gordan coefficients are determined up to a factor independent of x . Now the Clebsch-Gordan coefficients are given by $C_+ e^{x_2} \varphi_{2\rho}^{(2k_1-1, 2k_2-1)}(x/t)$ and $C_- e^{x_1} \varphi_{2\rho}^{(2k_2-1, 2k_1-1)}(-x/t)$, for $t > 0$, respectively $t < 0$, where C_+ and C_- are factors independent of x which need to be determined.

Theorem 3.8. *The Clebsch-Gordan coefficients for the parabolic bases are given by*

$$g(\rho) = \begin{cases} C_-(\rho) e^{x_2} \varphi_{2\rho}^{(2k_2-1, 2k_1-1)}\left(\frac{x_2}{x_1 - x_2}\right), & x_2 - x_1 < 0, \\ C_+(\rho) e^{x_1} \varphi_{2\rho}^{(2k_1-1, 2k_2-1)}\left(\frac{x_1}{x_2 - x_1}\right), & x_2 - x_1 > 0, \end{cases}$$

where

$$C_-(\rho) = \frac{(-1)^L}{\sqrt{2\pi}} (x_1 - x_2)^{\frac{1}{2}k_1 - k_2 - i\rho} \sqrt{\frac{\Gamma(2k_1)}{\Gamma(2k_2)}} \frac{\Gamma(k_1 + k_2 - \frac{1}{2} + i\rho)}{\Gamma(k_1 - k_2 + \frac{1}{2} + i\rho) \Gamma(2i\rho)},$$

$$C_+(\rho) = \frac{1}{\sqrt{2\pi}} (x_2 - x_1)^{\frac{1}{2}k_1 - k_2 + i\rho} \sqrt{\frac{\Gamma(2k_2)}{\Gamma(2k_1)}} \frac{\Gamma(k_1 + k_2 - \frac{1}{2} + i\rho)}{\Gamma(k_2 - k_1 + \frac{1}{2} + i\rho) \Gamma(2i\rho)}.$$

Proof. Recall from the beginning of this subsection that the Clebsch-Gordan coefficients are the unique functions g such that $\mathcal{Y} = \mathcal{Y}_g$, or equivalently, the functions g determined by

$$\langle v^+(x_1) \otimes v^-(x_2), e_{n_1} \otimes e_{n_2} \rangle = \left\langle \int_0^\infty g(\rho) v^{\rho, \varepsilon}(x_2 - x_1) d\rho, \int_0^\infty S_n(\rho) e_{n_1 - n_2 - L} d\mu^{\frac{1}{2}}(\rho) \right\rangle.$$

Here $d\mu$ is the orthogonality measure for the continuous dual Hahn polynomials as in Theorem 3.1. Put $x = x_1$, $t = x_2 - x_1$ and assume $t > 0$. Since g is independent of n_1 and n_2 , it is enough to verify that the function g in the theorem satisfies the above identity in case $n_1 = n_2 = 0$. Explicitly, we must verify the following identity

$$1 = \frac{(-1)^L}{\sqrt{2\pi}} \int_0^\infty e^{x_2} \varphi_{2\rho}^{(2k_1-1, 2k_2-1)}\left(\frac{x}{t}\right) \psi_{-L}(t; \rho, k_1 - k_2 + L) \\ \times \frac{C_+(\rho)}{\sqrt{\Gamma(2k_1)\Gamma(2k_2)}} \left| \frac{\Gamma(k_1 + k_2 - \frac{1}{2} + i\rho) \Gamma(k_2 - k_1 + \frac{1}{2} + i\rho) \Gamma(k_1 - k_2 + \frac{1}{2} + i\rho)}{\Gamma(2i\rho)} \right| d\rho.$$

We use an integral representation for the second solution of the confluent hy-

pergeometric differential equation, see [16, (3.2.55)] (note that there is a misprint in the exponent of z in [16]),

$$U(a; b; z) = \frac{z^{c-a}}{\Gamma(c)} \int_0^\infty e^{-zy} y^{c-1} {}_2F_1\left(a, 1+a-b; c; -y\right) dy, \quad \Re c > 0, \Re z > 0,$$

with parameters given by

$$a = k_1 - k_2 + \frac{1}{2} + i\rho, \quad b = 1 + 2i\rho, \quad c = 2k_1, \quad y = \frac{x}{t}, \quad z = t.$$

By the definition of the Laguerre function $\psi_n(t)$ for $t > 0$, see §2, the definition of a Jacobi function (3.3) funct and Euler's transformation [1, (2.2.7)], we have

$$\psi_{-L}(t; \rho, k_1 - k_2 + L) = (-1)^L 2^{3-4k_1-4k_2} t^{k_1+k_2-\frac{1}{2}-i\rho} \frac{|\Gamma(k_1 - k_2 + \frac{1}{2} + i\rho)|}{\Gamma(2k_1)} \int_0^\infty e^{-x} \varphi_{2\rho}^{(2k_1-1, 2k_2-1)}\left(\frac{x}{t}\right) \Delta_{2k_1-1, 2k_2-1}\left(\frac{x}{t}\right) \frac{dx}{t}.$$

Taking the inverse Jacobi transform, which is allowed since $e^{-x} \in L^2([0, \infty), \Delta_{2k_1-1, 2k_2-2}(x/t) dx/t)$, we find

$$\begin{aligned} 1 &= (-1)^L \frac{t^{\frac{1}{2}-k_1-k_2}}{2\pi} \int_0^\infty e^x \varphi_{2\rho}^{(2k_1-1, 2k_2-1)}\left(\frac{x}{t}\right) t^{-i\rho} \psi_{-L}(t; \rho, k_1 - k_2 + L) \\ &\quad \times \frac{|\Gamma(k_1 - k_2 + \frac{1}{2} + i\rho)|}{\Gamma(2k_1)} \left| \frac{\Gamma(k_1 + k_2 - \frac{1}{2} + i\rho)}{\Gamma(2i\rho)} \right|^2 d\rho \end{aligned}$$

This is the desired identity. Note that it follows from the unitarity of the Jacobi transform, that the factor $C_+(\rho)$ is unique. For $t = x_1 - x_2 < 0$ the theorem is proved similarly. \square

Remark 3.9. The explicit expressions for the Clebsch-Gordan coefficients as ${}_2F_1$ -series can also be found in Basu and Wolf [2]. The method used in [2] to compute the Clebsch-Gordan coefficients is different from the method used here.

Theorem 3.8 gives the following Clebsch-Gordan decomposition for the parabolic basis vectors

$$v^+(x_1) \otimes v^-(x_2) = \begin{cases} \int_0^\infty C_-(\rho) e^{x_2} \varphi_{2\rho}^{(2k_2-1, 2k_1-1)}\left(\frac{x_2}{x_1 - x_2}\right) v^{\rho, \varepsilon}(x_2 - x_1) d\rho, \\ \quad x_2 - x_1 < 0, \\ \int_0^\infty C_+(\rho) e^{x_1} \varphi_{2\rho}^{(2k_1-1, 2k_2-1)}\left(\frac{x_1}{x_2 - x_1}\right) v^{\rho, \varepsilon}(x_2 - x_1) d\rho, \\ \quad x_2 - x_1 > 0, \end{cases}$$

and the inversion of this can be found using the Jacobi transform.

From Theorem 3.8 we obtain a product formula for Laguerre polynomials.

Theorem 3.10. Let $|k_1 - k_2| \leq \frac{1}{2}$ and $k_1 + k_2 \geq \frac{1}{2}$, then the Laguerre polynomials satisfy the following product formula: for $x_1 > x_2$,

$$\begin{aligned} & L_{n_1}^{(2k_1-1)}(x_1)L_{n_2}^{(2k_2-1)}(x_2) \\ &= \frac{1}{2\pi} \int_0^\infty d(\rho) s_{n_1}(\rho^2; k_2 - k_1 + n_2 - n_1 + \frac{1}{2}, k_1 + k_2 - \frac{1}{2}, k_1 - k_2 + \frac{1}{2}) \\ & \quad \times e^{x_2} \varphi_{2\rho}^{(2k_2-1, 2k_1-1)}\left(\frac{x_2}{x_1 - x_2}\right) U(n_2 - n_1 + k_2 - k_1 + \frac{1}{2} - i\rho; 1 - 2i\rho; x_1 - x_2) d\rho, \\ &= \frac{1}{2\pi} \int_{-\infty}^\infty D(\rho) s_{n_1}(\rho^2; k_2 - k_1 + n_2 - n_1 + \frac{1}{2}, k_1 + k_2 - \frac{1}{2}, k_1 - k_2 + \frac{1}{2}) \\ & \quad \times e^{x_2} \varphi_{2\rho}^{(2k_2-1, 2k_1-1)}\left(\frac{x_2}{x_1 - x_2}\right) {}_1F_1(n_2 - n_1 + k_2 - k_1 + \frac{1}{2} + i\rho; 1 + 2i\rho; x_1 - x_2) d\rho, \end{aligned}$$

for $x_1 = x_2 = x$,

$$\begin{aligned} & L_{n_1}^{(2k_1-1)}(x)L_{n_2}^{(2k_2-1)}(x) \\ &= \frac{1}{2\pi} \int_0^\infty s_{n_1}(\rho^2; k_2 - k_1 + n_2 - n_1 + \frac{1}{2}, k_1 + k_2 - \frac{1}{2}, k_1 - k_2 + \frac{1}{2}) \\ & \quad \times e^x \left(D_0(\rho) x^{\frac{1}{2}-k_1-k_2-i\rho} + D_0(-\rho) x^{\frac{1}{2}-k_1-k_2+i\rho} \right) d\rho \\ &= \frac{1}{2\pi} \int_{-\infty}^\infty D_0(\rho) s_{n_1}(\rho^2; k_2 - k_1 + n_2 - n_1 + \frac{1}{2}, k_1 + k_2 - \frac{1}{2}, k_1 - k_2 + \frac{1}{2}) e^x x^{\frac{1}{2}-k_1-k_2-i\rho} d\rho, \end{aligned}$$

where

$$\begin{aligned} d(\rho) &= \frac{(x_1 - x_2)^{\frac{1}{2}-k_1-k_2-i\rho}}{n_1! n_2! \Gamma(2k_2)} \left| \frac{\Gamma(k_1 + k_2 - \frac{1}{2} + i\rho) \Gamma(k_2 - k_1 + n_2 - n_1 + \frac{1}{2} + i\rho)}{\Gamma(2i\rho)} \right|^2, \\ D(\rho) &= (x_1 - x_2)^{\frac{1}{2}-k_1-k_2+i\rho} \frac{|\Gamma(k_1 + k_2 - \frac{1}{2} + i\rho)|^2 \Gamma(n_2 - n_1 + k_2 - k_1 + \frac{1}{2} + i\rho)}{n_1! n_2! \Gamma(2k_2) \Gamma(2i\rho)}, \\ D_0(\rho) &= \frac{1}{n_1! n_2!} \Gamma(k_1 + k_2 - \frac{1}{2} - i\rho) (k_2 - k_1 + \frac{1}{2} + i\rho)_{n_2 - n_1}. \end{aligned}$$

The case $x_1 < x_2$ follows from the case $x_1 > x_2$ using the substitutions $(k_1, x_1, n_1) \leftrightarrow (k_2, x_2, n_2)$.

Proof. For $x_1 > x_2$ the first equality in the theorem follows from writing out explicitly $\mathcal{Y}(e_{n_1} \otimes e_{n_2}) = \mathcal{Y}_g(\int^\oplus S_n(\rho; n_1 - n_2) e_{n_1 - n_2 - L} d\mu^{\frac{1}{2}})$, $n = \min\{n_1, n_2\}$, where g is given in Theorem 3.8 and using [6, (3.13)]

$$\begin{aligned} & s_n(\rho^2; k_1 - k_2 + \frac{1}{2}, k_1 + k_2 - \frac{1}{2}, k_2 - k_1 - p + \frac{1}{2}) = \\ & (-1)^p |(k_1 - k_2 + \frac{1}{2} + i\rho)_p|^2 s_{n-p}(\rho^2; k_2 - k_1 + \frac{1}{2}, k_1 + k_2 - \frac{1}{2}, k_1 - k_2 + p + \frac{1}{2}). \end{aligned}$$

Using (1.1) we have

$$\begin{aligned} & t^{\frac{1}{2}-k_1-k_2-i\rho} U(p + k_2 - k_1 + \frac{1}{2} - i\rho; 1 - 2i\rho; t) = \\ & \frac{t^{\frac{1}{2}-k_1-k_2+i\rho} \Gamma(-2i\rho)}{\Gamma(p + k_2 - k_1 + \frac{1}{2} - i\rho)} {}_1F_1(p + k_2 - k_1 + \frac{1}{2} + i\rho; 1 + 2i\rho; t) + \text{idem}(\rho \leftrightarrow -\rho), \end{aligned}$$

and the first integral for the case $x_1 > x_2$ splits according to this into two integrals. Substituting $\rho \mapsto -\rho$ in the second integral and using that the continuous dual Hahn polynomial and the Jacobi function are even in ρ , we obtain the second equality.

The case $x_1 = x_2$ can be obtained from the case $x_1 > x_2$ by letting $x_1 \rightarrow x_2$. We use the c -functions expansion of the Jacobi function

$$\begin{aligned} \varphi_{2\rho}^{(2k_2-1, 2k_1-1)}\left(\frac{x_2}{x_1-x_2}\right) &= \\ & \frac{\Gamma(2k_2)\Gamma(2i\rho)}{\Gamma(k_2-k_1+\frac{1}{2}+i\rho)\Gamma(k_1+k_2-\frac{1}{2}+i\rho)}\left(\frac{x_2}{x_1-x_2}\right)^{\frac{1}{2}-k_1-k_2+i\rho} \\ & \times {}_2F_1\left(\begin{matrix} k_1+k_2-\frac{1}{2}-i\rho, k_1-k_2-\frac{1}{2}-i\rho \\ 1-2i\rho \end{matrix}; \frac{x_2-x_1}{x_2}\right) + \text{idem}(\rho \leftrightarrow -\rho), \end{aligned}$$

which can be obtained from [1, (2.3.12)]. Using this expression in the second integral for $x_1 > x_2$ and letting $x_1 \rightarrow x_2$ then gives the second equality for $x_1 = x_2$. The first equality for $x_1 = x_2$ is obtained directly from the second. \square

Remark 3.11. (i) The confluent hypergeometric U -function can be considered as a Whittaker function of the second kind, see [16, (1.9.6)];

$$W_{k,m}(x) = e^{-\frac{1}{2}x} x^{m+\frac{1}{2}} U\left(m-k+\frac{1}{2}; 2m+1; x\right).$$

These Whittaker functions are the kernel in the Whittaker function transform, given by

$$\begin{cases} (Wf)(\lambda) = \int_0^\infty f(x) W_{k,i\lambda}(x) x^{-\frac{3}{2}} dx, \\ f(x) = \frac{1}{2\pi} \int_0^\infty (Wf)(\lambda) x^{-\frac{1}{2}} W_{k,i\lambda}(x) \left| \frac{\Gamma(2i\lambda)}{\Gamma(\frac{1}{2}-k+i\lambda)} \right|^2 d\lambda, \end{cases}$$

where $k \leq \frac{1}{2}$. Using the Whittaker function transform we see that Theorem 3.10 is a generalization of Koornwinder's formula [13, (5.14)], stating that Laguerre polynomials are mapped onto continuous dual Hahn polynomials by the Whittaker function transform.

(ii) For general $k_1, k_2 > 0$ discrete mass points must be added to the integral in the theorem. In case the discrete mass points corresponds to discrete series in the tensor product decomposition, cf. Theorem 3.1, the confluent hypergeometric U -function can be written as a terminating ${}_1F_1$ -series, which is a Laguerre polynomial.

(iii) Theorem 3.10 can be obtained as a limit case of a bilinear summation formula for Meixner-Pollaczek polynomials, see [7, Rem.3.2(ii)].

(iv) The first case of the product formula for $x_1 = x_2$ corresponds to the formal definition of the Laguerre function $\psi_n(0; \rho, \varepsilon)$ given in Remark 2.10 as follows. The Clebsch-Gordan coefficient for the case $x = x_1 = x_2$ is the (\mathbb{C}^2 -valued) function g , such that

$$I_{n_1}^{(2k_1-1)}(x) I_{n_2}^{(2k_2-1)}(x) = (-1)^{n_1+L} \int_0^\infty S_{n_1}(\rho^2; n_1 - n_2) g^*(\rho) |\Gamma(2i\rho)| \left(\frac{\overline{t_{n_1-n_2-L}(0)}}{t_{n_1-n_2-L}(0)} \right) d\mu^{\frac{1}{2}}(\rho; n_1 - n_2).$$

Here g^* denotes the Hermitian transpose of g . Using the explicit expression for $t_n(0)$, see Proposition 2.2, the Clebsch-Gordan coefficients can now be computed from Theorem 3.10, and this gives

$$g = \begin{pmatrix} \overline{C} e^x x^{\frac{1}{2}-k_1-k_2+i\rho} \\ C e^x x^{\frac{1}{2}-k_1-k_2-i\rho} \end{pmatrix},$$

where C is a factor independent of x . The function g should be an eigenfunction of $\pi_{k_1}^+ \otimes \pi_{k_2}^-(\Delta(\Omega))$ in the realizations of Proposition 3.6. Indeed, setting $e^x \varphi(x) = p(x)q(x)$ in (3.15), it follows from Theorem 3.1 that the function $\varphi(x)$ must be a solution to the following Euler differential equation:

$$-x^2 y'' - (2k_1 + 2k_2)xy' + (k_1 + k_2)(1 - k_1 - k_2)y = (\rho^2 + \frac{1}{4})y.$$

The general solutions to this equation are given by $y = c_1 x^{\frac{1}{2}-k_1-k_2+i\rho} + c_2 x^{\frac{1}{2}-k_1-k_2-i\rho}$.

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