Constr Approx (2009) 29: 85–127
DOI 10.1007/s00365-008-9009-z

CONSTRUCTIVE APPROXIMATION

The Vector-Valued Big q-Jacobi Transform

Wolter Groenevelt

Received: 12 January 2007 / Accepted: 3 September 2007 / Published online: 26 February 2008

Abstract Big q-Jacobi functions are eigenfunctions of a second-order q-difference operator L. We study L as an unbounded self-adjoint operator on an L^2-space of functions on $\mathbb{R}$ with a discrete measure. We describe explicitly the spectral decomposition of L using an integral transform $\mathcal{F}$ with two different big q-Jacobi functions as a kernel, and we construct the inverse of $\mathcal{F}$.

Keywords Vector-valued big q-Jacobi transform · Difference operator · Spectral analysis · Big q-Jacobi polynomials · Integral transform

Mathematics Subject Classification (2000) Primary 33D45 · 47B25 · Secondary 44A15 · 47B39

1 Introduction

Integral transforms with a hypergeometric function as a kernel have been the subject of many papers in the literature. A famous example is the Jacobi transform, first studied by Weyl [18], which is an integral transform with a certain ${}_2F_1$-function, the Jacobi function, as a kernel. The inverse of the Jacobi transform can be obtained from spectral analysis of the hypergeometric differential operator D, which is an unbounded self-adjoint operator on a weighted L^2-space of functions on $[0, \infty)$. We refer to [14] for a survey on Jacobi functions. In a recent paper [16] Neretin studied the hypergeometric differential operator D as a self-adjoint operator on a weighted L^2-space of functions on $\mathbb{R}$. In this setting the spectral analysis of D leads to an integral transform with two different Jacobi functions (vector-valued Jacobi functions) as a kernel, corresponding to the multiplicity two of the continuous spectrum.

Communicated by Mourad Ismail.

W. Groenevelt (✉)
Technische Universiteit Delft, EWI-DIAM, Postbus 5031, 2600 GA Delft, The Netherlands
e-mail: w.g.m.groenevelt@tudelft.nl

In this paper we obtain a q-analogue of Neretin's vector-valued Jacobi transform (or double index hypergeometric transform). There exist several q-analogues of Jacobi functions, namely the little and big q-Jacobi functions and the Askey–Wilson functions, see [9, 10, 12, 13]. Here we consider the big q-Jacobi function, which is a basic hypergeometric ${}_3\phi_2$-function that is the kernel in the big q-Jacobi transform by Koelink and Stokman [13]. The big q-Jacobi transform and its inverse arise from spectral analysis of a second-order q-difference operator L that is an unbounded self-adjoint operator on an L^2-space consisting of square integrable functions with respect to a discrete measure on $[-1,\infty)$. In this paper we study the same q-difference operator L as an unbounded self-adjoint operator on a different Hilbert space, namely an L^2-space of functions on $\mathbb{R}$ with a discrete measure. The continuous spectrum of L has multiplicity two, thus leading to an integral transform pair with two different big q-Jacobi functions as a kernel. We call this the vector-valued big q-Jacobi transform.

The vector-valued Jacobi transform has an interpretation in the representation theory of the Lie algebra $\mathfrak{su}(1,1)$ (or equivalently $\mathfrak{sl}(2,\mathbb{R})$) as follows, see [16, Sect. 4]. The hypergeometric differential operator D arises from a suitable restriction of the Casimir operator in the tensor product of two principal unitary series. The spectral analysis of D now gives the decomposition into irreducible representations, and the vector-valued Jacobi transform can be used to construct explicitly the intertwiner for these representations. The multiplicity two of the continuous spectrum corresponds to the multiplicity of the principal unitary series occurring in the decomposition (see [15, 17] for the precise decomposition). There is a similar interpretation of the vector-valued big q-Jacobi transform in the representation theory of the quantized universal enveloping algebra $\mathcal{U}_q(\mathfrak{su}(1,1))$. However, the corresponding representation is no longer a tensor product representation, but a sum of two tensor products of principal unitary series. This will be the subject of a future paper.

The big q-Jacobi functions are nonpolynomial extensions of the big q-Jacobi polynomials [1], but they can also be considered as extensions of the continuous dual q^{-1}-Hahn polynomials (see [13]). In this light, the vector-valued big q-Jacobi transform may also be considered as a q-analogue of the integral transform corresponding to the ${}_3F_2$-functions $(\Xi_n^{(1)}, \Xi_n^{(2)})$ from [16, Thm. 1.3], and of the continuous Hahn transform from [7]. In both transforms the kernel consists of two ${}_3F_2$-functions that are extensions of the continuous dual Hahn polynomials.

The organization of this paper is as follows. In Sect. 2 we introduce the second-order q-difference operator L and a weighted L^2-space of functions on $\mathbb{R}_q$, a q-analogue of the real line. The difference operator L is an unbounded operator on this L^2-space. We define the Casorati determinant, a difference analogue of the Wronskian, and with the Casorati determinant we determine a dense domain on which L is self-adjoint. In Sect. 3 we introduce the big q-Jacobi functions as eigenfunctions of L given by a specific ${}_3\phi_2$-series. We also give the asymptotic solutions, which are ${}_3\phi_2$-series with nice asymptotic behavior at $+\infty$ or $-\infty$. A crucial point here is the fact that all eigenfunctions we consider can uniquely be extended to functions on $\mathbb{R}_q$. In Sect. 4 we define the Green kernel using the asymptotic solutions, and we determine the spectral decomposition for L. In Sect. 5 we define the vector-valued big q-Jacobi transform $\mathcal{F}$, and we determine its inverse. A left inverse $\mathcal{G}$ of $\mathcal{F}$ follows immediately from the spectral analysis done in Sect. 4. To show that $\mathcal{G}$ is also a right inverse, we use a classical method that essentially comes down to approximating with the Fourier transform. Finally, in the Appendix two lemmas are proved which involve rather long computations.

Notations We use standard notations for q-shifted factorials, θ-functions and basic hypergeometric series [5]. We fix a number $q \in (0,1)$. The q-shifted factorials are defined by

$$(x;q)_\infty = \prod_{k=0}^{\infty}(1-xq^k), \qquad (x;q)_n = \frac{(x;q)_\infty}{(xq^n;q)_\infty}, \quad n \in \mathbb{Z}.$$

The (normalized) Jacobi θ-function is defined by

$$\theta(x) = (x, q/x; q)_\infty, \quad x \notin q^{\mathbb{Z}}.$$

From this definition it follows that the θ-function satisfies

$$\theta(x) = \theta(q/x) = -x\theta(qx) = -x\theta(1/x).$$

We often use these identities without mentioning them. For products of q-shifted factorials and products of θ-functions we use the shorthand notations

$$\begin{aligned}(x_1, x_2, \dots, x_k; q)_n &= (x_1;q)_n(x_2;q)_n \cdots (x_k;q)_n, \quad n \in \mathbb{Z} \cup \{\infty\},\\ \theta(x_1, x_2, \dots, x_k) &= \theta(x_1)\theta(x_2)\cdots\theta(x_k),\end{aligned}$$

and

$$\left(xy^{\pm 1};q\right)_\infty = (xy, x/y;q)_\infty, \qquad \theta\left(xy^{\pm 1}\right) = \theta(xy, x/y).$$

An identity for θ-functions that we frequently use is

$$\theta(xv, x/v, yw, y/w) - \theta(xw, x/w, yv, y/v) = \frac{y}{v}\theta(xy, x/y, vw, v/w) \tag{1.1}$$

(see [5, Exer. 2.16(i)]). The basic hypergeometric function ${}_r\phi_s$ is defined by

$${}_r\phi_s\left(\begin{matrix} x_1, x_2, \dots, x_r \\ y_1, y_2, \dots, y_s \end{matrix}; q, z\right) = \sum_{k=0}^{\infty} \frac{(x_1, x_2, \dots, x_r; q)_k}{(q, y_1, y_2, \dots, y_s; q)_k}\left((-1)^k q^{k(k-1)/2}\right)^{1+s-r} z^k.$$

2 The Second-Order q-Difference Operator

In this section we introduce an unbounded second-order q-difference operator L acting on functions on a q-analogue of the real line, and we determine a dense domain on which L is self-adjoint.

2.1 The Difference Operator

We fix two real numbers $z_+ > 0$ and $z_- < 0$. Let $\mathbb{R}_q^+$ and $\mathbb{R}_q^-$ be the two sets

$$\mathbb{R}_q^+ = \{z_+ q^n \mid n \in \mathbb{Z}\}, \qquad \mathbb{R}_q^- = \{z_- q^n \mid n \in \mathbb{Z}\},$$

and define

$$\mathbb{R}_q = \mathbb{R}_q^- \cup \mathbb{R}_q^+,$$

which we consider as a q-analogue of the real line. For $x \in \mathbb{R}_q$ we sometimes write $x = zq^k$, which means that $z = z_-$ or $z = z_+$, and $k \in \mathbb{Z}$. We denote by $F(\mathbb{R}_q)$ the linear space of complex-valued functions on $\mathbb{R}_q$.

The second-order difference operator L we are going to study depends on four parameters. Let P_{q,z_-,z_+} be the set consisting of pairs of parameters $(\alpha, \beta) \in \mathbb{C}^2$ such that $\alpha, \beta \notin z_\pm^{-1} q^{\mathbb{Z}}$, and one of the following conditions is satisfied:

- $\alpha = \overline{\beta}$.
- $\alpha, \beta \in \mathbb{R}$ and there exists a $k_0 \in \mathbb{Z}$ such that $z_+ q^{k_0} < \beta^{-1} < \alpha^{-1} < z_+ q^{k_0 - 1}$.
- $\alpha, \beta \in \mathbb{R}$ and there exists a $k_0 \in \mathbb{Z}$ such that $z_- q^{k_0 - 1} < \alpha^{-1} < \beta^{-1} < z_- q^{k_0}$.

In particular, this implies that $q < |\alpha/\beta| \le 1$, and α and β have the same sign in case they are real. We define the parameter domain P to be the following set:

$$P = \{(a, b, c, d) \in \mathbb{C}^4 \mid (a, b) \in P_{q,z_-,z_+},\ (c, d) \in P_{q,z_-,z_+},\ a \neq b\}.$$

From here on we assume that $(a, b, c, d) \in P$, unless explicitly stated otherwise.

We define a linear operator $L = L_{a,b,c,d} : F(\mathbb{R}_q) \to F(\mathbb{R}_q)$ by

$$L = A(\cdot) T_{q^{-1}} + B(\cdot) T_q + C(\cdot)\,\mathrm{id},$$

where T_α is the shift operator $(T_\alpha f)(x) = f(\alpha x)$ for $\alpha \in \mathbb{C}$, id denotes the identity operator, and

$$A(x) = s^{-1}\left(1 - \frac{q}{ax}\right)\left(1 - \frac{q}{bx}\right),$$

$$B(x) = s\left(1 - \frac{1}{cx}\right)\left(1 - \frac{1}{dx}\right),$$

$$C(x) = s^{-1} + s - A(x) - B(x),$$

where $s = \sqrt{cdq/ab}$. Here we use the usual branch of $\sqrt{\cdot}$ that is positive on $\mathbb{R}_+$. Note that the conditions on the parameters ensure that $A(x) \neq 0$ and $B(x) \neq 0$ for all $x \in \mathbb{R}_q$.

Remark 2.1 (a) There is an obvious symmetry in the parameters;

$$L_{a,b,c,d} = L_{b,a,c,d} = L_{a,b,d,c}.$$

This will be useful when we study eigenfunctions of L later on.

(b) Let $a = z_+^{-1} q^{1-m}$, $m \in \mathbb{Z}$ fixed (so $(a, b, c, d) \notin P$). Then the coefficient $A(x)$ vanishes at the point $x = z_+ q^m$. In this case certain restrictions of the difference operator L are well-known in the literature. Let L_- (respectively L_+) denote the operator L restricted to functions on $\{z_+ q^k \mid k \in \mathbb{Z}_{\geq m}\}$ (respectively $\{z_+ q^k \mid k \in \mathbb{Z}_{\leq m-1}\}$). Then $L_\pm$ is equivalent to the Jacobi operator for the continuous dual $q^{\pm 1}$-Hahn polynomials, which are Askey–Wilson polynomials with one of the parameters equal to zero (see [2, 11]). Moreover, the operator L restricted to functions on $\mathbb{R}_q^- \cup \{z_+ q^k \mid k \in \mathbb{Z}_{\geq m}\}$ is equivalent to the difference operator studied by Koelink and Stokman [13] to obtain the big q-Jacobi function transform.

If we also set $c = z_-^{-1} q^{-n}$, $n \in \mathbb{Z}$ fixed, then the coefficient $B(x)$ vanishes at the point $x = z_- q^n$. In this case the operator L restricted to functions on $\{z_- q^k \mid k \in \mathbb{Z}_{\geq n}\} \cup \{z_+ q^k \mid k \in \mathbb{Z}_{\geq m}\}$ is equivalent to the difference operator for big q-Jacobi polynomials (see [13, Sect. 10]), where the orthogonality relations for the big q-Jacobi polynomials are obtained from spectral analysis of L. See also [1] and [11] for the big q-Jacobi polynomials.

2.2 The Casorati Determinant

The Jackson q-integral is defined by

$$\int_0^\alpha f(x)\, d_q x = (1-q) \sum_{k=0}^\infty f\big(\alpha q^k\big) \alpha q^k,$$

$$\int_\alpha^\beta f(x)\, d_q x = \int_0^\beta f(x)\, d_q x - \int_0^\alpha f(x)\, d_q x,$$

$$\int_0^{\infty(\alpha)} f(x)\, d_q x = (1-q) \sum_{k=-\infty}^\infty f\big(\alpha q^k\big) \alpha q^k,$$

for $\alpha, \beta \in \mathbb{C}^*$, and f is a function such that the sums converge absolutely. We will denote

$$\int_{\mathbb{R}_q} f(x)\, d_q x = \int_0^{\infty(z_+)} f(x)\, d_q x - \int_0^{\infty(z_-)} f(x)\, d_q x.$$

We define a weight function w on $\mathbb{R}_q$ by

$$w(x) = w(x; a, b, c, d; q) = \frac{(ax, bx; q)_\infty}{(cx, dx; q)_\infty}. \tag{2.1}$$

Note that for $(a, b, c, d) \in P$ the weight function w is positive on $\mathbb{R}_q$, and w is continuous at the origin. Let $\mathcal{L}^2 = \mathcal{L}^2(\mathbb{R}_q, w(x)\, d_q x)$ be the Hilbert space consisting of functions $f \in F(\mathbb{R}_q)$ that have finite norm with respect to the inner product

$$\langle f, g \rangle_{\mathcal{L}^2} = \int_{\mathbb{R}_q} f(x) \overline{g(x)} w(x)\, d_q x.$$

For $k, l, m, n \in \mathbb{Z}$ we define a truncated inner product by

$$\langle f, g\rangle_{k,l;m,n} = \int_{z_-q^k}^{z_-q^{l+1}} f(x)\overline{g(x)}w(x)\, d_qx + \int_{z_+q^{m+1}}^{z_+q^n} f(x)\overline{g(x)}w(x)\, d_qx.$$

If $f, g \in \mathcal{L}^2$, we have

$$\lim_{\substack{l,m\to\infty\\ k,n\to-\infty}} \langle f, g\rangle_{k,l;m,n} = \langle f, g\rangle_{\mathcal{L}^2}.$$

We define a function u on $\mathbb{R}_q$, closely related to the weight function w, by

$$u(x) = (1-q)^2 B(x)x^2 w(x) = (1-q)^2\sqrt{q/abcd}\,\frac{(ax, bx; q)_\infty}{(cqx, dqx; q)_\infty}.$$

Definition 2.2 For $f, g \in F(\mathbb{R}_q)$ we define the Casorati determinant $D(f, g) \in F(\mathbb{R}_q)$ by

$$\begin{aligned} D(f, g)(x) &= \big(f(x)g(qx) - f(qx)g(x)\big)\frac{u(x)}{(1-q)x} \\ &= \big((D_q f)(x)g(x) - f(x)(D_q g)(x)\big)u(x). \end{aligned}$$

Here $D_q : F(\mathbb{R}_q) \to F(\mathbb{R}_q)$ is the q-difference operator given by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{x(1-q)}.$$

Proposition 2.3 *For $f, g \in F(\mathbb{R}_q)$ we have*

$$\begin{aligned} &\langle Lf, g\rangle_{k,l;m,n} - \langle f, Lg\rangle_{k,l;m,n} \\ &\quad = D(f, \overline{g})\big(z_-q^l\big) - D(f, \overline{g})\big(z_-q^{k-1}\big) + D(f, \overline{g})\big(z_+q^{n-1}\big) - D(f, \overline{g})\big(z_+q^m\big). \end{aligned}$$

Proof For $f, g \in F(\mathbb{R}_q)$,

$$\begin{aligned} &\big((Lf)(x)g(x) - f(x)(Lg)(x)\big)xw(x) \\ &\quad = A(x)\big(f(x/q)g(x) - f(x)g(x/q)\big)xw(x) \\ &\qquad + B(x)\big(f(xq)g(x) - f(x)g(xq)\big)xw(x) \\ &\quad = \big(f(x/q)g(x) - f(x)g(x/q)\big)\frac{(ax/q, bx/q; q)_\infty}{(cx, dx; q)_\infty}\frac{q^2}{xsab} \\ &\qquad + \big(f(xq)g(x) - f(x)g(xq)\big)\frac{(ax, bx; q)_\infty}{(cqx, dqx; q)_\infty}\frac{s}{cdx}. \end{aligned}$$

Note that $abs/q = \sqrt{abcd/q} = cd/s$, so we obtain

$$\big((Lf)(x)g(x) - f(x)(Lg)(x)\big)(1-q)xw(x) = D(f, g)(x/q) - D(f, g)(x).$$

Now the sums of the truncated inner products in the lemma become telescoping, and the result follows. □

In order to determine a suitable domain on which L is self-adjoint, we need to find the limit behavior of Casorati determinants. First, to find the asymptotic behavior of $D(f,g)(x)$ for large x, we need the behavior of $u(x)$ for large x.

Lemma 2.4 *Let $x = zq^{-k} \in \mathbb{R}_q$. Then for $k \to \infty$*

$$xw(x) = (1-q)^{-1} K_z s^{-2k}\big(1 + \mathcal{O}(q^k)\big),$$
$$\frac{u(x)}{x} = (1-q) K_z s^{1-2k}\big(1 + \mathcal{O}(q^k)\big),$$

where

$$K_z = K_z(a,b,c,d;q) = z(1-q)\frac{\theta(az,bz)}{\theta(cz,dz)}.$$

Proof Let $x = zq^{-k} \in \mathbb{R}_q$. Using the identity

$$\frac{(\alpha q^{-n};q)_n}{(\beta q^{-n};q)_n} = \frac{(q/\alpha;q)_n}{(q/\beta;q)_n}\left(\frac{\alpha}{\beta}\right)^n,$$

and the definition (2.1) of the weight function w we obtain

$$w\big(zq^{-k}\big) = \frac{(q/az, q/bz;q)_k (az,bz;q)_\infty}{(q/cz, q/dz;q)_k (cz,dz;q)_\infty}\left(\frac{ab}{cd}\right)^k,$$

and

$$u\big(zq^{-k}\big) = (1-q)^2\sqrt{q/abcd}\,\frac{(q/az, q/bz;q)_k (az,bz;q)_\infty}{(1/cz, 1/dz;q)_k (zcq,zdq;q)_\infty}\left(\frac{ab}{cdq^2}\right)^k.$$

From this the asymptotic behavior of $xw(x)$ and $u(x)/x$ for large x follows. □

Lemma 2.5 *Let $f, g \in \mathcal{L}^2$. Then*

$$\lim_{x\to\pm\infty} D(f,g)(x) = 0.$$

Proof Let $f, g \in \mathcal{L}^2$. Using the asymptotic behavior of $xw(x)$ for large x (see Lemma 2.4), we find that f and g satisfy

$$\lim_{k\to\infty} s^{-k} f\big(zq^{-k}\big) = \lim_{k\to\infty} s^{-k} g\big(zq^{-k}\big) = 0.$$

From Definition 2.2 and the asymptotic behavior of $u(x)/x$ from Lemma 2.4 we now see that $\lim_{k\to\infty} D(f,g)(zq^{-k}) = 0$. □

2.3 Self-Adjointness

For $f \in F(\mathbb{R}_q)$ we denote

$$f(0^-) = \lim_{k\to\infty} f\big(z_- q^k\big), \qquad f\big(0^+\big) = \lim_{k\to\infty} f\big(z_+ q^k\big),$$
$$f'(0^-) = \lim_{k\to\infty} (D_q f)\big(z_- q^k\big), \qquad f'\big(0^+\big) = \lim_{k\to\infty} (D_q f)\big(z_+ q^k\big),$$

provided that all these limits exist.

Definition 2.6 We define the subspace $\mathcal{D} \subset \mathcal{L}^2$ by

$$\mathcal{D} = \big\{ f \in \mathcal{L}^2 \mid Lf \in \mathcal{L}^2,\ f(0^-) = f\big(0^+\big),\ f'(0^-) = f'\big(0^+\big) \big\}.$$

The domain $\mathcal{D}$ contains the finitely supported functions in $\mathcal{L}^2$; hence $\mathcal{D}$ is dense in $\mathcal{L}^2$.

Proposition 2.7 *The operator* $(L, \mathcal{D})$ *is self-adjoint.*

The proposition is proved in the same way as [13, Prop. 2.7]. For convenience we repeat the proof here.

Proof First we need to show that $(L, \mathcal{D})$ is symmetric. Let $f, g \in \mathcal{D}$. Using the second expression in Definition 2.2 we find

$$\begin{aligned} D(f, \overline{g})(0^-) &= u(0)\big((D_q f)(0^-)\overline{g(0^-)} - f(0^-)\overline{(D_q g)(0^-)}\big) \\ &= u(0)\big((D_q f)\big(0^+\big)\overline{g\big(0^+\big)} - f\big(0^+\big)\overline{(D_q g)\big(0^+\big)}\big) \\ &= D(f, \overline{g})\big(0^+\big). \end{aligned}$$

By Proposition 2.3 and Lemma 2.5 this leads to

$$\begin{aligned} \langle Lf, g\rangle_{\mathcal{L}^2} - \langle f, Lg\rangle_{\mathcal{L}^2} &= \lim_{\substack{l,m\to\infty \\ k,n\to-\infty}} \big(\langle Lf, g\rangle_{k,l;m,n} - \langle f, Lg\rangle_{k,l;m,n}\big) \\ &= D(f, \overline{g})\big(0^+\big) - D(f, \overline{g})(0^-) = 0; \end{aligned}$$

hence $(L, \mathcal{D})$ is symmetric with respect to $\langle\cdot,\cdot\rangle_{\mathcal{L}^2}$.

Now we know that $(L, \mathcal{D}) \subset (L^*, \mathcal{D}^*)$, where $(L^*, \mathcal{D}^*)$ is the adjoint of the operator $(L, \mathcal{D})$. Observe that

$$L^* = L|_{\mathcal{D}^*}.$$

Indeed, let f be a nonzero function with support at only one point $x \in \mathbb{R}_q$ and let $g \in F(\mathbb{R}_q)$. Then

$$\langle Lf, g\rangle_{\mathcal{L}^2} = \langle f, Lg\rangle_{\mathcal{L}^2}.$$

In particular, for $g \in \mathcal{D}^*$ we then have $\langle f, Lg\rangle_{\mathcal{L}^2} = \langle f, L^* g\rangle_{\mathcal{L}^2}$, so $(Lg)(x) = (L^* g)(x)$. This holds for all $x \in \mathbb{R}_q$; hence $L^* = L|_{\mathcal{D}^*}$.

Finally we show that $\mathcal{D}^* \subset \mathcal{D}$. Let $f \in \mathcal{D}$ and let $g \in \mathcal{D}^*$. Using Proposition 2.3 and Lemma 2.5,

$$D(f,\overline{g})(0^-) - D(f,\overline{g})\big(0^+\big) = \langle Lf, g\rangle_{\mathcal{L}^2} - \langle f, L^*g\rangle_{\mathcal{L}^2} = 0.$$

Since this holds for all $f \in \mathcal{D}$, we find that the limits $g(0^-)$, $g(0^+)$, $g'(0^-)$ and $g'(0^+)$ exist, and

$$g(0^-) = g\big(0^+\big), \qquad g'(0^-) = g'\big(0^+\big);$$

hence $g \in \mathcal{D}$, which proves the proposition. □

Remark 2.8 Let $f \in \mathcal{D}$ and let α be a complex number with $|\alpha| = 1$. We define

$$\tilde{f}(x) = \begin{cases} f(x), & x \in \mathbb{R}_q^-, \\ \overline{\alpha} f(x), & x \in \mathbb{R}_q^+, \end{cases}$$

and then it is easy to verify that $\tilde{f}(0^-) = \alpha \tilde{f}(0^+)$ and $\tilde{f}'(0^-) = \alpha \tilde{f}'(0^+)$. So we have a family of dense domains

$$\mathcal{D}_\alpha = \big\{ f \in \mathcal{L}^2 \mid Lf \in \mathcal{L}^2,\ f(0^-) = \alpha f\big(0^+\big),\ f'(0^-) = \alpha f'\big(0^+\big)\big\},$$

such that $(L, \mathcal{D}_\alpha)$ is self-adjoint. Without loss of generality we may work with the dense domain $\mathcal{D} = \mathcal{D}_1$.

3 Eigenfunctions

In this section we study eigenfunctions of the second-order difference operator L.

3.1 Spaces of Eigenfunctions

For $\mu \in \mathbb{C}$ we introduce the spaces

$$\begin{aligned} V_\mu^- &= \{ f : \mathbb{R}_q^- \to \mathbb{C} \mid Lf = \mu f \}, \\ V_\mu^+ &= \{ f : \mathbb{R}_q^+ \to \mathbb{C} \mid Lf = \mu f \}, \\ V_\mu &= \big\{ f : \mathbb{R}_q \to \mathbb{C} \mid Lf = \mu f,\ f(0^-) = f\big(0^+\big),\ f'(0^-) = f'\big(0^+\big)\big\}. \end{aligned}$$

Lemma 3.1 *Let* $\mu \in \mathbb{C}$.

(a) *For* $f, g \in V_\mu^\pm$ *the Casorati determinant* $D(f,g)$ *is constant on* $\mathbb{R}_q^\pm$.
(b) *For* $f, g \in V_\mu$ *the Casorati determinant* $D(f,g)$ *is constant on* $\mathbb{R}_q$.
(c) $\dim V_\mu^\pm = 2$.
(d) $\dim V_\mu \le 2$.

Proof For (a) let $f, g \in F(\mathbb{R}_q)$. From the proof of Proposition 2.3 we have

$$\big((Lf)(x)g(x) - f(x)(Lg)(x)\big)(1-q)xw(x) = D(f,g)(x/q) - D(f,g)(x).$$

Now if f and g satisfy $(Lf)(x) = \mu f(x)$ and $(Lg)(x) = \mu g(x)$, we find $D(f,g)(x/q) = D(f,g)(x)$; hence $D(f,g)$ is constant on $\mathbb{R}_q^+$ and $\mathbb{R}_q^-$.

Let $f, g \in V_\mu$. Statement (b) follows from (a) and the fact that $D(f,g)(0^-) = D(f,g)(0^+)$.

For (c) we write $f(zq^k) = f_k$, and we see that $Lf = \mu f$ gives a recurrence relation of the form $\alpha_k f_{k+1} + \beta_k f_k + \gamma_k f_{k-1} = \mu f_k$, with $\alpha_k, \gamma_k \neq 0$ for all $k \in \mathbb{Z}$. Solutions of such a recurrence relation are uniquely determined by specifying f_k at two different points $k = l$ and $k = m$. So there are two independent solutions, which means that $\dim V_\mu^\pm = 2$.

Finally, suppose that $f_1, f_2 \in V_\mu$ are such that the restrictions $f_i^{\mathrm{res}} = f_i|_{\mathbb{R}_q^+}$ are linearly independent in V_μ^+. By (a) the Casorati determinant $D(f_1^{\mathrm{res}}, f_2^{\mathrm{res}})(x)$ is nonzero and constant on $\mathbb{R}_q^+$; hence $D(f_1, f_2)$ is nonzero and constant on $\mathbb{R}_q$. Therefore f_1 and f_2 are linearly independent. Now choose a function $f_3 \in V_\mu$. Since $\dim V_\mu^+ = 2$, we have $f_3^{\mathrm{res}} = \alpha f_1^{\mathrm{res}} + \beta f_2^{\mathrm{res}}$ for some constants $\alpha, \beta \in \mathbb{C}$. This shows that

$$\begin{aligned} D(f_3, f_1) &= D\big(f_3^{\mathrm{res}}, f_1^{\mathrm{res}}\big) = \beta D\big(f_2^{\mathrm{res}}, f_1^{\mathrm{res}}\big) = \beta D(f_2, f_1), \\ D(f_3, f_2) &= D\big(f_3^{\mathrm{res}}, f_2^{\mathrm{res}}\big) = \alpha D\big(f_1^{\mathrm{res}}, f_2^{\mathrm{res}}\big) = \alpha D(f_1, f_2); \end{aligned}$$

hence $f_3 = \alpha f_1 + \beta f_2$. So $\dim V_\mu \leq 2$. □

3.2 Big q-Jacobi Functions

Let P_{gen} be the dense subset of P given by

$$P_{\mathrm{gen}} = \big\{(a,b,c,d) \in P \mid c \neq d,\ c/a, c/b, d/a, d/b, cd/ab \notin q^{\mathbb{Z}}\big\}.$$

From here on we assume that $(a,b,c,d) \in P_{\mathrm{gen}}$, unless stated otherwise.

The difference operator L is equivalent to the difference operator studied in [13]. To see this, set

$$X = -\frac{ax}{q}, \qquad A = s, \qquad B = \frac{qc}{sa}, \qquad C = \frac{qd}{sa}, \tag{3.1}$$

where the capitals stand for the parameters in [13]. We have the following eigenfunction, which is called a big q-Jacobi function,

$$\varphi_\gamma(x) = \varphi_\gamma(x; a,b,c,d|q) = {}_3\phi_2\left(\begin{matrix} q/ax, s\gamma, s/\gamma \\ cq/a, dq/a \end{matrix}; q, bx\right), \quad |x| < \frac{1}{|b|}. \tag{3.2}$$

If $|x| < |q/b|$, the function φ_γ is a solution of the eigenvalue equation

$$(Lf)(x) = \mu(\gamma) f(x), \quad \mu(\gamma) = \gamma + \gamma^{-1}, \tag{3.3}$$

where $\gamma \in \mathbb{C}^*$ and $x \in \mathbb{R}_q$. This can be obtained from [13], or directly from the contiguous relation [8, (2.10)]. For a function f depending on the parameters a, b, c, d, $f = f(\cdot; a, b, c, d)$, we write

$$f^\dagger = f^\dagger(\cdot; a, b, c, d) = f(\cdot; b, a, c, d).$$

Clearly, we have $(f^\dagger)^\dagger = f$. Since $L_{a,b,c,d} = L_{b,a,c,d}$ (cf. Remark 2.1(a)), it is immediately clear that $\varphi_\gamma^\dagger$ is also a solution for the eigenvalue equation (3.3). If $a = \overline{b}$, we have $\varphi_\gamma^\dagger(x) = \overline{\varphi_{\overline{\gamma}}(x)}$. The symmetry $L_{a,b,c,d} = L_{a,b,d,c}$ does not give rise to different eigenfunctions.

So far, the functions $\varphi_\gamma(x)$ and $\varphi_\gamma^\dagger(x)$ are defined for small $x \in \mathbb{R}_q$. Using the eigenvalue equation (3.3), the functions φ_γ and $\varphi_\gamma^\dagger$ can uniquely be extended to functions on whole $\mathbb{R}_q$ (that we also denote by φ_γ and $\varphi_\gamma^\dagger$) that also satisfy (3.3). Later on we give explicit expressions for the functions $\varphi_\gamma(x)$ and $\varphi_\gamma^\dagger(x)$ for $|x| > q/|b|$.

First we establish the q-differentiability at the origin of the functions φ_γ and $\varphi_\gamma^\dagger$.

Proposition 3.2 *The functions φ_γ and $\varphi_\gamma^\dagger$ are continuous q-differentiable at the origin. At $x = 0$ we have*

$$\varphi_\gamma(0; a, b, c, d|q) = {}_2\phi_2\left(\begin{matrix} \gamma\sqrt{cdq/ab}, \sqrt{cdq/ab}/\gamma \\ cq/a, dq/a \end{matrix}; q, \frac{bq}{a}\right),$$

$$\varphi_\gamma'(0; a, b, c, d|q) = \frac{b(1-s\gamma)(1-s/\gamma)}{(1-q)(1-cq/a)(1-dq/a)}\varphi_\gamma\big(0; aq^{-\frac{1}{2}}, bq^{-\frac{1}{2}}, cq^{\frac{1}{2}}, dq^{\frac{1}{2}}|q\big).$$

Proof The expression for $\varphi_\gamma(0)$ follows from letting $x \to 0$ in (3.2). If $|x|$ is small enough, we find from the explicit expression (3.2) for φ_γ,

$$\begin{aligned}
\varphi_\gamma(x) - \varphi_\gamma(qx) &= \sum_{n=1}^{\infty} \frac{(s\gamma, s/\gamma; q)_n}{(q, cq/a, dq/a; q)_n}(bx)^n\big[(q/ax; q)_n - (1/ax; q)_n q^n\big] \\
&= \sum_{n=1}^{\infty} \frac{(s\gamma, s/\gamma; q)_n}{(q, cq/a, dq/a; q)_n}(bx)^n(q/ax; q)_{n-1}\big(1-q^n\big) \\
&= \frac{bx(1-s\gamma)(1-s/\gamma)}{(1-cq/a)(1-dq/a)}\, {}_3\phi_2\left(\begin{matrix} q/ax, sq\gamma, sq/\gamma \\ cq^2/a, dq^2/a \end{matrix}; q, bx\right).
\end{aligned}$$

Now it follows that the q-derivative of φ_γ is given by

$$(D_q\varphi_\gamma)(x) = \frac{b(1-s\gamma)(1-s/\gamma)}{(1-q)(1-cq/a)(1-dq/a)}\varphi_\gamma\big(xq^{\frac{1}{2}}; aq^{-\frac{1}{2}}, bq^{-\frac{1}{2}}, cq^{\frac{1}{2}}, dq^{\frac{1}{2}}|q\big).$$

Letting $x \to 0$ gives the result. ☐

3.3 Asymptotic Solutions

We define the set of regular spectral values

$$\mathcal{S}_{\mathrm{reg}} = \mathbb{C}^* \setminus \{\pm q^{\frac{1}{2}k} \mid k \in \mathbb{Z}\}.$$

For $\gamma \in \mathcal{S}_{\mathrm{reg}} \cup \{\pm 1\}$ another solution for the eigenvalue equation (3.3) is the function

$$\begin{aligned}\Phi_\gamma(x) &= \Phi_\gamma(x; a, b, c, d|q) \\ &= (s\gamma)^k \frac{(q/bx, q^2\gamma/asx; q)_\infty}{(q/cx, q/dx; q)_\infty} {}_3\phi_2\left(\begin{matrix} q\gamma/s, cq\gamma/sa, dq\gamma/sa \\ q^2\gamma/asx, q\gamma^2 \end{matrix}; q, \frac{q}{bx}\right), \\ &\qquad |x| > \frac{q}{|b|},\end{aligned} \tag{3.4}$$

where $x = zq^{-k}$ (see [13]). For $x \to \pm\infty$ we have

$$\Phi_\gamma(zq^{-k}) = (s\gamma)^k \big(1 + \mathcal{O}(q^k)\big), \quad k \to \infty. \tag{3.5}$$

Clearly, $\Phi_{\gamma^{-1}}$, $\Phi_\gamma^\dagger$ and $\Phi_{\gamma^{-1}}^\dagger$ are also solutions to (3.3). We remark that it follows from applying the transformation [5, (III.9)] for ${}_3\phi_2$-series, that

$$\begin{aligned}\Phi_\gamma(x) &= (s\gamma)^k \frac{(q/ax, q^2\gamma/bsx; q)_\infty}{(q/cx, q/dx; q)_\infty} {}_3\phi_2\left(\begin{matrix} q\gamma/s, cq\gamma/sb, dq\gamma/sb \\ q^2\gamma/bsx, q\gamma^2 \end{matrix}; q, \frac{q}{ax}\right), \\ &\quad x = zq^{-k}.\end{aligned}$$

So we see that $\Phi_\gamma = \Phi_\gamma^\dagger$, and if $\gamma \in \mathbb{R}$, we see that Φ_γ is real-valued. Using the eigenvalue equation $L\Phi_\gamma = \mu(\gamma)\Phi_\gamma$, we can extend Φ_γ to single-valued functions Φ_γ^+ on $\mathbb{R}_q^+$ and Φ_γ^- on $\mathbb{R}_q^-$. We call Φ_γ^+ and Φ_γ^- the asymptotic solutions of $Lf = \mu(\gamma)f$ on $\mathbb{R}_q^+$ and $\mathbb{R}_q^-$, respectively. The following lemma shows that $\Phi_\gamma^\pm$ and $\Phi_{\gamma^{-1}}^\pm$ are linear independent; hence they form linear bases for the eigenspaces $V_{\mu(\gamma)}^\pm$.

Lemma 3.3 *For $\gamma \in \mathcal{S}_{\mathrm{reg}}$ we have*

$$D\big(\Phi_\gamma^\pm, \Phi_{1/\gamma}^\pm\big)\big(z_\pm q^{-k}\big) = (\gamma - 1/\gamma)K_{z_\pm}.$$

Proof Since Φ_γ^+ lies in $V_{\mu(\gamma)}^+$, the Casorati determinant $D(\Phi_\gamma^+, \Phi_{1/\gamma}^+)$ is constant on $\mathbb{R}_q^+$, so we can find the Casorati determinant by taking the limit $x \to \infty$. From Lemma 2.4 we find

$$\lim_{k\to\infty} \frac{s^{2k-1}u(z_+q^{-k})}{z_+q^{-k}(1-q)} = K_{z_+},$$

and then it follows from the first expression in Definition 2.2 and (3.5) that

$$\lim_{k\to\infty} D\big(\Phi_\gamma^+, \Phi_{1/\gamma}^+\big)\big(z_+q^{-k}\big) = (\gamma - 1/\gamma)K_{z_+}.$$

The proof for Φ_γ^- is similar. □

Now we can expand the functions φ_γ and $\varphi_\gamma^\dagger$ on $V^\pm_{\mu(\gamma)}$ in terms of $\Phi^\pm_{\gamma^{\pm 1}}$. The expansion of φ_γ in terms of $\Phi^+_{\gamma^{\pm 1}}$ (respectively $\Phi^-_{\gamma^{\pm 1}}$) gives an explicit expression for φ_γ for $x > q/|b|$ (respectively $x < -q/|b|$). For $\gamma \in \mathcal{S}_{\mathrm{reg}}$ we define a function $c_z(\gamma)$ by

$$c_z(\gamma) = c_z(\gamma; a, b, c, d|q) = \frac{(s/\gamma, cq/as\gamma, dq/as\gamma; q)_\infty \theta(bsz\gamma)}{(cq/a, dq/a, 1/\gamma^2; q)_\infty \theta(bz)}.$$

The desired expansion uses the c-function (see [13, Prop. 4.4]), with parameters as in (3.1) and $Z = za/q$, or use the three-term transformation for ${}_3\phi_2$-functions [5, (III.33)].

Proposition 3.4 *For $\gamma \in \mathcal{S}_{\mathrm{reg}}$ and $x = z_\pm q^k \in \mathbb{R}_q$,*

$$\varphi_\gamma(x) = c_{z_\pm}(\gamma)\Phi^\pm_\gamma(x) + c_{z_\pm}\big(\gamma^{-1}\big)\Phi^\pm_{\gamma^{-1}}(x),$$
$$\varphi^\dagger_\gamma(x) = c^\dagger_{z_\pm}(\gamma)\Phi^\pm_\gamma(x) + c^\dagger_{z_\pm}\big(\gamma^{-1}\big)\Phi^\pm_{\gamma^{-1}}(x).$$

The spaces $V^\pm_2$ and $V^\pm_{-2}$ are 2-dimensional by Lemma 3.1, but they are clearly not spanned by $\Phi^\pm_\gamma$ and $\Phi^\pm_{1/\gamma}$, since $\gamma = \pm 1$ here. In the following lemma we give linear bases for the spaces $V^\pm_2$ and $V^\pm_{-2}$ that will be useful later on.

Lemma 3.5 *For $\gamma = 1$ or $\gamma = -1$, the functions $\Phi^\pm_\gamma$ and $\frac{d\Phi^\pm_{\gamma'}}{d\gamma'}|_{\gamma'=\gamma}$ form a linear basis for the spaces $V^\pm_2$ and $V^\pm_{-2}$, respectively.*

Proof Differentiating the equation $L\Phi^+_\gamma = \mu(\gamma)\Phi^+_\gamma$ with respect to γ and setting $\gamma = \pm 1$ shows that $\frac{d\Phi^+_\gamma}{d\gamma}|_{\gamma=\pm 1}$ is an eigenfunction of L for eigenvalue ± 2. From the asymptotic behavior (3.5) of Φ^+_γ we find

$$\frac{d\Phi^+_\gamma}{d\gamma}\big(zq^{-k}\big) = sk(s\gamma)^{k-1}\big(1 + \mathcal{O}\big(q^k\big)\big), \quad k \to \infty,$$

and then using Lemma 2.4 it follows that

$$D\left(\Phi^+_\gamma, \frac{d\Phi^+_\gamma}{d\gamma}\right)(x) = \gamma^{2k-2} K_{z_+}, \quad x \in \mathbb{R}^+_q.$$

For $\gamma = \pm 1$ we see that $D(\Phi^+_{\pm 1}, \frac{d\Phi^+_\gamma}{d\gamma}|_{\gamma=\pm 1}) = K_{z_+} \neq 0$. This proves the lemma for Φ^+_γ. For Φ^-_γ the proof is the same. □

3.4 A Basis for V_μ

We are going to show that, under certain conditions on γ, the solutions φ_γ and $\varphi^\dagger_\gamma$ form a linear basis for $V_{\mu(\gamma)}$. We do this by computing the Casorati determinant $D(\varphi_\gamma, \varphi^\dagger_\gamma)$.

Lemma 3.6 *For $x \in \mathbb{R}_q$ and $\gamma \in \mathbb{C}^*$ we have*

$$D(\varphi_\gamma, \varphi_\gamma^\dagger)(x) = \frac{(1-q)q}{as} \frac{(s\gamma, s/\gamma; q)_\infty \theta(a/b)}{(cq/a, cq/b, dq/a, dq/b; q)_\infty}.$$

Proof Let $\gamma \in \mathcal{S}_{\mathrm{reg}}$. From Proposition 3.2 we know that $\varphi_\gamma, \varphi_\gamma^\dagger \in V_{\mu(\gamma)}$; hence by Lemma 3.1 the Casorati determinant $D(\varphi_\gamma, \varphi_\gamma^\dagger)$ is constant on $\mathbb{R}_q$. In order to calculate $D(\varphi_\gamma, \varphi_\gamma^\dagger)(x)$ we use the c-function expansions from Proposition 3.4,

$$D(\varphi_\gamma, \varphi_\gamma^\dagger)(x) = \sum_{\epsilon,\eta \in \{-1,1\}} c_{z_+}(\gamma^\epsilon) c_{z_+}^\dagger(\gamma^\eta) D(\Phi_{\gamma^\epsilon}^+, \Phi_{\gamma^\eta}^+)(x), \quad x = z_+ q^{-k}.$$

We apply Lemma 3.3. Then

$$D(\varphi_\gamma, \varphi_\gamma^\dagger)(z_+ q^{-k}) = (\gamma - 1/\gamma) K_{z_+} \big(c_{z_+}(\gamma) c_{z_+}^\dagger(1/\gamma) - c_{z_+}(1/\gamma) c_{z_+}^\dagger(\gamma)\big).$$

Using $cq/as = bs/d$ and $dq/as = bs/c$, we find

$$\begin{aligned}
&c_{z_+}(\gamma) c_{z_+}^\dagger(1/\gamma) - c_{z_+}(1/\gamma) c_{z_+}^\dagger(\gamma) \\
&\quad = \frac{(s\gamma, s/\gamma; q)_\infty}{(\gamma^2, 1/\gamma^2; q)_\infty (cq/a, cq/b, dq/a, dq/b; q)_\infty \theta(az_+, bz_+)} \\
&\qquad \times \big(\theta(q/bsz_+\gamma, q\gamma/asz_+, cq\gamma/bs, cq/as\gamma) \\
&\qquad - \theta(q\gamma/bsz_+, q/asz_+\gamma, cq/bs\gamma, cq\gamma/as)\big).
\end{aligned}$$

Now we use the θ-product identity (1.1) with

$$x = \frac{qe^{\frac{1}{2}i\kappa}}{bs}\sqrt{\frac{|c|}{z_+}}, \qquad v = \gamma e^{\frac{1}{2}i\kappa}\sqrt{|c|z_+},$$

$$y = \frac{qe^{\frac{1}{2}i\kappa}}{as}\sqrt{\frac{|c|}{z_+}}, \qquad w = \gamma e^{-\frac{1}{2}i\kappa}\sqrt{\frac{1}{|c|z_+}},$$

where $c = |c|e^{i\kappa}$. We then obtain

$$\begin{aligned}
&c_{z_+}(\gamma) c_{z_+}^\dagger(1/\gamma) - c_{z_+}(1/\gamma) c_{z_+}^\dagger(\gamma) \\
&\quad = \frac{q}{asz_+(\gamma - 1/\gamma)} \frac{(s\gamma, s/\gamma; q)_\infty \theta(a/b, cz_+, dz_+)}{(cq/a, cq/b, dq/a, dq/b; q)_\infty \theta(az_+, bz_+)}.
\end{aligned}$$

With the explicit expression for K_{z_+} we find the desired result for $\gamma \in \mathcal{S}_{\mathrm{reg}}$. By continuity in γ the result holds for all $\gamma \in \mathbb{C}^*$. □

Let $\mathcal{S}_{\mathrm{pol}}$ be the set of zeros of $\gamma \mapsto D(\varphi_\gamma, \varphi_\gamma^\dagger)$, i.e.,

$$\mathcal{S}_{\mathrm{pol}} = \{sq^k \mid k \in \mathbb{Z}_{\geq 0}\} \cup \{s^{-1}q^{-k} \mid k \in \mathbb{Z}_{\geq 0}\}.$$

Proposition 3.7 *Let $\gamma \in \mathbb{C}^* \setminus \mathcal{S}_{\mathrm{pol}}$. Then* $\dim V_{\mu(\gamma)} = 2$ *and the set* $\{\varphi_\gamma, \varphi_\gamma^\dagger\}$ *is a linear basis of* $V_{\mu(\gamma)}$.

Proof From Lemma 3.6 it follows that φ_γ and $\varphi_\gamma^\dagger$ are linearly independent if $\gamma \notin \mathcal{S}_{\mathrm{pol}}$. Since both functions are continuously q-differentiable at the origin (see Proposition 3.2), and since $\dim V_{\mu(\gamma)} \le 2$ by Lemma 3.1, we have $\dim V_{\mu(\gamma)} = 2$, and φ_γ and $\varphi_\gamma^\dagger$ form a linear basis for $V_{\mu(\gamma)}$. □

Corollary 3.8 *For $\gamma \in \mathbb{C}^* \setminus \mathcal{S}_{\mathrm{pol}}$ every function in $V^+_{\mu(\gamma)}$ (and $V^-_{\mu(\gamma)}$, respectively) has a unique extension to $V_{\mu(\gamma)}$.*

Proof Fix a $\gamma \in \mathbb{C}^* \setminus \mathcal{S}_{\mathrm{pol}}$ and denote $\mu = \mu(\gamma)$. We consider the restriction map $\mathrm{res} : V_\mu \to V_\mu^+$ defined by $f^{\mathrm{res}} = f|_{\mathbb{R}_q^+}$. Let f and g be linearly independent in V_μ. As in the proof of Lemma 3.1, it follows that f^{res} and g^{res} are linearly independent in V_μ^+. Since $\dim V_\mu = \dim V_\mu^+ = 2$, the map res is a linear isomorphism. In a similar way, a linear isomorphism between V_μ and V_μ^- can be constructed. □

For $\gamma \in \mathcal{S}_{\mathrm{pol}}$ the big q-Jacobi functions φ_γ and $\varphi_\gamma^\dagger$ are actually multiples of big q-Jacobi polynomials (see [13, Prop. 5.3]). The big q-Jacobi polynomials (see [1, 11]) are defined by

$$P_k(x;\alpha,\beta,\delta;q) = {}_3\phi_2\left(\begin{matrix} q^{-k}, \alpha\beta q^{k+1}, x \\ \alpha q, \delta q \end{matrix} ; q, q \right), \quad k \in \mathbb{Z}_{\ge 0}.$$

Lemma 3.9 *Let $\gamma_k = sq^k \in \mathcal{S}_{\mathrm{pol}}$ or $\gamma_k = s^{-1}q^{-k} \in \mathcal{S}_{\mathrm{pol}}$. Then*

$$\varphi_{\gamma_k}(x) = \frac{(cq/b, dq/b; q)_k}{(cq/a, dq/a; q)_k} \left(\frac{b}{a}\right)^k \varphi_{\gamma_k}^\dagger(x),$$

and

$$\varphi_{\gamma_k}^\dagger(x) = q^{-\frac{1}{2}k(k+1)} \left(-\frac{a}{c}\right)^k \frac{(cq/a;q)_k}{(dq/b;q)_k} P_k(cx; c/b, d/a, c/a; q),$$

for $x \in \mathbb{R}_q$.

Proof Let $k \in \mathbb{Z}_{\ge 0}$ and $\gamma_k = sq^k$. From Lemma 3.6 we see that the Casorati determinant $D(\varphi_{\gamma_k}, \varphi_{\gamma_k}^*)(x)$, $x \in \mathbb{R}_q$, is equal to zero, hence $\varphi_{\gamma_k}(x) = C_k \varphi_{\gamma_k}^\dagger(x)$, for some constant C_k independent of x. To find the constant C_k we use Proposition 3.4. We have $c_z(\gamma_k) = 0$ and $c_z^\dagger(\gamma_k) = 0$; hence

$$\varphi_{\gamma_k}(x) = c_{z_+}(1/\gamma_k)\Phi^+_{1/\gamma_k}(x) = \frac{c_{z_+}(1/\gamma_k)}{c^\dagger_{z_+}(1/\gamma_k)} \varphi^\dagger_{\gamma_k}(x), \quad x = z_+q^n \in \mathbb{R}_q.$$

Using $\theta(q^k x) = (-x)^{-k} q^{-\frac{1}{2}k(k-1)}\theta(x)$ we find

$$C_k = \frac{c_{z_+}(1/\gamma_k)}{c^\dagger_{z_+}(1/\gamma_k)} = \frac{(cq/b, dq/b; q)_k}{(cq/a, dq/a; q)_k} \left(\frac{b}{a}\right)^k.$$

Since $\varphi_\gamma = \varphi_{1/\gamma}$, the result also holds for $\gamma_k = s^{-1}q^{-k}$, $k \in \mathbb{Z}_{\ge 0}$.

Finally, writing out $\varphi^\dagger_{\gamma_k}(x)$ explicitly as a ${}_3\phi_2$-series using (3.2) and then applying the transformation formula [5, (III.13)] shows that $\varphi^\dagger_{\gamma_k}(x)$ is a multiple of a big q-Jacobi polynomial in the variable cx. □

3.5 Extensions of the Asymptotic Solutions

By Corollary 3.8 the asymptotic solutions $\Phi^+_\gamma \in V^+_{\mu(\gamma)}$ and $\Phi^-_\gamma \in V^-_{\mu(\gamma)}$ have unique extensions to $V_{\mu(\gamma)}$, provided that $\gamma \in \mathbb{C}^* \setminus \mathcal{S}_{\mathrm{pol}}$. We denote these extensions again by Φ^+_γ and Φ^-_γ. Propositions 3.7 and 3.4 enable us to expand $\Phi^\pm_{\gamma^{\pm 1}}$ in terms of the basis $\{\varphi_\gamma, \varphi^\dagger_\gamma\}$ of $V_{\mu(\gamma)}$.

Proposition 3.10 *For $x \in \mathbb{R}_q$ and $\gamma \in \mathcal{S}_{\mathrm{reg}} \setminus \{sq^k \mid k \in \mathbb{Z}_{\geq 0}\}$,*

$$\Phi^+_\gamma(x) = d_{z_+}(\gamma)\varphi_\gamma(x) + d^\dagger_{z_+}(\gamma)\varphi^\dagger_\gamma(x),$$
$$\Phi^-_\gamma(x) = d_{z_-}(\gamma)\varphi_\gamma(x) + d^\dagger_{z_-}(\gamma)\varphi^\dagger_\gamma(x),$$

where

$$\begin{aligned} d_z(\gamma) &= d_z(\gamma; a,b,c,d|q) \\ &= \frac{(cq/a, dq/a; q)_\infty \theta(bz)}{\theta(a/b, cz, dz)} \frac{(cq\gamma/sb, dq\gamma/sb; q)_\infty \theta(asz/q\gamma)}{(q\gamma^2, s/\gamma; q)_\infty}. \end{aligned}$$

Proof Let $\gamma \in \mathcal{S}_{\mathrm{reg}} \setminus \mathcal{S}_{\mathrm{pol}}$. By Proposition 3.7 we may expand

$$\Phi^\pm_\gamma(x) = d_{z_\pm}(\gamma)\varphi_\gamma(x) + e_{z_\pm}(\gamma)\varphi^\dagger_\gamma(x), \quad x \in \mathbb{R}_q,$$

for some coefficients $d_z(\gamma)$ and $e_z(\gamma)$ independent of x. In order to compute the coefficients $d_z(\gamma)$ and $e_z(\gamma)$, we observe that it follows from $\Phi^{\pm\dagger}_\gamma = \Phi^\pm_\gamma$ that $e_z(\gamma) = d^\dagger_z(\gamma)$. To compute $d_z(\gamma)$ we use

$$d_{z_\pm}(\gamma) = \frac{D(\Phi^\pm_\gamma, \varphi^\dagger_\gamma)(x)}{D(\varphi_\gamma, \varphi^\dagger_\gamma)(x)}.$$

From the c-function expansion, Proposition 3.4, we find

$$D\big(\Phi^\pm_\gamma, \varphi^\dagger_\gamma\big)(x) = c^\dagger_{z_\pm}(1/\gamma) D\big(\Phi^\pm_\gamma, \Phi^\pm_{1/\gamma}\big)(x), \quad x \in \mathbb{R}_q,$$

and then it follows from Lemmas 3.3 and 3.6 that

$$\begin{aligned} d_z(\gamma) &= \frac{(\gamma - 1/\gamma) K_z c^\dagger_z(1/\gamma)}{D(\varphi_\gamma, \varphi^\dagger_\gamma)} \\ &= \frac{asz(cq/a, dq/a; q)_\infty \theta(bz)}{q\,\theta(a/b, cz, dz)} \frac{(\gamma - 1/\gamma)(cq\gamma/bs, dq\gamma/bs; q)_\infty \theta(asz/\gamma)}{(\gamma^2, s/\gamma; q)_\infty}. \end{aligned}$$

Here we also used the explicit expression for K_z from Lemma 2.4. This is the desired result for $\gamma \in \mathcal{S}_{\mathrm{reg}} \setminus \mathcal{S}_{\mathrm{pol}}$. By continuity in γ it holds also for $\gamma \in \{s^{-1}q^{-k} \mid k \in \mathbb{Z}\}$. □

For $\gamma = s^{-1}q^{-k}$, $k \in \mathbb{Z}_{\geq 0}$, the Casorati determinant $D(\varphi_\gamma, \varphi_\gamma^\dagger)$ is equal to zero; hence φ_γ is a multiple of $\varphi_\gamma^\dagger$. In this case Proposition 3.10 states that $\Phi_\gamma^\pm$ is also a multiple of $\varphi_\gamma^\dagger$.

Corollary 3.11 *For $\gamma_k = s^{-1}q^{-k}$, $k \in \mathbb{Z}_{\geq 0}$,*

$$\Phi_{\gamma_k}^\pm(x) = q^{\frac{1}{2}k(k-1)}\left(\frac{-1}{az_\pm}\right)^k \frac{(cq/b, dq/b; q)_k}{(s^2; q)_k}\varphi_{\gamma_k}^\dagger(x).$$

Proof Using Proposition 3.10 and Lemma 3.9, we find

$$\Phi_{\gamma_k}^+(x) = \frac{b}{aq}\left(\frac{b}{qcdz_+}\right)^{k-1} \frac{(cq/b, dq/b; q)_k F_{z_+}}{(q^{1-k}/s^2; q)_k \theta(b/a, cz_+, dz_+, s^2)}\varphi_{\gamma_k}^\dagger(x),$$

with

$$F_{z_+} = \theta(cq/a, dq/a, q/bz_+, cdqz_+/b) - \theta(cq/b, dq/b, q/az_+, cdqz_+/a).$$

Applying the θ-product identity (1.1) with

$$x = \frac{qe^{i(\kappa+\delta)}\sqrt{|cd|}}{b}, \qquad y = \frac{qe^{i(\kappa+\delta)}\sqrt{|cd|}}{a},$$
$$v = \frac{e^{-i(\kappa+\delta)}}{z_+\sqrt{|cd|}}, \qquad w = e^{i(\kappa-\delta)}\sqrt{\left|\frac{c}{d}\right|},$$

where $c = |c|e^{i\kappa}$ and $d = |d|e^{i\delta}$, we obtain

$$F_{z_+} = \frac{cdqz_+}{a}\theta\big(cdq^2/ab, a/b, 1/dz_+, 1/cz_+\big).$$

Applying $(q^{1-k}/y; q)_k = (-y)^{-k}q^{-\frac{1}{2}k(k-1)}(y; q)_k$, identities for θ-functions, and $s^2 = cdq/ab$, the result follows for $\Phi_{\gamma_k}^+$. Replacing z_+ by z_- gives the result for $\Phi_{\gamma_k}^-$. □

In the expansion of $\Phi_\gamma^\pm$ in Proposition 3.10 we have assumed that $\gamma \notin \{sq^k \mid k \in \mathbb{Z}_{\geq 0}\}$. At first sight, it seems that the functions $\Phi_\gamma^\pm(x)$, considered as functions of γ and with $x \in \mathbb{R}_q$ fixed, have simple poles at the points $\gamma = sq^k$, $k \in \mathbb{Z}_{\geq 0}$, which are the poles of the function $d_z(\gamma)$. It turns out that the functions $\Phi_\gamma^\pm(x)$ are actually analytic at these points.

Proposition 3.12 *For a given $x \in \mathbb{R}_q$ the functions $\gamma \mapsto \Phi_\gamma^\pm(x)$ are analytic on $\mathcal{S}_{\mathrm{reg}}$. In particular, for $\gamma_k = sq^k$, $k \in \mathbb{Z}_{\geq 0}$,*

$$\Phi_{\gamma_k}^\pm(x) = \operatorname*{Res}_{\gamma=\gamma_k}\big(d_{z_\pm}(\gamma)\big)\frac{d}{d\gamma}\varphi_\gamma(x)\Big|_{\gamma=\gamma_k} + \operatorname*{Res}_{\gamma=\gamma_k}\big(d_{z_\pm}^\dagger(\gamma)\big)\frac{d}{d\gamma}\varphi_\gamma^\dagger(x)\Big|_{\gamma=\gamma_k}$$
$$+ \tilde{d}_{z_\pm}(\gamma_k)\varphi_{\gamma_k}^\dagger(x),$$

where

$$\tilde{d}_z(\gamma_k) = \lim_{\gamma\to\gamma_k}\left(\frac{(cq/b, dq/b; q)_k}{(cq/a, dq/a; q)_k}\left(\frac{b}{a}\right)^k d_z(\gamma) + d_z^\dagger(\gamma)\right).$$

Proof The expansion from Proposition 3.10 shows that the functions $\gamma \mapsto \Phi_\gamma^\pm(x)$, for a given $x \in \mathbb{R}_q$, are analytic functions on $\mathcal{S}_{\text{reg}} \setminus \{sq^k \mid k \in \mathbb{Z}_{\geq 0}\}$. So we only have to consider the functions $\Phi_\gamma^\pm(x)$ at the points $\gamma_k = sq^k$, $k \in \mathbb{Z}_{\geq 0}$.

Fix a $k \in \mathbb{Z}_{\geq 0}$ and a $x \in \mathbb{R}_q$. The function $\gamma \mapsto d_z(\gamma)$ has a simple pole at $\gamma = \gamma_k$ coming from the zero of the infinite product $(s/\gamma; q)_\infty$, and the functions $\gamma \mapsto \varphi_\gamma(x)$ and $\gamma \mapsto \varphi_\gamma^\dagger(x)$ are analytic at $\gamma = \gamma_k$. From Proposition 3.10 and Lemma 3.9 it follows that

$$\begin{aligned}\Phi_\gamma^+(x) &= (\gamma-\gamma_k)d_{z_+}(\gamma)\frac{\varphi_\gamma(x)-\varphi_{\gamma_k}(x)}{\gamma-\gamma_k} + (\gamma-\gamma_k)d_{z_+}^\dagger(\gamma)\frac{\varphi_\gamma^\dagger(x)-\varphi_{\gamma_k}^\dagger(x)}{\gamma-\gamma_k}\\ &\quad + \left(\frac{(cq/b, dq/b; q)_k}{(cq/a, dq/a; q)_k}\left(\frac{b}{a}\right)^k d_{z_+}(\gamma) + d_{z_+}^\dagger(\gamma)\right)\varphi_{\gamma_k}^\dagger(x).\end{aligned}$$

We see that the limit $\lim_{\gamma\to\gamma_k}\Phi_\gamma^+(x)$ exists if $\tilde{d}_{z_+}(\gamma_k)$, as defined in the proposition, exists. Let us define

$$\hat{d}_z(\gamma) = (s/\gamma; q)_\infty d_z(\gamma).$$

Then $\gamma \mapsto \hat{d}_z(\gamma)$ is regular at $\gamma = \gamma_k$. By a straightforward computation we obtain

$$\frac{(cq/b, dq/b; q)_k}{(cq/a, dq/a; q)_k}\left(\frac{b}{a}\right)^k \hat{d}_z(\gamma_k) + \hat{d}_z^\dagger(\gamma_k) = 0,$$

and then it follows that $\tilde{d}_z(\gamma_k)$ exists. □

We now have the following properties of the functions $\Phi_\gamma^\pm$.

Theorem 3.13 *For $\gamma \in \mathcal{S}_{\text{reg}}$ the functions $\Phi_\gamma^\pm$ satisfy the following properties*:

(a) $\Phi_\gamma^\pm \in V_{\mu(\gamma)}$.
(b) *For $|\gamma| < 1$ we have*

$$\int_{\infty(z_-)}^0 \left|\Phi_\gamma^-(x)\right|^2 w(x)\,d_qx < \infty, \qquad \int_0^{\infty(z_+)} \left|\Phi_\gamma^+(x)\right|^2 w(x)\,d_qx < \infty.$$

(c) *The Casorati determinant $v(\gamma) = D(\Phi_\gamma^+, \Phi_\gamma^-)$ is constant on $\mathbb{R}_q$, and*

$$\begin{aligned}v(\gamma) &= v(\gamma; a, b, c, d; z_-, z_+|q)\\ &= -\frac{z_+(1-q)\theta(z_-/z_+)}{\theta(cz_-, dz_-, cz_+, dz_+)}\\ &\quad \times \frac{(cq\gamma/as, dq\gamma/as, cq\gamma/bs, dq\gamma/bs, s\gamma, q\gamma/s; q)_\infty \theta(absz_-z_+/q\gamma)}{\gamma(q\gamma^2; q)_\infty^2}.\end{aligned}$$

Proof Properties (a) and (b) follow directly from Proposition 3.10 and the asymptotic behavior of $\Phi_\gamma^\pm(x)$ for $|x|\to\infty$, so we only need to check the third property.

Let $\gamma \in \mathcal{S}_{\text{reg}} \setminus \{sq^k \mid k \in \mathbb{Z}_{\geq 0}\}$. Since $\Phi_\gamma^\pm \in V_{\mu(\gamma)}$, the Casorati determinant $D(\Phi_\gamma^-, \Phi_\gamma^+)$ is constant on $\mathbb{R}_q$ by Lemma 3.1. To calculate the determinant we use Proposition 3.10. Then

$$D\big(\Phi_\gamma^+, \Phi_\gamma^-\big) = d_{z_-}(\gamma) D\big(\Phi_\gamma^+, \varphi_\gamma\big) + d_{z_-}^\dagger(\gamma) D\big(\Phi_\gamma^+, \varphi_\gamma^\dagger\big).$$

We find from Proposition 3.4 and Lemma 3.3:

$$\begin{aligned} D\big(\Phi_\gamma^+, \varphi_\gamma\big) &= c_{z_+}(1/\gamma) D\big(\Phi_\gamma^+, \Phi_{1/\gamma}^+\big) = (\gamma - 1/\gamma) c_{z_+}(1/\gamma) K_{z_+}, \\ D\big(\Phi_\gamma^+, \varphi_\gamma^\dagger\big) &= c_{z_+}^\dagger(1/\gamma) D\big(\Phi_\gamma^+, \Phi_{1/\gamma}^+\big) = (\gamma - 1/\gamma) c_{z_+}^\dagger(1/\gamma) K_{z_+}, \end{aligned}$$

so we have

$$D\big(\Phi_\gamma^+, \Phi_\gamma^-\big) = (\gamma - 1/\gamma) K_{z_+}\big(d_{z_-}(\gamma) c_{z_+}(1/\gamma) + d_{z_-}^\dagger(\gamma) c_{z_+}^\dagger(1/\gamma)\big).$$

From the explicit expression for $d_{z_-}(\gamma)$ and $c_{z_+}(\gamma)$ we obtain

$$\begin{aligned} &d_{z_-}(\gamma) c_{z_+}(1/\gamma) + d_{z_-}^\dagger(\gamma) c_{z_+}^\dagger(1/\gamma) \\ &\quad = \frac{bsz_-(cq\gamma/as, dq\gamma/as, cq\gamma/bs, dq\gamma/bs, s\gamma; q)_\infty}{q\gamma(\gamma^2, q\gamma^2, s/\gamma; q)_\infty \theta(cz_-, dz_-, b/a, az_+, bz_+)} \\ &\qquad \times \big[\theta(bz_-, az_+, asz_-/\gamma, bsz_+/\gamma) - \theta(az_-, bz_+, bsz_-/\gamma, asz_+/\gamma)\big]. \end{aligned}$$

Using the θ-product identity (1.1) with

$$x = \frac{ise^{i(\alpha+\beta)/2}\sqrt{|abz_+z_-|}}{\gamma}, \qquad y = ie^{i(\alpha+\beta)/2}\sqrt{|abz_+z_-|},$$

$$v = ie^{i(\alpha-\beta)/2}\sqrt{\left|\frac{az_-}{bz_+}\right|}, \qquad w = ie^{i(\beta-\alpha)/2}\sqrt{\left|\frac{bz_-}{az_+}\right|},$$

where $a = |a|e^{i\alpha}$ and $b = |b|e^{i\beta}$, the term between square bracket equals

$$bz_+\theta(z_-/z_+, a/b, s/\gamma, absz_-z_+/\gamma).$$

Using the explicit expression for K_{z_+}, we now find the Casorati determinant given in the theorem. By continuity in γ, the result holds for all $\gamma \in \mathcal{S}_{\text{reg}}$. □

4 The Spectral Measure

In this section we calculate explicitly the spectral measure E for the self-adjoint operator $(L,\mathcal{D})$ using the formula (see [4, Thm. XII.2.10])

$$\big\langle E(\lambda_1,\lambda_2)f,g\big\rangle_{\mathcal{L}^2}=\lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_{\lambda_1+\delta}^{\lambda_2-\delta}\big(\big\langle R(\mu+i\varepsilon)f,g\big\rangle_{\mathcal{L}^2}-\big\langle R(\mu-i\varepsilon)f,g\big\rangle_{\mathcal{L}^2}\big)\,d\mu, \tag{4.1}$$

for $\lambda_1<\lambda_2$ and $f,g\in\mathcal{L}^2$. Here $R(\mu)=(L-\mu)^{-1}$, $\mu\in\mathbb{C}\setminus\mathbb{R}$, denotes the resolvent operator. Our first goal is to find a useful description for the resolvent $R(\mu)$.

4.1 The Resolvent

Let $\lambda\in\mathbb{C}\setminus\mathbb{R}$, and let γ_λ be the unique complex number such that $|\gamma_\lambda|<1$ and $\lambda=\mu(\gamma_\lambda)$. Note that $\gamma_\lambda\notin\mathbb{R}$, so $\gamma_\lambda\in\mathcal{S}_{\mathrm{reg}}$. Let $\mathcal{V}$ denote the set of zeros of $v(\gamma)$, i.e.,

$$\mathcal{V}=\Bigg(\bigcup_{\alpha\in\{\frac{cq}{as},\frac{dq}{as},\frac{cq}{bs},\frac{dq}{bs},s,\frac{q}{s}\}}\left\{\frac{1}{\alpha q^k}\;\middle|\;k\in\mathbb{Z}_{\geq 0}\right\}\cup\left\{z_-z_+q^k\sqrt{abcd/q}\;\middle|\;k\in\mathbb{Z}\right\}\Bigg).$$

If $v(\gamma)=D(\Phi_\gamma^+,\Phi_\gamma^-)\neq 0$, the functions Φ_γ^+ and Φ_γ^- are linearly independent; hence for $\gamma\in\mathcal{S}_{\mathrm{reg}}\setminus\mathcal{V}$ they form a basis for the solution space $V_{\mu(\gamma)}$.

For $\lambda\in\mathbb{C}\setminus(\mathbb{R}\cup\mu(\mathcal{V}))$, we define the operator $R_\lambda:\mathcal{D}\to F(\mathbb{R}_q)$ by

$$(R_\lambda f)(y)=\big\langle f,\overline{K_\lambda(\cdot,y)}\big\rangle_{\mathcal{L}^2},\qquad f\in\mathcal{D},\ y\in\mathbb{R}_q,$$

where $K_\lambda:\mathbb{R}_q\times\mathbb{R}_q\to\mathbb{C}$ is the Green kernel defined by

$$K_\lambda(x,y)=\begin{cases}\dfrac{\Phi_{\gamma_\lambda}^-(x)\Phi_{\gamma_\lambda}^+(y)}{v(\gamma_\lambda)}, & x\leq y,\\[2ex] \dfrac{\Phi_{\gamma_\lambda}^-(y)\Phi_{\gamma_\lambda}^+(x)}{v(\gamma_\lambda)}, & x>y.\end{cases}$$

Observe that by Theorem 3.13 we have $K_\lambda(x,\cdot)\in\mathcal{D}$ as well as $K_\lambda(\cdot,y)\in\mathcal{D}$ for $x,y\in\mathbb{R}_q$. So R_λ is well-defined as an operator mapping from $\mathcal{D}$ to $F(\mathbb{R}_q)$. From Propositions 3.10 and 3.12 we know that the functions $\Phi_\gamma^\pm(x)$, considered as functions in γ, are analytic on $\mathcal{S}_{\mathrm{reg}}$. Now we see that, for $x,y\in\mathbb{R}_q$, the Green kernel $K_{\mu(\gamma)}(x,y)$ is a meromorphic function in γ, with poles coming from the zeros of $v(\gamma)$.

Proposition 4.1 *For $\lambda\in\mathbb{C}\setminus(\mathbb{R}\cup\mu(\mathcal{V}))$, the operator R_λ is the resolvent of $(L,\mathcal{D})$.*

Proof The operator $(L,\mathcal{D})$ is self-adjoint; hence the spectrum is contained in $\mathbb{R}$. So for $\lambda\in\mathbb{C}\setminus\mathbb{R}$ the resolvent $R(\lambda)$ is a bounded linear operator mapping from $\mathcal{L}^2$ to $\mathcal{D}$, and therefore for a given $y\in\mathbb{R}_q$, the assignment $f\mapsto(R(\lambda)f)(y)$ defines a bounded linear functional on $\mathcal{L}^2$. By the Riesz representation theorem there exists a kernel $K'_\lambda(\cdot, y) \in \mathcal{L}^2$ such that $(R(\lambda)f)(y) = \langle f, K'_\lambda(\cdot, y)\rangle_{\mathcal{L}^2}$. So it suffices to show that $(L-\lambda)R_\lambda f = f$ for $f \in \mathcal{D}$.

Suppose that $y > 0$. Then

$$
\begin{aligned}
(L-\lambda)R_\lambda f(y) &= \int_{\mathbb{R}_q} f(x)\big(A(y)K_\lambda(x, y/q) + B(y)K_\lambda(x, yq) \\
&\quad + \big(C(y)-\lambda\big)K_\lambda(x,y)\big)w(x)\,d_qx \\
&= \frac{1}{v(\gamma_\lambda)} \int_{\infty(z_-)}^{yq} f(x)\Phi^-_{\gamma_\lambda}(x)\big(A(y)\Phi^+_{\gamma_\lambda}(y/q) + B(y)\Phi^+_{\gamma_\lambda}(yq) \\
&\quad + \big(C(y)-\lambda\big)\Phi^+_{\gamma_\lambda}(y)\big)w(x)\,d_qx \\
&\quad + \frac{1}{v(\gamma_\lambda)} \int_{y/q}^{\infty(z_+)} f(x)\Phi^+_{\gamma_\lambda}(x)\big(A(y)\Phi^-_{\gamma_\lambda}(y/q) + B(y)\Phi^-_{\gamma_\lambda}(yq) \\
&\quad + \big(C(y)-\lambda\big)\Phi^-_{\gamma_\lambda}(y)\big)w(x)\,d_qx \\
&\quad + \frac{(1-q)yw(y)}{v(\gamma_\lambda)} f(y)\big(A(y)\Phi^-_{\gamma_\lambda}(y)\Phi^+_{\gamma_\lambda}(y/q) \\
&\quad + B(y)\Phi^-_{\gamma_\lambda}(yq)\Phi^+_{\gamma_\lambda}(y) + \big(C(y)-\lambda\big)\Phi^-_{\gamma_\lambda}(y)\Phi^+_{\gamma_\lambda}(y)\big) \\
&= \frac{(1-q)yB(y)w(y)}{v(\gamma_\lambda)} f(y)\big(\Phi^+_{\gamma_\lambda}(y)\Phi^-_{\gamma_\lambda}(yq) - \Phi^+_{\gamma_\lambda}(yq)\Phi^-_{\gamma_\lambda}(y)\big) \\
&= f(y).
\end{aligned}
$$

Here we used that $\Phi^\pm_{\gamma_\lambda}$ are solutions of $Lf = \lambda f$, $v(\gamma) = D(\Phi^+_\gamma, \Phi^-_\gamma)$, and Definition 2.2 of the Casorati determinant. The proof for $y < 0$ runs along the same lines. □

4.2 The Continuous Spectrum

We are going to investigate the integrand in (4.1). Using the definition of the Green kernel we have

$$
\begin{aligned}
\langle R_\mu f, g\rangle_{\mathcal{L}^2} &= \iint_{\mathbb{R}_q\times\mathbb{R}_q} f(x)\overline{g(y)}K_\mu(x,y)w(x)w(y)\,d_qx\,d_qy \\
&= \iint_{\substack{(x,y)\in\mathbb{R}_q\times\mathbb{R}_q \\ x\le y}} \frac{\Phi^-_{\gamma_\mu}(x)\Phi^+_{\gamma_\mu}(y)}{v(\gamma_\mu)} \\
&\quad \times \big(f(x)\overline{g(y)} + f(y)\overline{g(x)}\big)\left(1 - \frac{1}{2}\delta_{xy}\right)w(x)w(y)\,d_qx\,d_qy. \qquad (4.2)
\end{aligned}
$$

The Kronecker-delta function δ_{xy} is needed here to prevent the terms on the diagonal $x = y$ from being counted twice.

We define two functions v_1 and v_2 that we need to describe the spectral measure E;

$$v_1(\gamma) = \frac{(cq/a, dq/a; q)_\infty^2 \theta(bz_+, bz_-)}{(1-q)abz_-^2 z_+^2 \theta(z_-/z_+, z_+/z_-, a/b, b/a)}$$
$$\times \frac{(\gamma^{\pm 2}; q)_\infty}{(s\gamma^{\pm 1}, cq\gamma^{\pm 1}/as, dq\gamma^{\pm 1}/as; q)_\infty \theta(s\gamma^{\pm 1}, absz_-z_+\gamma^{\pm 1})}$$
$$\times \big(z_-\theta\big(az_+, cz_+, dz_+, bz_-, asz_-\gamma^{\pm 1}\big)$$
$$- z_+\theta\big(az_-, cz_-, dz_-, bz_+, asz_+\gamma^{\pm 1}\big)\big),$$
$$v_2(\gamma) = \frac{(cq/a, dq/a, cq/b, dq/b; q)_\infty \theta(az_+, az_-, bz_+, bz_-, cdz_-z_+)}{abz_-^2 z_+(1-q)\theta(z_+/z_-, a/b, b/a)}$$
$$\times \frac{(\gamma^{\pm 2}; q)_\infty}{(s\gamma^{\pm 1}; q)_\infty \theta(s\gamma^{\pm 1}, absz_-z_+\gamma^{\pm 1})}.$$

Note that v_1 and v_2 are both invariant under $\gamma \leftrightarrow 1/\gamma$. Let $\mathcal{D}_{\text{fin}} \subset \mathcal{D}$ be the subspace consisting of finitely supported functions in $\mathcal{L}^2$. To a function $f \in \mathcal{D}_{\text{fin}}$ we associate two functions $\mathcal{F}_c f$ and $\mathcal{F}_c^\dagger f$ on the unit circle $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ defined by

$$(\mathcal{F}_c f)(\gamma) = \int_{\mathbb{R}_q} f(x)\varphi_\gamma(x)w(x)\,d_q x,$$
$$\big(\mathcal{F}_c^\dagger f\big)(\gamma) = \int_{\mathbb{R}_q} f(x)\varphi_\gamma^\dagger(x)w(x)\,d_q x,$$

where $\gamma \in \mathbb{T}$.

We are now almost ready to describe the spectral measure $E((\lambda_1, \lambda_2))$ for $(\lambda_1, \lambda_2) \subset (-2, 2)$. First we give a preliminary result. The proof is an easy but rather tedious computation that we carry out in the Appendix.

Lemma 4.2 *For $x, y \in \mathbb{R}_q$ and $\gamma, \gamma^{-1} \in \mathcal{S}_{\text{reg}} \setminus \mathcal{V}$ we have*

$$\frac{\Phi_{1/\gamma}^-(x)\Phi_{1/\gamma}^+(y)}{v(1/\gamma)} - \frac{\Phi_\gamma^-(x)\Phi_\gamma^+(y)}{v(\gamma)}$$
$$= \frac{1}{\gamma - 1/\gamma}\big[v_1(\gamma)\varphi_\gamma(x)\varphi_\gamma(y) + v_2(\gamma)\big(\varphi_\gamma(x)\varphi_\gamma^\dagger(y) + \varphi_\gamma^\dagger(x)\varphi_\gamma(y)\big)$$
$$+ v_1^\dagger(\gamma)\varphi_\gamma^\dagger(x)\varphi_\gamma^\dagger(y)\big].$$

Proposition 4.3 *Let $(a, b, c, d) \in P_{\text{gen}}$, let $0 < \psi_1 < \psi_2 < \pi$, and let $\lambda_1 = \mu(e^{i\psi_2})$ and $\lambda_2 = \mu(e^{i\psi_1})$. Then for $f, g \in \mathcal{D}_{\text{fin}}$,*

$$\big\langle E(\lambda_1, \lambda_2)f, g\big\rangle_{\mathcal{L}^2} = \frac{1}{2\pi}\int_{\psi_1}^{\psi_2} \big(\big(\mathcal{F}_c^\dagger \overline{g}\big)\big(e^{i\psi}\big)\,(\mathcal{F}_c \overline{g})\big(e^{i\psi}\big)\big)$$
$$\times \begin{pmatrix} v_2(e^{i\psi}) & v_1^\dagger(e^{i\psi}) \\ v_1(e^{i\psi}) & v_2(e^{i\psi}) \end{pmatrix} \begin{pmatrix} (\mathcal{F}_c f)(e^{i\psi}) \\ (\mathcal{F}_c^\dagger f)(e^{i\psi}) \end{pmatrix} d\psi.$$

Proof Let $\lambda \in (-2,2)$. Then $\lambda = \mu(e^{i\psi})$ for a unique $\psi \in (0,\pi)$. In this case we have

$$\lim_{\varepsilon \downarrow 0} \gamma_{\lambda \pm i\varepsilon} = e^{\mp i\psi}.$$

Now we obtain

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}&\left(\frac{\Phi^-_{\gamma_{\lambda+i\varepsilon}}(x)\Phi^+_{\gamma_{\lambda+i\varepsilon}}(y)}{v(\gamma_{\lambda+i\varepsilon})} - \frac{\Phi^-_{\gamma_{\lambda-i\varepsilon}}(x)\Phi^+_{\gamma_{\lambda-i\varepsilon}}(y)}{v(\gamma_{\lambda-i\varepsilon})}\right)\\ &= \frac{\Phi^-_{e^{-i\psi}}(x)\Phi^+_{e^{-i\psi}}(y)}{v(e^{-i\psi})} - \frac{\Phi^-_{e^{i\psi}}(x)\Phi^+_{e^{i\psi}}(y)}{v(e^{i\psi})},\end{aligned}$$

which is symmetric in x and y by Lemma 4.2. Symmetrizing the double q-integral from (4.2) then gives

$$\begin{aligned}&\lim_{\varepsilon\downarrow 0}\big(\langle R_{\lambda+i\varepsilon} f, g\rangle_{\mathcal{L}^2} - \langle R_{\lambda-i\varepsilon} f, g\rangle_{\mathcal{L}^2}\big)\\ &\quad= \iint_{\mathbb{R}_q\times\mathbb{R}_q} \frac{1}{\gamma - 1/\gamma}\big[v_1(\gamma)\varphi_\gamma(x)\varphi_\gamma(y) + v_2(\gamma)\big(\varphi_\gamma(x)\varphi^\dagger_\gamma(y) + \varphi^\dagger_\gamma(x)\varphi_\gamma(y)\big)\\ &\qquad + v_1^\dagger(\gamma)\varphi^\dagger_\gamma(x)\varphi^\dagger_\gamma(y)\big] f(x)\overline{g(y)} w(x)w(y)\, d_qx\, d_qy,\end{aligned}$$

where $\gamma = e^{i\psi}$. Rewriting this expression in vector notation and using formula (4.1), we obtain the desired result. □

The previous proposition implies that $(-2,2)$ is contained in the spectrum $\sigma(L)$ of L. Since $\varphi_\gamma, \varphi^\dagger_\gamma \notin \mathcal{L}^2$ for $\gamma \in \mathbb{T}$, $(-2,2)$ is part of the continuous spectrum. Observe that the spectral projection is on a 2-dimensional space of eigenvectors, so $(-2,2)$ has multiplicity two. Because the spectrum is a closed set, the points -2 and 2 must be elements of the spectrum $\sigma(L)$.

Lemma 4.4 *The points -2 and 2 are elements of the continuous spectrum of L.*

Proof Since the residual spectrum of a self-adjoint operator is empty, $\mu(-1) = -2$ and $\mu(1) = 2$ must either be elements of the point spectrum or the continuous spectrum. We show that 2 is not in the point spectrum of L. The proof for -2 is the same.

Suppose that there exists a function $f \in \mathcal{L}^2$ that satisfies $Lf = 2f$. Then the restriction f^{res} of f to $\mathbb{R}_q^+$ is an element of V_2^+. From Lemma 3.5 it follows that $f^{\text{res}} = \alpha\Phi_1^+ + \beta\frac{d\Phi^+_\gamma}{d\gamma}|_{\gamma=1}$ for some coefficients α and β. But neither of the functions Φ_1^+ and $\frac{d\Phi^+_\gamma}{d\gamma}|_{\gamma=1}$ is integrable with respect to $w(x)$ on $\mathbb{R}_q^+$ (see (3.5) and Lemma 2.4), which contradicts the fact that $f \in \mathcal{L}^2$. □

4.3 The Point Spectrum

Let $\mu \in \mathbb{R} \setminus [-2, 2]$. Then

$$\lim_{\varepsilon \downarrow 0} \gamma_{\mu \pm i\varepsilon} = \gamma_\mu.$$

From (4.1) and (4.2) we see that in this case the only contribution to the spectral measure E comes from the real poles of the Green kernel $K_{\mu(\gamma)}(x, y)$, $x, y \in \mathbb{R}_q$, considered as a function of γ. Let $\Gamma \subset \mathcal{V}$ denote the set of poles of the Green kernel inside the interval $(-1, 1)$. We now have the following property for the spectral measure.

Proposition 4.5 *For real numbers $\mu_1 < \mu_2$ satisfying $(\mu_1, \mu_2) \cap (\mu(\Gamma) \cup [-2, 2]) = \emptyset$, we have $E((\mu_1, \mu_2)) = 0$.*

The set Γ of real poles of the Green kernel inside the unit disc is given by

$$\begin{aligned} \Gamma &= \Gamma_s^{\mathrm{fin}} \cup \Gamma_{q/s}^{\mathrm{fin}} \cup \Gamma_{dq/as}^{\mathrm{fin}} \cup \Gamma^{\mathrm{inf}}, \\ \Gamma_\alpha^{\mathrm{fin}} &= \left\{ \frac{1}{\alpha q^k} \;\middle|\; k \in \mathbb{Z}_{\geq 0},\ \alpha q^k > 1 \right\}, \\ \Gamma^{\mathrm{inf}} &= \left\{ z_- z_+ q^k \sqrt{abcd/q} \;\middle|\; k \in \mathbb{Z},\ -z_- z_+ q^k \sqrt{abcd/q} < 1 \right\}. \end{aligned}$$

The superscripts 'fin' and 'inf' refer to the finite or infinite cardinality of the sets. Recall that for $a, b, c, d \in \mathbb{R}$ we have assumed that $q < a/b < 1$ and $q < c/d < 1$; therefore

$$\frac{acq}{bd}, \frac{bcq}{ad}, \frac{adq}{bc} < 1, \qquad 1 < \frac{bdq}{ac} < \frac{1}{q}.$$

So the factor $(cq\gamma/as, dq\gamma/as, cq\gamma/bs, dq\gamma/bs; q)_\infty$ of the function $v(\gamma)$ has at most one zero inside the interval $(-1, 1)$, and this zero only occurs when $a, b, c, d \in \mathbb{R}$. This shows that the set $\Gamma_{dq/as}^{\mathrm{fin}}$ has at most one element. We remark that for $(a, b, c, d) \in P_{\mathrm{gen}}$ the real poles of the Green kernel are simple.

Next we need to find the spectral measure on the set $\mu(\Gamma)$. For this the following lemma is useful.

Lemma 4.6 *For $\gamma \in \Gamma$ we have*

$$\Phi_\gamma^+(x) = b(\gamma)\Phi_\gamma^-(x), \qquad x \in \mathbb{R}_q,$$

where $b(\gamma) = b(\gamma; a, b, c, d; z_-, z_+; q)$ *is given by*

$$b(\gamma) = \begin{cases} \left(\frac{z_+}{z_-}\right)^{k+1} \frac{\theta(cz_-, dz_-)}{\theta(cz_+, dz_+)}, & \gamma = z_- z_+ q^k \sqrt{abcd/q} \in \Gamma^{\text{inf}}, \\ \left(\frac{z_-}{z_+}\right)^{k}, & \gamma = s^{-1} q^{-k} \in \Gamma_s^{\text{fin}}, \\ \left(\frac{z_-}{z_+}\right)^{k} \frac{\theta(az_+, bz_+, cz_-, dz_-)}{\theta(az_-, bz_-, cz_+, dz_+)}, & \gamma = sq^{-1-k} \in \Gamma_{q/s}^{\text{fin}}, \\ \frac{\theta(bz_+, cz_-)}{\theta(bz_-, cz_+)}, & \gamma = \frac{as}{dq} \in \Gamma_{dq/as}^{\text{fin}}. \end{cases}$$

Proof If $\gamma \in \Gamma$, then $v(\gamma) = D(\Phi_\gamma^-, \Phi_\gamma^+) = 0$; hence $\Phi_\gamma^+ = b(\gamma)\Phi_\gamma^-$ for some nonzero factor $b(\gamma)$. For $\gamma_k = s^{-1}q^{-k} \in \Gamma_s^{\text{fin}}$ the value of $b(\gamma_k)$ follows from Corollary 3.11. For the other cases it is enough by Proposition 3.10 to show that $d_{z_+}(\gamma) = b(\gamma) d_{z_-}(\gamma)$ and $d_{z_+}^\dagger(\gamma) = b(\gamma) d_{z_-}^\dagger(\gamma)$. This is verified by a straightforward calculation. Note that for $\gamma = as/dq \in \Gamma_{dq/as}^{\text{fin}}$ we have $d_{z_+}^\dagger(\gamma) = d_{z_-}^\dagger(\gamma) = 0$. □

We are now ready to calculate the spectral measure E of $(L, \mathcal{D})$ for the discrete part of the spectrum $\sigma(L)$. We will write $E(\{\mu(\gamma)\})$ for the spectral measure $E((a,b))$ if (a,b) is an interval such that $(a,b) \cap \mu(\Gamma) = \{\mu(\gamma)\}$. For $f \in \mathcal{L}^2$ we define a function $\mathcal{F}_p f$ on Γ by

$$(\mathcal{F}_p f)(\gamma) = \langle f, \Phi_\gamma^+ \rangle_{\mathcal{L}^2}, \quad \gamma \in \Gamma.$$

Note that Theorem 3.13(b) and Lemma 4.6 imply that $\Phi_\gamma^+ \in \mathcal{L}^2$ for $\gamma \in \Gamma$, so the inner product above exists for all $f \in \mathcal{L}^2$.

Proposition 4.7 *Let* $(a,b,c,d) \in P_{\text{gen}}$. *For* $f, g \in \mathcal{L}^2$ *and* $\gamma \in \Gamma$, *the spectral measure* $E(\{\mu(\gamma)\})$ *is given by*

$$\langle E(\{\mu(\gamma)\}) f, g \rangle_{\mathcal{L}^2} = (\mathcal{F}_p f)(\gamma) \overline{(\mathcal{F}_p g)(\gamma)} N(\gamma),$$

where

$$N(\gamma) = N(\gamma; a, b, c, d; z_-, z_+ | q) = b(\gamma)^{-1} \operatorname*{Res}_{\lambda=\gamma} \left(\frac{1/\lambda - \lambda}{\lambda v(\lambda)} \right),$$

and $b(\gamma)$ *is given in Lemma 4.6.*

Proof Let $\gamma\in\Gamma$, and $f,g\in\mathcal{L}^2$. We use (4.1) and (4.2) to calculate the spectral measure $E(\{\mu(\gamma)\})$. By the residue theorem we find

$$\begin{aligned}\big\langle E\big(\{\mu(\gamma)\}\big)f,g\big\rangle_{\mathcal{L}^2}&=\frac{1}{2\pi i}\int_{\mathcal{C}}\langle R_\mu f,g\rangle_{\mathcal{L}^2}\,d\mu\\&=\iint_{\substack{(x,y)\in\mathbb{R}_q\times\mathbb{R}_q\\ x\le y}}-\operatorname*{Res}_{\lambda=\gamma}\left(\big(1-1/\lambda^2\big)\frac{\Phi_\lambda^-(x)\Phi_\lambda^+(y)}{v(\lambda)}\right)\\&\quad\times\big(f(x)\overline{g(y)}+f(y)\overline{g(x)}\big)\left(1-\frac{1}{2}\delta_{x,y}\right)w(x)w(y)\,d_qx\,d_qy.\end{aligned}$$

Here $\mathcal{C}$ is a clockwise oriented contour encircling $\mu(\gamma)$ once, and $\mathcal{C}$ does not encircle any other points in Γ. The factor $1-1/\lambda^2$ comes from changing the integration variable $\mu=\mu(\lambda)$ to λ. By Lemma 4.6 we have $\Phi_\gamma^+=b(\gamma)\Phi_\gamma^-$, so we may symmetrize the double q-integral, and then

$$\big\langle E\big(\{\mu(\gamma)\}\big)f,g\big\rangle_{\mathcal{L}^2}=b(\gamma)^{-1}\big\langle f,\Phi_\gamma^+\big\rangle_{\mathcal{L}^2}\big\langle\Phi_\gamma^+,g\big\rangle_{\mathcal{L}^2}\operatorname*{Res}_{\lambda=\gamma}\left(\frac{1/\lambda-\lambda}{\lambda v(\lambda)}\right).$$

This proves the proposition. □

It is an easy exercise to calculate the weight $N(\gamma)$, $\gamma\in\Gamma$, explicitly. The result is as follows. For $\gamma=as/dq\in\Gamma^{\mathrm{fin}}_{dq/as}$ we have

$$N(\gamma)=\frac{\theta(bz_-,cz_+,cz_+,dz_-,dz_+)(ac/bdq;q)_\infty}{z_+(1-q)\theta(bz_+,bdz_-z_+,z_-/z_+)(q,a/b,a/d,c/b,c/d;q)_\infty},$$

for $\gamma=s^{-1}q^{-k}\in\Gamma^{\mathrm{fin}}_s$

$$\begin{aligned}N(\gamma)&=\frac{\theta(cz_-,cz_+,dz_-,dz_+)(ab/cdq;q)_\infty}{z_+(1-q)\theta(z_-/z_+,cdz_-z_+)(q,a/c,a/d,b/c,b/d;q)_\infty}\\&\quad\times\frac{(q^2cd/ab,qcd/ab;q)_{2k}}{(q,cq/a,cq/b,dq/a,dq/b,qcd/ab;q)_k}\left(-\frac{abz_+^2}{q}\right)^k q^{\frac{3}{2}k(k-1)},\end{aligned}$$

for $\gamma=sq^{-1-k}\in\Gamma^{\mathrm{fin}}_{q/s}$

$$\begin{aligned}N(\gamma)&=\frac{\theta(az_-,bz_-,cz_+,cz_+,dz_+,dz_+)(cd/abq;q)_\infty}{z_+(1-q)\theta(az_+,bz_+,z_-/z_+,abz_-z_+)(q,c/a,c/b,d/a,d/b;q)_\infty}\\&\quad\times\frac{(qab/cd,ab/cd;q)_{2k}}{(q,aq/c,aq/d,bq/d,bq/d,abq/cd;q)_k}\left(-\frac{z_+^2q^4}{cd}\right)^k q^{\frac{3}{2}k(k-1)},\end{aligned}$$

and finally for $\gamma = absz_-z_+q^{k-1} \in \Gamma^{\inf}$ we have

$$N(\gamma) = \frac{\theta(cz_+, dz_+)^2(abcdz_-^2z_+^2/q, abcdz_-^2z_+^2; q)_\infty}{z_+(1-q)\theta(z_-/z_+)(q, q, abz_-z_+, acz_-z_+, adz_-z_+, bcz_-z_+, bdz_-z_+, cdz_-z_+; q)_\infty} \times \frac{(abz_-z_+, acz_-z_+, adz_-z_+, bcz_-z_+, bdz_-z_+, cdz_-z_+; q)_k}{(abcdz_-^2z_+^2/q, abcdz_-^2z_+^2; q)_{2k}} \times \left(\frac{z_-}{z_+}\right)^{k+1}(-1)^k q^{\frac{1}{2}k(k+1)}.$$

As a result of Proposition 4.7 we obtain orthogonality relations for Φ_γ^+, $\gamma \in \Gamma$.

Corollary 4.8 *Let $\gamma, \gamma' \in \Gamma$. Then*

$$\langle \Phi_\gamma^+, \Phi_{\gamma'}^+\rangle_{\mathcal{L}^2} = \frac{\delta_{\gamma\gamma'}}{N(\gamma)}.$$

Proof Eigenfunctions corresponding to different eigenvalues of a self-adjoint operator are pairwise orthogonal. Since for $\gamma, \gamma' \in \Gamma$, $\gamma \neq \gamma'$, the functions Φ_γ^+ and $\Phi_{\gamma'}^+$ are eigenfunctions of $(L, \mathcal{D})$ with distinct eigenvalues $\mu(\gamma)$ and $\mu(\gamma')$, orthogonality follows.

Let $\gamma \in \Gamma$. By Proposition 4.7,

$$\langle \Phi_\gamma^+, \Phi_\gamma^+\rangle_{\mathcal{L}^2} = \langle E(\{\mu(\gamma)\})\Phi_\gamma^+, \Phi_\gamma^+\rangle_{\mathcal{L}^2} = N(\gamma)\langle \Phi_\gamma^+, \Phi_\gamma^+\rangle_{\mathcal{L}^2}^2,$$

from which the squared norm of Φ_γ^+ follows. □

Remark 4.9 For $\gamma, \gamma' \in \Gamma_s^{\text{fin}}$ Corollary 4.8 gives orthogonality relations for a finite number of big q-Jacobi polynomials (see Proposition 3.12 and Lemma 3.9).

Since $\Phi_{s^{-1}}^+(x) = 1$, Corollary 4.8 gives an evaluation of the integral $\langle 1, 1\rangle_{\mathcal{L}^2}$ in case Γ_s^{fin} is not empty, i.e., if $s > 1$.

Corollary 4.10 *For $\sqrt{ab/cdq} < 1$ we have*

$$\begin{aligned}
\frac{1}{1-q}\int_{\mathbb{R}_q} \frac{(ax, bx; q)_\infty}{(cx, dx; q)_\infty} d_qx
&= z_+\frac{(az_+, bz_+; q)_\infty}{(cz_+, dz_+; q)_\infty} {}_2\psi_2\left(\begin{matrix} cz_+, dz_+ \\ az_+, bz_+ \end{matrix}; q, q\right) \\
&\quad - z_-\frac{(az_-, bz_-; q)_\infty}{(cz_-, dz_-; q)_\infty} {}_2\psi_2\left(\begin{matrix} cz_-, dz_- \\ az_-, bz_- \end{matrix}; q, q\right) \\
&= z_+\frac{(q, a/c, a/d, b/c, b/d; q)_\infty\theta(z_-/z_+, cdz_-z_+)}{(ab/cdq; q)_\infty\theta(cz_-, dz_-, cz_+, dz_+; q)_\infty}.
\end{aligned}$$

Here ${}_2\psi_2$ denotes the usual bilateral series as defined in [5].

Remark 4.11 This is the summation formula from [5, Exer. 5.10], and it is actually valid without the restrictions on a, b, c, d as long as the denominator of the integrand is nonzero for all $x \in \mathbb{R}_q$. Note that there is a misprint in [5, Exer. 5.10]: the factors $(e/ab, q^2 f/e; q)_\infty$ on the left hand side must be replaced by $(c/qf, q^2 f/c; q)_\infty$.

Corollary 4.12 *Let* $(a, b, c, d) \in P_{\mathrm{gen}}$. *The spectrum of the self-adjoint operator* $(L, \mathcal{D})$ *consists of the continuous spectrum* $\sigma_c(L) = [-2, 2]$, *with multiplicity two, and the point spectrum* $\sigma_p(L) = \mu(\Gamma)$, *with multiplicity one.*

Proof This follows from Propositions 4.3, 4.5, 4.7 and Lemma 4.4. □

5 The Vector-Valued Big q-Jacobi Function Transform

In this section we define the vector-valued big q-Jacobi function transform $\mathcal{F}$, which is closely related to the maps $\mathcal{F}_c$, $\mathcal{F}_c^\dagger$ and $\mathcal{F}_p$. We show that $\mathcal{F}$ is an isometric isomorphism mapping from $\mathcal{L}^2$ into a certain Hilbert space $\mathcal{H}$, and we also determine $\mathcal{F}^{-1}$. The vector-valued big q-Jacobi function transform diagonalizes the second-order difference operator L; let M be the multiplication operator defined by $(Mf)(\gamma) = \mu(\gamma) f(\gamma)$ for all $\mu(\gamma) \in \sigma(L)$, then

$$\left(\mathcal{F} \circ L \circ \mathcal{F}^{-1}\right) f = Mf,$$

for all $f \in \mathcal{H}$ such that $Mf \in \mathcal{H}$.

We still assume that $(a, b, c, d) \in P_{\mathrm{gen}}$, and we distinguish between the cases $a \neq \overline{b}$ and $a = \overline{b}$. For a vector $y = \binom{y_1}{y_2} \in \mathbb{C}^2$ we denote

$$y^T = \begin{cases} (\overline{y_1}\ \overline{y_2}), & \text{if } a = \overline{b}, \\ (\overline{y_2}\ \overline{y_1}), & \text{if } a \neq \overline{b}. \end{cases}$$

With this convention we have for $\gamma \in \mathbb{R} \cup \mathbb{T}$ and $x \in \mathbb{R}_q$

$$\begin{pmatrix} \varphi_\gamma(x) \\ \varphi_\gamma^\dagger(x) \end{pmatrix}^T = \left(\varphi_\gamma^\dagger(x) \quad \varphi_\gamma(x)\right),$$

since $\overline{\varphi_\gamma(x)} = \varphi_\gamma^\dagger(x)$ if $a = \overline{b}$, and $\overline{\varphi_\gamma(x)} = \varphi_\gamma(x)$ if $a \neq \overline{b}$.

5.1 The Vector-Valued Big q-Jacobi Function Transform $\mathcal{F}$

Let $F(\mathbb{T} \cup \Gamma)$ be the linear space consisting of functions that are complex-valued on Γ and $\mathbb{C}^2$-valued on $\mathbb{T}$. With the maps $\mathcal{F}_c$, $\mathcal{F}_c^\dagger$ and $\mathcal{F}_p$ we define an integral transform $\mathcal{F} : \mathcal{D}_{\mathrm{fin}} \to F(\mathbb{T} \cup \Gamma)$.

Definition 5.1 For $f \in \mathcal{D}_{\mathrm{fin}}$ we define the vector-valued big q-Jacobi function $\mathcal{F}$ by

$$(\mathcal{F} f)(\gamma) = \begin{cases} \begin{pmatrix} (\mathcal{F}_c f)(\gamma) \\ (\mathcal{F}_c^\dagger f)(\gamma) \end{pmatrix}, & \gamma \in \mathbb{T}, \\ (\mathcal{F}_p f)(\gamma), & \gamma \in \Gamma. \end{cases}$$

We define a kernel $\Psi(x,\gamma)$, $x\in\mathbb{R}_q$, $\gamma\in\mathbb{T}\cup\Gamma$, by

$$\Psi(x,\gamma)=\begin{cases}\begin{pmatrix}\varphi_\gamma(x)\\ \varphi_\gamma^\dagger(x)\end{pmatrix}, & \gamma\in\mathbb{T},\\ \Phi_\gamma^+(x), & \gamma\in\Gamma.\end{cases} \tag{5.1}$$

We may write $\mathcal{F}$ as an integral transform with kernel Ψ,

$$(\mathcal{F}f)(\gamma)=\int_{\mathbb{R}_q} f(x)\Psi(x,\gamma)w(x)\,d_qx,\qquad f\in\mathcal{D}_{\text{fin}},\ \gamma\in\mathbb{T}\cup\Gamma.$$

We are going to show that $\mathcal{F}$ extends to a continuous operator mapping from $\mathcal{L}^2$ into a Hilbert space $\mathcal{H}$, which we now define.

We define a matrix-valued function $\mathbf{v}$ on $\mathbb{T}$ by

$$\gamma\mapsto\mathbf{v}(\gamma)=\begin{pmatrix}v_2(\gamma) & v_1^\dagger(\gamma)\\ v_1(\gamma) & v_2(\gamma)\end{pmatrix}.$$

We remark that $\mathbf{v}(\gamma)$, $\gamma\in\mathbb{T}$, is positive-definite. Let $\mathcal{H}_c'$ be the Hilbert space consisting of $\mathbb{C}^2$-valued functions on $\mathbb{T}$ that have finite norm with respect to the inner product

$$\langle g_1,g_2\rangle_{\mathcal{H}_c'}=\frac{1}{4\pi i}\int_{\mathbb{T}} g_2(\gamma)^T\mathbf{v}(\gamma)g_1(\gamma)\frac{d\gamma}{\gamma},$$

where the unit circle $\mathbb{T}$ is oriented in the counter-clockwise direction. Let r denote the reflection operator defined by $(rg)(\gamma)=g(\gamma^{-1})$. We define the Hilbert space $\mathcal{H}_c$ to be the subspace of $\mathcal{H}_c'$ consisting of functions g that satisfy $rg=g$ in $\mathcal{H}_c'$. We denote the inner product on $\mathcal{H}_c$ by $\langle\cdot,\cdot\rangle_{\mathcal{H}_c}$. Furthermore, let $\mathcal{H}_p$ be the Hilbert space consisting of complex-valued functions on Γ that have finite norm with respect to the inner product

$$\langle g_1,g_2\rangle_{\mathcal{H}_p}=\sum_{\gamma\in\Gamma} g_1(\gamma)\overline{g_2(\gamma)}N(\gamma).$$

We define the Hilbert space $\mathcal{H}\subset F(\mathbb{T}\cup\Gamma)$ by $\mathcal{H}=\mathcal{H}_c\oplus\mathcal{H}_p$.

Proposition 5.2 *The map $\mathcal{F}$ extends uniquely to an operator $\mathcal{F}:\mathcal{L}^2\to\mathcal{H}$, satisfying*

$$\langle\mathcal{F}f_1,\mathcal{F}f_2\rangle_{\mathcal{H}}=\langle f_1,f_2\rangle_{\mathcal{L}^2},\qquad f_1,f_2\in\mathcal{L}^2.$$

Hence, $\mathcal{F}$ is an isometric isomorphism onto its range $\mathcal{R}(\mathcal{F})\subset\mathcal{H}$.

Proof Let $f_1,f_2\in\mathcal{D}_{\text{fin}}$. Combining Propositions 4.3 and 4.7 we find

$$\langle f_1,f_2\rangle_{\mathcal{L}^2}=\big\langle E(\mathbb{R})f_1,f_2\big\rangle_{\mathcal{L}^2}=\langle\mathcal{F}f_1,\mathcal{F}f_2\rangle_{\mathcal{H}}.$$

Here we used

$$\big((\mathcal{F}_c^\dagger\overline{f})(\gamma)\quad(\mathcal{F}_c\overline{f})(\gamma)\big)=\begin{pmatrix}(\mathcal{F}_c^\dagger f)(\gamma)\\ (\mathcal{F}_c f)(\gamma)\end{pmatrix}^T.$$

Since $\mathcal{D}_{\text{fin}}$ is dense in $\mathcal{L}^2$, the map $\mathcal{F}$ extends uniquely to a continuous operator, also denoted by $\mathcal{F}$, mapping isometrically into $\mathcal{R}(\mathcal{F}) \subset \mathcal{H}$. □

Lemma 5.3 *Let $y \in \mathbb{R}_q$ and let $f_y(x) = \delta_{xy}/w(y) \in \mathcal{L}^2$. Then $\mathcal{F} f_y = \Psi(y, \cdot) \in \mathcal{H}$.*

Proof We have

$$(\mathcal{F} f_y)(\gamma) = \begin{cases} \begin{pmatrix} \varphi_\gamma(y) \\ \varphi_\gamma^\dagger(y) \end{pmatrix}, & \gamma \in \mathbb{T}, \\ \Phi_\gamma^+(y), & \gamma \in \Gamma, \end{cases}$$

so $\mathcal{F} f_y = \Psi(y, \cdot)$. By Proposition 5.2 this lies in $\mathcal{H}$. □

We define an integral transform $\mathcal{G} : \mathcal{H} \to F(\mathbb{R}_q)$ by

$$(\mathcal{G} g)(x) = \langle g, \Psi(x, \cdot) \rangle_{\mathcal{H}}, \quad g \in \mathcal{H},\ x \in \mathbb{R}_q.$$

By Lemma 5.3 this inner product exists for all $g \in \mathcal{H}$. We denote by $\mathcal{G}_c$ and $\mathcal{G}_p$ the integral transform $\mathcal{G}$ restricted to $\mathcal{H}_c$ and $\mathcal{H}_p$, respectively.

Proposition 5.4 $\mathcal{G}\mathcal{F} = \mathrm{id}_{\mathcal{L}^2}$.

Proof Let $f \in \mathcal{L}^2$ and let $f_y \in \mathcal{L}^2$ be defined as in Lemma 5.3. Then it follows from Proposition 5.2 that

$$f(y) = \langle f, f_y \rangle_{\mathcal{L}^2} = \langle \mathcal{F} f, \mathcal{F} f_y \rangle_{\mathcal{H}} = \big(\mathcal{G}(\mathcal{F} f)\big)(y). \qquad □$$

We showed that $\mathcal{G}$ is a left inverse of $\mathcal{F}$. Next we are going show that $\mathcal{G}$ is also a right inverse. We do this for the transforms $\mathcal{G}_c$ and $\mathcal{G}_p$ separately. First a preliminary result: we denote by $\langle f, g \rangle_{k;n}$ the limit of the truncated inner product $\lim_{l,m\to\infty} \langle f, g \rangle_{k,l;m,n}$, provided that this limit exists.

Lemma 5.5 *Let $\gamma_1, \gamma_2 \in \mathbb{C}^*$ such that $\mu(\gamma_1) \neq \mu(\gamma_2)$. Then for $\phi \in V_{\mu(\gamma_1)}$ and $\psi \in V_{\mu(\gamma_2)}$,*

$$\langle \phi, \overline{\psi} \rangle_{k;n} = \frac{D(\phi, \psi)(z_+ q^{n-1}) - D(\phi, \psi)(z_- q^{k-1})}{\mu(\gamma_1) - \mu(\gamma_2)}.$$

Proof Functions in V_μ, $\mu \in \mathbb{C}$, are continuously q-differentiable at the origin; therefore

$$\lim_{l\to\infty} D(\phi, \psi)\big(z_\pm q^l\big) = 0.$$

Since ϕ and ψ are eigenfunctions of L for eigenvalue $\mu(\gamma_1)$ and $\mu(\gamma_2)$, respectively, we obtain from Proposition 2.3

$$\big(\mu(\gamma_1) - \mu(\gamma_2)\big) \langle \phi, \overline{\psi} \rangle_{k;n} = D(\phi, \psi)\big(z_+ q^{n-1}\big) - D(\phi, \psi)\big(z_- q^{k-1}\big). \qquad □$$

We are going to apply the previous lemma to the functions φ_γ, $\varphi_\gamma^\dagger$ and $\Phi_\gamma^\pm$, which are functions in $V_{\mu(\gamma)}$ by Propositions 3.7 and 3.10. The following lemma will be useful.

Lemma 5.6 *For* $k \to \infty$,

$$D\big(\Phi_{\gamma_1}^\pm, \Phi_{\gamma_2}^\pm\big)\big(z_\pm q^{-k}\big) = (\gamma_1 - \gamma_2) K_{z_\pm} (\gamma_1\gamma_2)^{k-1}\big(1 + \mathcal{O}\big(q^k\big)\big).$$

Proof This follows from the definition of the Casorati determinant (2.2) and from the asymptotic behavior of $\Phi_\gamma^\pm(x)$ and $u(x)/x$ for large $|x|$ (see (3.5) and Lemma 2.4). See also the proof of Lemma 3.3. □

As a consequence we obtain the following orthogonality relation.

Lemma 5.7 *Let* $\gamma \in \mathbb{T}$ *and* $\gamma' \in \Gamma$. *Then*

$$\big\langle \varphi_\gamma, \Phi_{\gamma'}^+\big\rangle_{\mathcal{L}^2} = 0, \qquad \big\langle \varphi_\gamma^\dagger, \Phi_{\gamma'}^+\big\rangle_{\mathcal{L}^2} = 0.$$

Proof From Lemmas 5.5, 5.6 and the c-function expansions from Proposition 3.4 we find

$$\big\langle \varphi_\gamma, \Phi_{\gamma'}^+\big\rangle_{\mathcal{L}^2} = \lim_{k,n\to-\infty} \big(c_{z_+}(\gamma)\big\langle \Phi_\gamma^+, \Phi_{\gamma'}^+\big\rangle_{k;n} + c_{z_+}(1/\gamma)\big\langle \Phi_{1/\gamma}^+, \Phi_{\gamma'}^+\big\rangle_{k;n}\big) = 0,$$

since $|\gamma'| < 1$. In the same way it follows that $\langle \varphi_\gamma^\dagger, \Phi_{\gamma'}^+\rangle_{\mathcal{L}^2} = 0$. □

We are now ready to show that $\mathcal{G}_p$ is a partial right inverse of the map $\mathcal{F}$.

Proposition 5.8 *The map* $\mathcal{G}_p : \mathcal{H}_p \to F(\mathbb{R}_q)$ *satisfies*

$$\langle \mathcal{G}_p g_1, \mathcal{G}_p g_2\rangle_{\mathcal{L}_2} = \langle g_1, g_2\rangle_{\mathcal{H}_p}, \quad g_1, g_2 \in \mathcal{H}_p.$$

Moreover, for $g \in \mathcal{H}_p$ *we have* $\mathcal{F}(\mathcal{G}_p g) = \mathbf{0} + g$ *in* $\mathcal{H}$, *where* $\mathbf{0}$ *denotes the zero function in* $\mathcal{H}_c$.

Proof Let g, h be finitely supported functions in $\mathcal{H}_p$. Then we find from Corollary 4.8,

$$\begin{aligned}\langle \mathcal{G}_p g, \mathcal{G}_p h\rangle_{\mathcal{L}^2} &= \int_{\mathbb{R}_q} \bigg(\sum_{\gamma\in\Gamma} g(\gamma)\Phi_\gamma^+(x)N(\gamma)\bigg)\overline{\bigg(\sum_{\gamma'\in\Gamma} h(\gamma')\Phi_{\gamma'}^+(x)N(\gamma')\bigg)} w(x)\, d_q x \\ &= \sum_{\gamma,\gamma'\in\Gamma} \big\langle \Phi_\gamma^+, \Phi_{\gamma'}^+\big\rangle_{\mathcal{L}^2} g(\gamma)\overline{h(\gamma')}N(\gamma)N(\gamma') \\ &= \sum_{\gamma\in\Gamma} g(\gamma)\overline{h(\gamma)}N(\gamma) = \langle g, h\rangle_{\mathcal{H}_p}.\end{aligned}$$

In order to prove the identity $\mathcal{F}(\mathcal{G}_p g) = \mathbf{0} + g$, we split this identity into three different cases:

$$\mathcal{F}_p(\mathcal{G}_p g) = g, \qquad \mathcal{F}_c(\mathcal{G}_p g) = 0 \quad \text{and} \quad \mathcal{F}_c^\dagger(\mathcal{G}_p g) = 0.$$

The identity $\mathcal{F}_p(\mathcal{G}_p g) = g$ is proved in a similar way as in the proof of Proposition 5.4, and the other two identities follow from Lemma 5.7. Since the set of finitely supported functions is dense in $\mathcal{H}_p$, the proposition follows. □

Next we are going to show that $\mathcal{G}_c$ is also a partial right inverse of $\mathcal{F}$. For this we apply a classical method used by Götze [6] and by Braaksma and Meulenbeld [3] for the Jacobi function transform.

We define for $\gamma \in \mathbb{T}$,

$$\begin{aligned} u_1(\gamma) &= u_1(\gamma; a, b, c, d; z_-, z_+|q) = K_{z_+} c_{z_+}(\gamma) c_{z_+}(1/\gamma) - K_{z_-} c_{z_-}(\gamma) c_{z_-}(1/\gamma), \\ u_2(\gamma) &= u_2(\gamma; a, b, c, d; z_-, z_+|q) = K_{z_+} c_{z_+}(\gamma) c_{z_+}^\dagger(1/\gamma) - K_{z_-} c_{z_-}(\gamma) c_{z_-}^\dagger(1/\gamma). \end{aligned} \tag{5.2}$$

Explicitly, using the expressions for c_z and K_z, we have

$$\begin{aligned} u_1(\gamma) = {} & \frac{(1-q)(s\gamma^{\pm 1}, cq\gamma^{\pm 1}/as, dq\gamma^{\pm 1}/as; q)_\infty}{(cq/a, cq/a, dq/a, dq/a, \gamma^{\pm 2}; q)_\infty \theta(bz_+, bz_-, cz_+, cz_-, dz_+, dz_-)} \\ & \times \big(z_+\theta\big(az_+, bz_-, cz_-, dz_-, bsz_+\gamma^{\pm 1}\big) \\ & - z_-\theta\big(az_-, bz_+, cz_+, dz_+, bsz_-\gamma^{\pm 1}\big)\big). \end{aligned}$$

For u_2 we have

$$\begin{aligned} u_2(\gamma) = {} & \frac{(1-q)q\gamma}{as} \frac{(s\gamma^{\pm 1}, cq/as\gamma, cq\gamma/bs, dq/as\gamma, dq/bs; q)_\infty}{(cq/a, cq/b, dq/a, dq/b, \gamma^{\pm 2}; q)_\infty} \\ & \times \left(\frac{\theta(bsz_-\gamma, asz_-/q\gamma)}{\theta(cz_-, dz_-)} - \frac{\theta(bsz_+\gamma, asz_+/q\gamma)}{\theta(cz_+, dz_+)} \right). \end{aligned}$$

Using the θ-product identity (1.1) with

$$x = z_- e^{i(\kappa+\delta)/2}\sqrt{|cd|}, \qquad y = z_+ e^{i(\kappa+\delta)/2}\sqrt{|cd|},$$
$$v = \frac{bs\gamma e^{-i(\kappa+\delta)/2}}{\sqrt{|cd|}}, \qquad w = e^{i(\kappa-\delta)/2}\sqrt{\left|\frac{c}{d}\right|},$$

where $c = |c|e^{i\kappa}$ and $d = |d|e^{i\delta}$, we obtain

$$\begin{aligned} u_2(\gamma) = {} & z_+(1-q) \\ & \times \frac{(s\gamma^{\pm 1}, cq\gamma^{\pm 1}/as, cq\gamma^{\pm 1}/bs, dq\gamma^{\pm 1}/as, dq\gamma^{\pm 1}/bs; q)_\infty \theta(z_-/z_+, cdz_-z_+)}{(cq/a, cq/b, dq/a, dq/b, \gamma^{\pm 2}; q)_\infty \theta(cz_-, cz_+, dz_+, dz_-)}. \end{aligned}$$

Observe that $u_2 = u_2^\dagger$ and $u_2(\gamma) = u_2(1/\gamma)$, and that u_2 is real-valued on $\mathbb{T} \setminus \{-1, 1\}$.

Let $C_0(\mathbb{T})$ be the set of functions defined by

$$C_0(\mathbb{T}) = \{g : \mathbb{T} \to \mathbb{C} \mid g \text{ is continuous}, \ g(-1) = g(1) = 0, \ g(\gamma) = g(1/\gamma)\}.$$

Proposition 5.9 *Let $g \in C_0(\mathbb{T})$ and let $\gamma' \in \mathbb{T} \setminus \{-1, 1\}$. Then*

$$\lim_{k,n \to -\infty} \frac{1}{4\pi i} \int_{\mathbb{T}} g(\gamma) \langle \varphi_\gamma, \overline{\varphi_{\gamma'}} \rangle_{k;n} \frac{d\gamma}{\gamma} = g(\gamma') u_1(\gamma'),$$
$$\lim_{k,n \to -\infty} \frac{1}{4\pi i} \int_{\mathbb{T}} g(\gamma) \big\langle \varphi_\gamma^\dagger, \overline{\varphi_{\gamma'}} \big\rangle_{k;n} \frac{d\gamma}{\gamma} = g(\gamma') u_2(\gamma').$$

Proof We prove the first identity in the proposition. The second identity is proved in the same way. Let us fix a $g \in C_0(\mathbb{T})$, and let us define

$$I_z^m(\theta') = \frac{1}{2\pi} \int_0^\pi g\big(e^{i\theta}\big) \frac{D(\varphi_{e^{i\theta}}, \varphi_{e^{i\theta'}})(zq^{-m})}{2\cos(\theta) - 2\cos(\theta')} \, d\theta.$$

From Lemma 5.5 we find

$$\frac{1}{4\pi i} \int_{\mathbb{T}} g(\gamma) \langle \varphi_\gamma, \overline{\varphi_{\gamma'}} \rangle_{k;n} \frac{d\gamma}{\gamma} = I_{z_+}^{1-n}(\theta') - I_{z_-}^{1-k}(\theta'),$$

where $\gamma' = e^{i\theta'}$ with $\theta' \in (0, \pi)$. We see that we need to investigate the limit of $I_z^m(\theta')$ when $m \to \infty$. Using the c-function expansion from Proposition 3.4 and Lemma 5.6 we obtain, for large m,

$$I_z^m(\theta') = \frac{K_z}{2\pi} \sum_{\xi, \eta \in \{-1, 1\}} \int_0^\pi g\big(e^{i\theta}\big) \big(\psi_z^m(\theta, \theta'; \xi, \eta) + \mathcal{O}\big(q^m\big)\big) \, d\theta,$$

where

$$\psi_z^m(\theta, \theta'; \xi, \eta) = \frac{(e^{i\xi\theta} - e^{i\eta\theta'}) e^{i(m-1)(\xi\theta + \eta\theta')} c_z(\xi\theta) c_z(\eta\theta')}{2\cos(\theta) - 2\cos(\theta')}.$$

Since $c_z(\gamma)$ is continuous on $\mathbb{T} \setminus \{-1, 1\}$, the functions ψ_z^m, considered as functions of θ, are continuous on $(0, \pi) \setminus \{\theta'\}$. We see immediately that $\psi_z^m(\theta, \theta'; 1, 1)$ and $\psi_z^m(\theta, \theta'; -1, -1)$ have a removable singularity at $\theta = \theta'$. Now by the Riemann–Lebesgue lemma the terms with these two functions vanish in the limit, and this leaves us with

$$\lim_{m \to \infty} I_z^m(\theta') = \lim_{m \to \infty} \frac{K_z}{2\pi} \int_0^\pi g\big(e^{i\theta}\big) \big(\psi_z^m(\theta, \theta'; 1, -1) + \psi_z^m(\theta, \theta'; -1, 1)\big) \, d\theta.$$

Here we applied dominated convergence to get rid of the $\mathcal{O}(q^m)$-terms. Using the identity $\cos(\alpha) - \cos(\beta) = 2\sin(\frac{\alpha+\beta}{2})\sin(\frac{\alpha-\beta}{2})$ we find

$$
\begin{aligned}
&\psi_z^m(\theta,\theta';1,-1)+\psi_z^m(\theta,\theta';-1,1)\\
&\quad=\frac{1}{4\sin\bigl(\frac{\theta+\theta'}{2}\bigr)\sin\bigl(\frac{\theta-\theta'}{2}\bigr)}\Bigl(c_z\bigl(e^{i\theta}\bigr)c_z\bigl(e^{-i\theta'}\bigr)e^{i(m-1)(\theta-\theta')}\bigl(e^{i\theta}-e^{-i\theta'}\bigr)\\
&\qquad+c_z\bigl(e^{-i\theta}\bigr)c_z\bigl(e^{i\theta'}\bigr)e^{i(m-1)(\theta'-\theta)}\bigl(e^{-i\theta}-e^{i\theta'}\bigr)\Bigr)\\
&\quad=\frac{1}{4\sin\bigl(\frac{\theta+\theta'}{2}\bigr)\sin\bigl(\frac{\theta-\theta'}{2}\bigr)}\Bigl(\bigl[c_z\bigl(e^{-i\theta}\bigr)c_z\bigl(e^{i\theta'}\bigr)\\
&\qquad-c_z\bigl(e^{i\theta}\bigr)c_z\bigl(e^{-i\theta'}\bigr)\bigr]e^{i(m-1)(\theta'-\theta)}\bigl(e^{-i\theta}-e^{i\theta'}\bigr)\\
&\qquad+c_z\bigl(e^{i\theta}\bigr)c_z\bigl(e^{-i\theta'}\bigr)\psi^m(\theta,\theta')\Bigr),
\end{aligned}
\tag{5.3}
$$

where

$$
\begin{aligned}
\psi^m(\theta,\theta')&=e^{i(m-1)(\theta'-\theta)}\bigl(e^{-i\theta}-e^{i\theta'}\bigr)+e^{i(m-1)(\theta-\theta')}\bigl(e^{i\theta}-e^{-i\theta'}\bigr)\\
&=2\cos\bigl(m\theta-(m-1)\theta'\bigr)-2\cos\bigl((m-1)\theta-m\theta'\bigr).
\end{aligned}
$$

The first term in (5.3) has a removable singularity, so by the Riemann–Lebesgue lemma this term also vanishes in the limit, and now we have

$$
\begin{aligned}
\lim_{m\to\infty}I_z^m(\theta')&=\lim_{m\to\infty}\frac{K_z}{2\pi}\int_0^\pi g\bigl(e^{i\theta}\bigr)c_z\bigl(e^{i\theta}\bigr)c_z\bigl(e^{-i\theta'}\bigr)\frac{\psi^m(\theta,\theta')}{4\sin\bigl(\frac{\theta+\theta'}{2}\bigr)\sin\bigl(\frac{\theta-\theta'}{2}\bigr)}\,d\theta\\
&=\lim_{m\to\infty}\frac{K_z}{2\pi}\int_0^\pi g\bigl(e^{i\theta}\bigr)c_z\bigl(e^{i\theta}\bigr)c_z\bigl(e^{-i\theta'}\bigr)D_m(\theta;\theta')\,d\theta,
\end{aligned}
$$

where $D_m(\theta;\theta')$ is the Dirichlet kernel

$$
D_m(\theta;\theta')=\frac{\sin\bigl((m-\frac12)(\theta-\theta')\bigr)}{\sin\bigl(\frac12(\theta-\theta')\bigr)}.
$$

From the well-known properties of the Dirichlet kernel we obtain

$$
\lim_{m\to\infty}I_z^m(\theta')=K_z\,g\bigl(e^{i\theta'}\bigr)c_z\bigl(e^{i\theta'}\bigr)c_z\bigl(e^{-i\theta'}\bigr),
$$

and from this the result follows. □

Proposition 5.10 *Let $g_1,g_2\in C_0(\mathbb{T})$ and let $\gamma'\in\mathbb{T}\setminus\{-1,1\}$. Then*

$$
\int_{\mathbb{R}_q}\left[\frac{1}{4\pi i}\int_{\mathbb{T}}\begin{pmatrix}\varphi_\gamma(x)\\ \varphi_\gamma^\dagger(x)\end{pmatrix}^T\begin{pmatrix}g_1(\gamma)\\ g_2(\gamma)\end{pmatrix}\frac{d\gamma}{\gamma}\right]\begin{pmatrix}\varphi_{\gamma'}(x)\\ \varphi_{\gamma'}^\dagger(x)\end{pmatrix}w(x)\,d_qx=\mathbf{u}(\gamma')\begin{pmatrix}g_1(\gamma')\\ g_2(\gamma')\end{pmatrix},
$$

where $\mathbf{u}$ is the matrix-valued function on $\mathbb{T}\setminus\{-1,1\}$ defined by

$$
\gamma\mapsto\mathbf{u}(\gamma)=\begin{pmatrix}u_2(\gamma) & u_1(\gamma)\\ u_1^\dagger(\gamma) & u_2(\gamma)\end{pmatrix}.
$$

Proof Let $g_1, g_2 \in C_0(\mathbb{T})$ and $\gamma, \gamma' \in \mathbb{T} \setminus \{-1, 1\}$. From Proposition 5.9 we find

$$g_2(\gamma')u_1(\gamma') = \lim_{k,n\to-\infty} \frac{1}{4\pi i} \int_{\mathbb{T}} g_2(\gamma)\left(\int_{z_-q^k}^{z_+q^n} \varphi_\gamma(x)\varphi_{\gamma'}(x)w(x)\,d_qx\right)\frac{d\gamma}{\gamma}$$
$$= \int_{\mathbb{R}_q} \varphi_{\gamma'}(x)\left(\frac{1}{4\pi i}\int_{\mathbb{T}} g_2(\gamma)\varphi_\gamma(x)\frac{d\gamma}{\gamma}\right)w(x)\,d_qx.$$

To justify the interchanging of the order of integration, we note that it follows from the explicit expressions for $\varphi_\gamma(x)$ and $w(x)$ that

$$\varphi_\gamma(x)\varphi_{\gamma'}(x)w(x) = 1 + \mathcal{O}\big(q^m\big), \quad x = z_\pm q^m \to 0,$$

so that the sums

$$\sum_{m=n}^{\infty} \varphi_\gamma\big(z_\pm q^m\big)\varphi_{\gamma'}\big(z_\pm q^m\big)q^m w\big(z_\pm q^m\big)$$

both converge uniformly on $\mathbb{T} \setminus \{-1, 1\}$. In the same way we find

$$g_2(\gamma')u_2(\gamma') = \int_{\mathbb{R}_q} \varphi^\dagger_{\gamma'}(x)\left(\frac{1}{4\pi i}\int_{\mathbb{T}} g_2(\gamma)\varphi_\gamma(x)\frac{d\gamma}{\gamma}\right)w(x)\,d_qx,$$
$$g_1(\gamma')u_1^\dagger(\gamma') = \int_{\mathbb{R}_q} \varphi^\dagger_{\gamma'}(x)\left(\frac{1}{4\pi i}\int_{\mathbb{T}} g_1(\gamma)\varphi^\dagger_\gamma(x)\frac{d\gamma}{\gamma}\right)w(x)\,d_qx,$$
$$g_1(\gamma')u_2(\gamma') = \int_{\mathbb{R}_q} \varphi_{\gamma'}(x)\left(\frac{1}{4\pi i}\int_{\mathbb{T}} g_1(\gamma)\varphi^\dagger_\gamma(x)\frac{d\gamma}{\gamma}\right)w(x)\,d_qx.$$

Now the proposition follows. □

The matrix-valued function $\mathbf{u}$ has the following useful property, which is proved in the Appendix.

Lemma 5.11 *For* $\gamma \in \mathbb{T} \setminus \{-1, 1\}$,

$$\mathbf{u}(\gamma)^{-1} = \mathbf{v}(\gamma).$$

We define

$$C_0\big(\mathbb{T}; \mathbb{C}^2\big) = \left\{ g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \;\middle|\; g_1, g_2 \in C_0(\mathbb{T}) \right\} \subset \mathcal{H}_c.$$

We are now in a position to show that $\mathcal{G}_c$ is a partial right inverse of $\mathcal{F}$.

Proposition 5.12 *The map* $\mathcal{G}_c : \mathcal{H}_c \to F(\mathbb{R}_q)$ *satisfies*

$$\langle \mathcal{G}_c g_1, \mathcal{G}_c g_2\rangle_{\mathcal{L}^2} = \langle g_1, g_2\rangle_{\mathcal{H}_c}, \quad g_1, g_2 \in \mathcal{H}_c.$$

Moreover, for $g \in \mathcal{H}_c$ *we have* $\mathcal{F}(\mathcal{G}_c g) = g + \mathbf{0}$ *in* $\mathcal{H}$, *where* $\mathbf{0}$ *denotes the zero function in* $\mathcal{H}_p$.

Proof Let $\gamma' \in \mathbb{T} \setminus \{-1, 1\}$, let $g^{(1)}, g^{(2)} \in C_0(\mathbb{T})$ and define $g = \binom{g^{(1)}}{g^{(2)}}$. Since v_1, $v_1^\dagger$ and v_2 are continuous on $\mathbb{T}$, both components of the $\mathbb{C}^2$-valued function

$$\gamma \mapsto \mathbf{v}(\gamma)\begin{pmatrix} g^{(1)}(\gamma) \\ g^{(2)}(\gamma) \end{pmatrix}$$

are in $C_0(\mathbb{T})$. Now by Proposition 5.10 and Lemma 5.11 we have

$$\begin{pmatrix} (\mathcal{F}_c(\mathcal{G}_c g))(\gamma') \\ (\mathcal{F}_c^\dagger(\mathcal{G}_c g))(\gamma') \end{pmatrix} = \mathbf{u}(\gamma')\mathbf{v}(\gamma')g(\gamma') = g(\gamma').$$

Moreover, for $\gamma' \in \Gamma$,

$$\begin{aligned} \big(\mathcal{F}_p(\mathcal{G}_c g)\big)(\gamma') &= \lim_{k,n\to-\infty} \langle \mathcal{G}_c g, \Phi^+_{\gamma'} \rangle_{k;n} \\ &= \lim_{k,n\to-\infty} \frac{1}{4\pi i} \int_{\mathbb{T}} \big(\langle \varphi^\dagger_\gamma, \Phi^+_{\gamma'} \rangle_{k;n} \, \langle \varphi_\gamma, \Phi^+_{\gamma'} \rangle_{k;n} \big) \mathbf{v}(\gamma) g(\gamma) \frac{d\gamma}{\gamma} \\ &= 0, \end{aligned}$$

by dominated convergence and Lemma 5.7. This shows that $\mathcal{F}(\mathcal{G}_c g) = g + \mathbf{0}$ in $\mathcal{H}$.

Let $g_1, g_2 \in C_0(\mathbb{T}; \mathbb{C}^2)$. Then it follows from Proposition 5.2 that

$$\langle g_1, g_2 \rangle_{\mathcal{H}_c} = \langle g_1 + \mathbf{0}, g_2 + \mathbf{0} \rangle_{\mathcal{H}} = \big\langle \mathcal{F}(\mathcal{G}_c g_1), \mathcal{F}(\mathcal{G}_c g_2) \big\rangle_{\mathcal{H}} = \langle \mathcal{G}_c g_1, \mathcal{G}_c g_2 \rangle_{\mathcal{L}^2}.$$

Since the set $C_0(\mathbb{T}; \mathbb{C}^2)$ is dense in $\mathcal{H}_c$, the proposition follows. □

Collecting the results of this subsection we come to the main theorem.

Theorem 5.13 *For $(a, b, c, d) \in P$, the map $\mathcal{F} : \mathcal{L}^2 \to \mathcal{H}$ is an isometric isomorphism with inverse $\mathcal{G}$.*

Proof Let $(a, b, c, d) \in P_{\text{gen}}$. Combining Propositions 5.8 and 5.12 gives $\mathcal{F}\mathcal{G} = \text{id}_{\mathcal{H}}$. Together with Proposition 5.4 this leads to the theorem. By continuity in the parameters, the result holds for all $(a, b, c, d) \in P$. □

Corollary 5.14 *The set $\{\Psi(x, \cdot) \mid x \in \mathbb{R}_q\}$ forms an orthogonal basis for $\mathcal{H}$ with squared norm $\|\Psi(x, \cdot)\|^2_{\mathcal{H}} = w(x)^{-1}$.*

Proof This follows from Lemma 5.3 and Theorem 5.13, since the functions f_y defined in Lemma 5.3 form an orthogonal basis for $\mathcal{L}^2$ with squared norm $w(y)^{-1}$. □

Remark 5.15 The Hilbert space $\mathcal{H}$ and the inverse $\mathcal{G}$ of the vector-valued big q-Jacobi function transform depend essentially on five parameters, namely az_-, bz_-, cz_-, dz_- and z_+/z_-.

5.2 An Equivalent Integral Transform

For $f \in \mathcal{D}_{\text{fin}}$ and $\gamma \in \Gamma$ we have $(\mathcal{F} f)(\gamma) = \langle f, \Phi_\gamma^+ \rangle_{\mathcal{L}^2}$. Since the function Φ_γ^+ can be expressed in terms of big q-Jacobi functions by Proposition 3.10, we can define an integral transform with only the big q-Jacobi functions φ_γ and $\varphi_\gamma^\dagger$ as a kernel, which is equivalent to $\mathcal{F}$. This new integral transform can of course also be extended to an isometric isomorphism. We only state the result here, and we leave the details to the reader.

For $f \in \mathcal{D}_{\text{fin}}$ we define an integral transform $\mathcal{J}$ that is closely related to the vector-valued big q-Jacobi function transform $\mathcal{F}$ by

$$(\mathcal{J} f)(\gamma) = \int_{\mathbb{R}_q} f(x) \begin{pmatrix} \varphi_\gamma(x) \\ \varphi_\gamma^\dagger(x) \end{pmatrix} w(x)\, d_q x, \quad \gamma \in \mathbb{T} \cup \Gamma.$$

For $(a, b, c, d) \in P$ we define an inner product on the vector space of $\mathbb{C}^2$-valued functions by

$$\langle f, g \rangle_{\mathcal{M}} = \frac{1}{4\pi i} \int_{\mathbb{T}} g(\gamma)^T \mathbf{v}(\gamma) f(\gamma) \frac{d\gamma}{\gamma} + \sum_{\gamma \in \Gamma} g(\gamma)^T \mathbf{v}_p(\gamma) f(\gamma).$$

Here $\mathbf{v}_p(\gamma)$ is the matrix-valued function on Γ given by

$$\gamma \mapsto \mathbf{v}_p(\gamma) = \begin{pmatrix} v_{3,p}(\gamma) & v_{4,p}(\gamma) \\ v_{1,p}(\gamma) & v_{2,p}(\gamma) \end{pmatrix},$$

where the matrix coefficients $v_{i,p}(\gamma) = v_{i,p}(\gamma; a, b, c, d; z_-, z_+|q)$, $i = 1, \ldots, 4$, are defined as follows:

For $\gamma \in \Gamma^{\text{inf}} \cup \Gamma_{q/s}^{\text{fin}}$, $v_{4,p}(\gamma) = v_{1,p}^\dagger(\gamma)$, $v_{3,p}(\gamma) = v_{2,p}(\gamma)$, and

$$v_{1,p}(\gamma) = d_{z_+}^2(\gamma) N(\gamma),$$
$$v_{2,p}(\gamma) = d_{z_+}(\gamma) d_{z_+}^\dagger(\gamma) N(\gamma).$$

For $\gamma \in \Gamma_{dq/as}^{\text{fin}} \cup \Gamma_s^{\text{fin}}$, $v_{2,p}(\gamma) = v_{4,p}(\gamma) = 0$, and

$$v_{1,p}(\gamma) = \begin{cases} 0, & \text{if } a = \overline{b}, \\ \dfrac{N(\gamma)}{(c_{z_+}(\gamma))^2}, & \text{if } a \neq \overline{b}, \end{cases}$$

$$v_{2,p}(\gamma) = \begin{cases} \dfrac{N(\gamma)}{c_{z_+}(\gamma) c_{z_+}^\dagger(\gamma)}, & \text{if } a = \overline{b}, \\ 0, & \text{if } a \neq \overline{b}. \end{cases}$$

Recall here that $\Gamma_{dq/as}^{\text{fin}}$ is only non-empty if $a \neq \overline{b}$. Now denote by $\mathcal{M} = \mathcal{M}(a, b, c, d; z_-, z_+|q)$ the closure of the set

$$\operatorname{span}\left\{ \gamma \to \begin{pmatrix} \varphi_\gamma(x) \\ \varphi_\gamma^\dagger(x) \end{pmatrix} \,\middle|\, x \in \mathbb{R}_q \right\}$$

with respect to the norm $\|\cdot\|_{\mathcal{M}}$. Note that a function $g \in \mathcal{M}$ satisfies $rg = g$.

Let $\Theta : \mathcal{M} \to \mathcal{H}$ be the operator defined by

$$(\Theta g)(\gamma) = \begin{cases} g(\gamma), & \gamma \in \mathbb{T}, \\ \big(d_{z_+}(\gamma)\, d^{\dagger}_{z_+}(\gamma)\big) g(\gamma), & \gamma \in \Gamma^{\mathrm{inf}} \cup \Gamma^{\mathrm{fin}}_{q/s}, \\ \big(c_{z_+}(\gamma)^{-1}\ 0\big) g(\gamma), & \gamma \in \Gamma^{\mathrm{fin}}_{s} \cup \Gamma^{\mathrm{fin}}_{dq/as}, \end{cases}$$

then

$$\langle \Theta g_1, \Theta g_2 \rangle_{\mathcal{H}} = \langle g_1, g_2 \rangle_{\mathcal{M}},$$

for functions $g_i \in \mathcal{M}$, $i = 1, 2$. In particular, we have

$$\left(\Theta \begin{pmatrix} \varphi_{\cdot}(x) \\ \varphi^{\dagger}_{\cdot}(x) \end{pmatrix}\right)(\gamma) = \Psi(x, \gamma), \quad x \in \mathbb{R}_q,\ \gamma \in \mathbb{T} \cup \Gamma,$$

so $\Theta : \mathcal{M} \to \mathcal{H}$ is an isomorphism. Also, $\mathcal{F} f = (\Theta \circ \mathcal{J}) f$ for $f \in \mathcal{D}_{\mathrm{fin}}$.

Theorem 5.16 *The map $\mathcal{J} : \mathcal{D}_{\mathrm{fin}} \to \mathcal{M}$ extends uniquely to an isometric isomorphism $\mathcal{J}_{\mathrm{ext}} : \mathcal{L}^2 \to \mathcal{M}$. Moreover, $\mathcal{I} = \mathcal{G} \circ \Theta : \mathcal{M} \to \mathcal{L}^2$ is the inverse of $\mathcal{J}_{\mathrm{ext}}$.*

Remark 5.17 (i) Let $f \in \mathcal{L}^2$ be a function for which $\mathcal{F} f$ can be written as an integral transform, i.e.,

$$(\mathcal{F} f)(\gamma) = \int_{\mathbb{R}_q} f(x) \Psi(x, \gamma) w(x)\, d_q x, \quad \gamma \in \mathbb{T} \cup \Gamma.$$

Then $\mathcal{J}_{\mathrm{ext}} f$ can in general *not* be written as the integral

$$\int_{\mathbb{R}_q} f(x) \begin{pmatrix} \varphi_{\gamma}(x) \\ \varphi^{\dagger}_{\gamma}(x) \end{pmatrix} w(x)\, d_q x,$$

when $f \notin \mathcal{D}_{\mathrm{fin}}$, since the integrals in the components of this vector might be divergent for $\gamma \in \Gamma$.

(ii) The inverse $\mathcal{I}$ of $\mathcal{J}_{\mathrm{ext}}$ can be given explicitly by

$$(\mathcal{I} g)(x) = \left\langle g, \begin{pmatrix} \varphi_{\cdot}(x) \\ \varphi^{\dagger}_{\cdot}(x) \end{pmatrix} \right\rangle_{\mathcal{M}}, \quad x \in \mathbb{R}_q,$$

for all $g \in \mathcal{M}$ for which the above inner product exists.

(iii) Like $\mathcal{F}$, the map $\mathcal{J}_{\mathrm{ext}}$ diagonalizes L;

$$\big(\mathcal{J}_{\mathrm{ext}} \circ L \circ \mathcal{J}^{-1}_{\mathrm{ext}}\big) g = M g,$$

for functions $g \in \mathcal{M}$ such that $Mg \in \mathcal{M}$.

Acknowledgements This research is done at the Korteweg-De Vries Institute for Mathematics at the University of Amsterdam. The author is supported by the Netherlands Organization for Scientific Research (NWO) for the Vidi-project "Symmetry and modularity in exactly solvable models."

Appendix

In this appendix we prove Lemmas 4.2 and 5.11.

A.1 Proof of Lemma 4.2

We prove the following statement:

For $x, y \in \mathbb{R}_q$ and $\gamma, \gamma^{-1} \in \mathcal{S}_{\mathrm{reg}} \setminus \mathcal{V}$ we have

$$\begin{aligned}&\frac{\Phi^-_{1/\gamma}(x)\Phi^+_{1/\gamma}(y)}{v(1/\gamma)} - \frac{\Phi^-_\gamma(x)\Phi^+_\gamma(y)}{v(\gamma)}\\&\quad= \frac{1}{\gamma - 1/\gamma}\big[v_1(\gamma)\varphi_\gamma(x)\varphi_\gamma(y) + v_2(\gamma)\big(\varphi_\gamma(x)\varphi^\dagger_\gamma(y) + \varphi^\dagger_\gamma(x)\varphi_\gamma(y)\big)\\&\qquad + v_1^\dagger(\gamma)\varphi^\dagger_\gamma(x)\varphi^\dagger_\gamma(y)\big],\end{aligned}$$

where

$$\begin{aligned}v_1(\gamma) &= \frac{(cq/a, dq/a; q)^2_\infty \theta(bz_+, bz_-)}{(1-q)abz_-^2 z_+^2 \theta(z_-/z_+, z_+/z_-, a/b, b/a)}\\&\quad\times \frac{(\gamma^{\pm 2}; q)_\infty}{(s\gamma^{\pm 1}, cq\gamma^{\pm 1}/as, dq\gamma^{\pm 1}/as; q)_\infty \theta(s\gamma^{\pm 1}, absz_-z_+\gamma^{\pm 1})}\\&\quad\times \big(z_-\theta\big(az_+, cz_+, dz_+, bz_-, asz_-\gamma^{\pm 1}\big)\\&\quad - z_+\theta\big(az_-, cz_-, dz_-, bz_+, asz_+\gamma^{\pm 1}\big)\big),\\v_2(\gamma) &= \frac{(cq/a, dq/a, cq/b, dq/b; q)_\infty \theta(az_+, az_-, bz_+, bz_-, cdz_-z_+)}{abz_-^2 z_+(1-q)\theta(z_+/z_-, a/b, b/a)}\\&\quad\times \frac{(\gamma^{\pm 2}; q)_\infty}{(s\gamma^{\pm 1}; q)_\infty \theta(s\gamma^{\pm 1}, absz_-z_+\gamma^{\pm 1})}.\end{aligned}$$

Proof Let $\gamma, \gamma^{-1} \in \mathcal{S}_{\mathrm{reg}} \setminus \mathcal{V}$. Note that $\mathcal{S}_{\mathrm{pol}} \subset \mathcal{V}$; hence $\gamma, \gamma^{-1} \notin \mathcal{S}_{\mathrm{pol}}$. We define

$$I_\gamma(x, y) = \frac{\Phi^-_{1/\gamma}(x)\Phi^+_{1/\gamma}(y)}{v(1/\gamma)} - \frac{\Phi^-_\gamma(x)\Phi^+_\gamma(y)}{v(\gamma)}.$$

Using $\varphi_\gamma = \varphi_{1/\gamma}$ and Proposition 3.10 we see that

$$I_\gamma(x,y) = v_1'(\gamma)\varphi_\gamma(x)\varphi_\gamma(y) + v_2'(\gamma)\varphi_\gamma(x)\varphi_\gamma^\dagger(y) + v_3'(\gamma)\varphi_\gamma^\dagger(x)\varphi_\gamma(y) + v_4'(\gamma)\varphi_\gamma^\dagger(x)\varphi_\gamma^\dagger(y),$$

where

$$v_1'(\gamma) = \frac{d_{z_-}(1/\gamma)d_{z_+}(1/\gamma)}{v(1/\gamma)} - \frac{d_{z_-}(\gamma)d_{z_+}(\gamma)}{v(\gamma)},$$

$$v_2'(\gamma) = \frac{d_{z_-}(1/\gamma)d_{z_+}^\dagger(1/\gamma)}{v(1/\gamma)} - \frac{d_{z_-}(\gamma)d_{z_+}^\dagger(\gamma)}{v(\gamma)},$$

$$v_3'(\gamma) = \frac{d_{z_-}^\dagger(1/\gamma)d_{z_+}(1/\gamma)}{v(1/\gamma)} - \frac{d_{z_-}^\dagger(\gamma)d_{z_+}(\gamma)}{v(\gamma)},$$

$$v_4'(\gamma) = \frac{d_{z_-}^\dagger(1/\gamma)d_{z_+}^\dagger(1/\gamma)}{v(1/\gamma)} - \frac{d_{z_-}^\dagger(\gamma)d_{z_+}^\dagger(\gamma)}{v(\gamma)}.$$

Since $v(\gamma) = v^\dagger(\gamma)$, it is immediately clear that $v_1'^\dagger(\gamma) = v_4'(\gamma)$ and $v_2'^\dagger(\gamma) = v_3'(\gamma)$.

Using the explicit expressions for $d_z(\gamma)$ and $v(\gamma)$ (see Proposition 3.10 and Theorem 3.13), we find

$$v_2'(\gamma) = \frac{bs(cq/a, dq/a, cq/b, dq/b; q)_\infty \theta(az_+, bz_-)}{q(1-q)(s\gamma, s/\gamma; q)_\infty \theta(z_-/z_+, a/b, b/a)} \times \left(\frac{\theta(q^2/asz_-\gamma, q/bsz_+\gamma)}{\theta(s\gamma, q^2/absz_-z_+\gamma)} - \frac{\theta(q^2\gamma/asz_-, q\gamma/bsz_+)}{\theta(s/\gamma, q^2\gamma/absz_-z_+)} \right).$$

From this we find the expression for $v_2(\gamma) = (\gamma - 1/\gamma)v_2'(\gamma)$ given in the lemma after using the θ-product identity (1.1) with

$$x = \frac{iqe^{-i\alpha/2}}{s}\sqrt{\left|\frac{q}{az_-}\right|}, \qquad y = \frac{iqe^{-i\alpha/2}}{bsz_+}\sqrt{\left|\frac{q}{az_-}\right|},$$

$$v = i\gamma e^{-i\alpha/2}\sqrt{\left|\frac{q}{az_-}\right|}, \qquad w = \frac{ie^{-i\alpha/2}}{\gamma}\sqrt{\left|\frac{q}{az_-}\right|},$$

where $a = |a|e^{i\alpha}$.

Next we compute $v_1(\gamma) = (\gamma - 1/\gamma)v_1'(\gamma)$;

$$v_1'(\gamma) = \frac{(cq/a, dq/a; q)_\infty^2 \theta(bz_+, bz_-)}{\gamma z_+(1-q)(s\gamma, s/\gamma; q)_\infty \theta(z_-/z_+)\theta(a/b)^2} \times \left(\frac{\gamma^2 (cq\gamma/bs, dq\gamma/bs; q)_\infty \theta(q^2\gamma/asz_+, q^2\gamma/asz_-)}{(cq\gamma/as, dq\gamma/as; q)_\infty \theta(s/\gamma, qs\gamma/cdz_-z_+)} - \frac{(qc/bs\gamma, dq/bs\gamma; q)_\infty \theta(q^2/\gamma asz_+, q^2/\gamma asz_-)}{(cq/as\gamma, dq/ad\gamma; q)_\infty \theta(s\gamma, qs/cdz_-z_+\gamma)} \right).$$

Since $cq/as = bs/c$, the expression between large brackets can be written as

$$\frac{-\gamma}{(cq\gamma^{\pm 1}/as, dq\gamma^{\pm 1}/as; q)_\infty \theta(s\gamma^{\pm 1}, cdz_-z_+\gamma^{\pm 1}/s)} \times \big(\gamma^{-1}\theta(\gamma asz_-/q, \gamma asz_+/q, dq/sb\gamma, cq/sb\gamma, s/\gamma, cdz_-z_+/s\gamma) - \gamma\theta(asz_-/q\gamma, asz_+/q\gamma, dq\gamma/bs, cq\gamma/bs, s\gamma, cdz_-z_+\gamma/s)\big). \tag{A.1}$$

We use the θ-product identity [5, Exer. 5.22]

$$\frac{1}{y}\theta(tx/p, ux/p, vx/p, wx/p, y/p, y/r, r/p) - \frac{1}{x}\theta(ty/p, uy/p, vy/p, wy/p, x/p, x/r, r/p) = \frac{1}{y}\theta(tr/p, ur/p, vr/p, wr/p, x/p, y/p, y/x) - \frac{x}{pr}\theta(t, u, v, w, r/x, r/y, y/x),$$

with parameters

$$p = \frac{q}{asz_+}, \qquad r = \frac{q}{asz_-}, \qquad t = \frac{q}{az_+}, \qquad u = \frac{q}{cz_+},$$
$$v = \frac{q}{dz_+}, \qquad w = bz_-, \qquad x = \frac{1}{\gamma}, \qquad y = \gamma.$$

Then (A.1) becomes

$$\frac{-\theta(\gamma^2)}{(cq\gamma^{\pm 1}/as, dq\gamma^{\pm 1}/as; q)_\infty \theta(s\gamma^{\pm 1}, cdz_-z_+\gamma^{\pm 1}/s, z_+/z_-)} \times \left(\theta\big(q/az_-, q/cz_-, q/dz_-, bz_+, asz_+\gamma^{\pm 1}/q\big) - \frac{a^2s^2z_-z_+}{q^2}\theta\big(q/az_+, q/cz_+, q/dz_+, bz_-, q\gamma^{\pm 1}/asz_-\big)\right).$$

The expression given in the lemma is obtained from this after using the identity $-x\theta(qx)=\theta(x)$ several times. □

A.2 Proof of Lemma 5.11

We show that

$$\mathbf{u}(\gamma)^{-1}=\mathbf{v}(\gamma),\quad \gamma\in\mathbb{T}\setminus\{-1,1\},$$

with

$$\mathbf{u}(\gamma)=\begin{pmatrix}u_2(\gamma) & u_1(\gamma)\\ u_1^\dagger(\gamma) & u_2(\gamma)\end{pmatrix},\qquad \mathbf{v}(\gamma)=\begin{pmatrix}v_2(\gamma) & v_1^\dagger(\gamma)\\ v_1(\gamma) & v_2(\gamma)\end{pmatrix}.$$

Proof By a direct verification, using the explicit expressions for v_1 and v_2 from Lemma 4.2 (see also Appendix A.1), and for u_1 and u_2, one sees that

$$v_2(\gamma)=\frac{u_2(\gamma)}{\delta(\gamma)},\qquad v_1^\dagger(\gamma)=-\frac{u_1(\gamma)}{\delta(\gamma)},$$

where $\delta(\gamma)$ is the function given by

$$\delta(\gamma)=\frac{(1-q)^2z_-^2z_+^2ab\,\theta(z_-/z_+,z_+/z_-,a/b,b/a)}{(cq/a,cq/b,dq/a,dq/b;q)_\infty^2\theta(az_-,az_+,bz_-,bz_+,cz_-,cz_+,dz_-,dz_+,)}\\ \times\frac{(s\gamma^{\pm1},s\gamma^{\pm1},cq\gamma^{\pm1}/as,cq\gamma^{\pm1}/bs,dq\gamma^{\pm1}/as,cq\gamma^{\pm1}/bs)_\infty\theta(s\gamma^{\pm1},absz_-z_+\gamma^{\pm1})}{(\gamma^{\pm2};q)_\infty^2}.$$

It remains to show that $\delta(\gamma)$ is the determinant of the matrix $\mathbf{u}(\gamma)$.

Using the definition (5.2) of the functions u_1 and u_2, and $u_2=u_2^\dagger$, the determinant of $\mathbf{u}(\gamma)$ becomes

$$\begin{aligned}\det\bigl(\mathbf{u}(\gamma)\bigr)\\ &=u_2^\dagger(\gamma)u_2(\gamma)-u_1^\dagger(\gamma)u_1(\gamma)\\ &=K_{z_-}K_{z_+}\bigl(c_{z_+}(\gamma)c_{z_+}(1/\gamma)c_{z_-}^\dagger(\gamma)c_{z_-}^\dagger(1/\gamma)-c_{z_+}(\gamma)c_{z_+}^\dagger(1/\gamma)c_{z_-}^\dagger(\gamma)c_{z_-}(1/\gamma)\\ &\quad+c_{z_+}^\dagger(\gamma)c_{z_+}^\dagger(1/\gamma)c_{z_-}(\gamma)c_{z_-}(1/\gamma)-c_{z_+}^\dagger(\gamma)c_{z_+}(1/\gamma)c_{z_-}(\gamma)c_{z_-}^\dagger(1/\gamma)\bigr)\\ &=K_{z_-}K_{z_+}\bigl(c_{z_+}(\gamma)c_{z_-}^\dagger(\gamma)-c_{z_+}^\dagger(\gamma)c_{z_-}(\gamma)\bigr)\\ &\quad\times\bigl(c_{z_+}(1/\gamma)c_{z_-}^\dagger(1/\gamma)-c_{z_+}^\dagger(1/\gamma)c_{z_-}(1/\gamma)\bigr).\end{aligned}$$

Explicitly, we have

$$\begin{aligned}&c_{z_+}(\gamma)c_{z_-}^\dagger(\gamma)-c_{z_+}^\dagger(\gamma)c_{z_-}(\gamma)\\ &\quad=\frac{(s/\gamma,s/\gamma,cq/as\gamma,cq/bs\gamma,dq/as\gamma,dq/bs\gamma;q)_\infty F(\gamma)}{(cq/a,cq/b,dq/a,dq/b,1/\gamma^2,1/\gamma^2;q)_\infty^2\theta(az_-,az_+,bz_-,bz_+)},\end{aligned}$$

where

$$F(\gamma) = \theta(asz_-\gamma, bsz_+\gamma, bz_-, az_+) - \theta(asz_+\gamma, bsz_-\gamma, bz_+, az_-)$$
$$= bz_+\theta(z_-/z_+, a/b, s\gamma, absz_-z_+\gamma).$$

See the proof of Theorem 3.13 for the last equality. By inspection it follows that indeed $\delta(\gamma) = \det(\mathbf{u}(\gamma))$. □

References

1. Andrews, G.E., Askey, R.: Classical orthogonal polynomials. In: Polynômes orthogonaux et applications. Lecture Notes in Math., vol. 1171, pp. 36–62. Springer, Berlin (1985)
2. Askey, R., Wilson, J.: Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials. Mem. Am. Math. Soc. **54**(319) (1985)
3. Braaksma, B.L.J., Meulenbeld, B.: Integral transforms with generalized Legendre functions as kernels. Compos. Math. **18**, 235–287 (1967)
4. Dunford, N., Schwartz, J.T.: Linear Operators. Part II. Interscience, New York (1963)
5. Gasper, G., Rahman, M.: Basic Hypergeometric Series, 2nd edn. Cambridge University Press, Cambridge (2004)
6. Götze, F.: Verallgemeinerung einer Integraltransformation von Mehler-Fock durch den von Kuipers und Meulenbeld eingefürten Kern $P_k^{m,n}(z)$. Indag. Math. **27**, 396–404 (1965)
7. Groenevelt, W., Koelink, E., Rosengren, H.: Continuous Hahn functions as Clebsch–Gordan coefficients. In: Ismail, M.E.H., Koelink, E. (eds.) Theory and Applications of Special Functions. Dev. Math., vol. 13, pp. 221–284. Springer, New York (2005)
8. Gupta, D.P., Ismail, M.E.H., Masson, D.R.: Contiguous relations, basic hypergeometric functions, and orthogonal polynomials. III. Associated continuous dual q-Hahn polynomials. J. Comput. Appl. Math. **68**, 115–149 (1996)
9. Kakehi, T.: Eigenfunction expansion associated with the Casimir operator on the quantum group $\mathrm{SU}_q(1,1)$. Duke Math. J. **80**, 535–573 (1995)
10. Kakehi, T., Masuda, T., Ueno, K.: Spectral analysis of a q-difference operator which arises from the quantum SU(1, 1) group. J. Oper. Theory **33**, 159–196 (1995)
11. Koekoek, R., Swarttouw, R.F.: The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue. Report 98-17, Technical University Delft, Delft (1998)
12. Koelink, E., Stokman, J.V.: The Askey–Wilson function transform. Int. Math. Res. Not., 1203–1227 (2001)
13. Koelink, E., Stokman, J.V.: The big q-Jacobi function transform. Constr. Approx. **19**, 191–235 (2003)
14. Koornwinder, T.H.: Jacobi functions and analysis on noncompact semisimple Lie groups. In: Askey, R.A., Koornwinder, T.H., Schempp, W. (eds.) Special Functions: Group Theoretical Aspects and Applications, pp. 1–85. Reidel, Dordrecht (1984)
15. Martin, R.P.: On the decomposition of tensor products of principal series representations for real-rank one semisimple groups. Trans. Am. Math. Soc. **201**, 177–211 (1975)
16. Neretin, Yu.A.: Some continuous analogs of the expansion in Jacobi polynomials and vector-valued orthogonal bases. Funct. Anal. Appl. **39**(2), 106–119 (2005)
17. Pukánszky, L.: On the Kronecker products of irreducible representations of the 2 × 2 real unimodular group. I. Trans. Am. Math. Soc. **100**, 116–152 (1961)
18. Weyl, H.: Über gewöhnliche lineare Differentialgleichungen mit singulare Stellen und ihre Eigenfunktionen (2. Note). Nachr. König. Gesell. Wiss. Göttingen. Math.-Phys. Kl., 442–467 (1910)