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THE UMBRAL METHOD: A SURVEY OF ELEMENTARY MNEMONIC AND MANIPULATIVE USES

ANDREW P. GUINAND

1. Introduction. The basic idea of the umbral method consists in the use of a notation where certain exponents can be interchanged with suffixes. Its primary use is as an aid in dealing with sequences whose properties somehow resemble the properties of integral powers of an algebraic symbol.

The case most often encountered in the literature is that of the Bernoulli numbers, $\{B_n\}$, when they are expressed in the even suffix notation. That is, for $|x|<2\pi$,

$$\frac{x}{e^x-1}=\sum_{n=0}^{\infty} B_n\frac{x^n}{n!}=1-\tfrac{1}{2}x+\frac{1}{6}\frac{x^2}{2!}-\frac{1}{30}\frac{x^4}{4!}+\frac{1}{42}\frac{x^6}{6!}-\cdots.$$

These numbers satisfy the recurrence formula

$$\sum_{r=0}^{n-1}\binom{n}{r}B_r=\begin{cases}1, & (n=1),\\ 0, & (n=0,2,3,4,\dots).\end{cases} \tag{1}$$

This formula can be written symbolically as

$$(B+1)^n-B^n\equiv\begin{cases}1, & (n=1),\\ 0, & (n=0,2,3,4,\dots),\end{cases} \tag{2}$$

with the understanding that the expression on the left is to be expanded in powers of B, and then each term B^m is to be replaced by B_m. The symbol B is referred to as an "umbra," and the symbol $\equiv$ is used to denote symbolic or umbral equivalences, in which we have put $B^m\equiv B_m$ (**[8]**, **[9]**, **[13]**).

In (2) this umbral method is only being used as a mnemonic for the recurrence (1), but the method can also be a great aid in simplifying manipulations.

The notation is sometimes called the Blissard notation **[15]**, but other writers have attributed it to Lucas **[11]** or to Sylvester **[16]**. The method has been used informally by many writers, sometimes with virtually no explanation **[14]**. At the other extreme, rigorous presentations as an algebraic system have been given by Bell [3] and Temple **[18]**, and it has been expressed in terms of linear operators by Rota and Mullin **[16]**.

The aim of the present survey is to show that there is an intermediate level, almost completely neglected in the literature, at which the method can be used to great advantage. Instead of regarding umbral symbols as new algebraic or operational entities, we treat the method as a matter of notation, subject to rules of interpretation and manipulation, and give examples to show how the method simplifies both proofs and expressions of results.

2. The rules of the umbral method.

I. RULE OF INTERPRETATION. *Expressions involving one or several umbrae are to be interpreted by expanding as power series in the umbrae and replacing exponents by suffixes.*

II. RULE OF MANIPULATION. *Additions or linear combinations of equations involving umbrae are permissible, but multiplication is only valid when the factors have no umbra in common. In general,*

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Rule I implies that we only give meanings to functions of umbrae which are analytic at the origin. The expressions B^{-1} and $\log B$ are meaningless, whereas e^{Bx} has the interpretation

$$e^{Bx} \equiv \sum_{n=0}^{\infty} \frac{(Bx)^n}{n!} \equiv \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = \frac{x}{e^x - 1}, \qquad (0<|x|<2\pi). \tag{3}$$

In Rule II the ban on some multiplications is needed to avoid such fallacious arguments as: $B^2 \times B^4 \equiv B^6$, therefore $B_2 B_4 = B_6$: or $e^{2Bx} \equiv \{e^{Bx}\}^2$, therefore

$$\frac{2x}{e^{2x}-1} = \left\{ \frac{x}{e^x - 1} \right\}^2.$$

The general principle contained in the latter part of Rule II ensures a check on such manipulations as

$$e^{Bx} \times e^x \equiv e^{(B+1)x}, \tag{4}$$

and

$$\frac{d}{dx}\{e^{Bx}\} \equiv B e^{Bx}. \tag{5}$$

Of these, (4) follows by Cauchy multiplication of exponential series. (Cf. [4], [19].) That is

$$e^{Bx} \times e^x \equiv \sum_{m=0}^{\infty} \frac{B^m x^m}{m!} \sum_{n=0}^{\infty} \frac{x^n}{n!} \equiv \sum_{p=0}^{\infty} \frac{x^p}{p!} \sum_{m+n=p} \frac{p!}{m!n!} B^m$$

$$\equiv \sum_{n=0}^{\infty} (B+1)^p \frac{x^p}{p!} \equiv e^{(B+1)x}.$$

Similarly (5) is validated by

$$\frac{d}{dx}\{e^{Bx}\} \equiv \frac{d}{dx}\left\{ \sum_{n=0}^{\infty} \left\{ B_n \frac{x^n}{n!} \right\} = \sum_{n=1}^{\infty} B_n \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} B_{n+1} \frac{x^n}{n!} \right.$$

$$\equiv B \sum_{n=0}^{\infty} \frac{B^n x^n}{n!} \equiv B e^{Bx}$$

3. Applications of umbral methods to the Bernoulli numbers. (i) The recurrence formula. From (3) we have

$$e^{Bx} \equiv \frac{x}{e^x - 1}. \tag{6}$$

Hence

$$e^{(B+1)x} - e^{Bx} \equiv x.$$

Expanding and equating coefficients of $x^n/n!$, we obtain the recurrence (2), as quoted earlier.

Changing the sign of x in (6), we have

$$e^{-Bx} \equiv \frac{-x}{e^{-x}-1} = \frac{xe^x}{e^x - 1} \equiv e^{(B+1)x}. \tag{7}$$

Hence for all integers $n \geqslant 0$,

$$(B+1)^n \equiv (-B)^n.$$

With (2) this gives $B^n \equiv (-B)^n$ for $n \neq 1$ and $B_1 = -B_1 - 1$. Hence $B_1 = -\frac{1}{2}$, $B_3 = B_5 = B_7 = \cdots = 0$.

(ii) Power series for certain trigonometric functions. From (6) and (7) it follows that

$$\cosh Bx \equiv \tfrac{1}{2}(e^{Bx}+e^{-Bx}) \equiv \tfrac{1}{2}\frac{x}{e^x-1}(1+e^x)=\tfrac{1}{2}x\coth\tfrac{1}{2}x.$$

Replacing x by $2ix$ we deduce that $\cot x \equiv (\cos 2Bx)/x$. This can be regarded as a mnemonic for the non-umbral formula

$$\cot x = \sum_{n=0}^{\infty}(-)^n 2^{2n}B_{2n}\frac{x^{2n-1}}{2n!}, \qquad (|x|<\pi).$$

From the identities

$$\operatorname{cosec} x = \cot\tfrac{1}{2}x - \cot x, \qquad \tan x = \cot x - 2\cot 2x,$$

we get

$$\operatorname{cosec} x \equiv \frac{2\cos Bx - \cos 2Bx}{x}, \qquad \tan x \equiv \frac{\cos 2Bx - \cos 4Bx}{x}.$$

That is,

$$\operatorname{cosec} x = \sum_{n=0}^{\infty}(-)^{n-1}(2^{2n}-2)B_{2n}\frac{x^{2n-1}}{2n!}, \qquad (|x|<\pi),$$

and

$$\tan x = \sum_{n=1}^{\infty}(-)^{n-1}(2^{4n}-2^{2n})B_{2n}(x^{2n-1}/2n!), \qquad \left(|x|<\tfrac{1}{2}\pi\right).$$

(iii) The Bernoulli polynomials. These are polynomials $B_n(x)$, defined as the coefficients in the expansion ([2])

$$\frac{ze^{xz}}{e^z-1}=\sum_{n=1}^{\infty}B_n(x)\frac{z^n}{n!}.$$

In umbral form the left-hand side is $e^{(B+x)z}$. Hence

$$B_n(x)\equiv(B+x)^n\equiv\sum_{r=0}^{n}\binom{n}{r}B_{n-r}x^r.$$

The basic properties of the Bernoulli polynomials follow readily from this umbral form. We have immediately

$$B_n(0)\equiv B^n\equiv B_n, \quad \text{and} \quad B_n'(x)\equiv n(B+x)^{n-1}\equiv nB_{n-1}(x).$$

Further

$$e^{-(B+x)z}\equiv\frac{-ze^{-xz}}{e^{-z}-1}\equiv\frac{ze^{(1-x)z}}{e^z-1}\equiv e^{(B+1-x)z},$$

whence

$$(-)^n(B+x)^n\equiv(B+1-x)^n.$$

That is, the symmetry property ([2]) $(-)^nB_n(x)\equiv B_n(1-x)$. Also

$$\sum_{r=0}^{m-1}e^{(mB+mx+r)z}\equiv e^{m(B+x)z}\sum_{r=0}^{m-1}e^{rz}\equiv\frac{mze^{mxz}}{e^{mz}-1}\times\frac{e^{mz}-1}{e^z-1}$$
$$=\frac{z}{e^z-1}\times me^{mxz}\equiv me^{(B+mx)z}.$$

Hence, by equating coefficients of $z^n/n!$,

$$m(B+mx)^n \equiv \sum_{r=0}^{n-1} (mB+mx+r)^n \equiv m^n \sum_{r=0}^{n-1} \left(B+x+\frac{r}{m}\right)^n.$$

That is, the multiplication formula

$$B_n(mx) = m^{n-1} \sum_{r=0}^{m-1} B_n\left(x+\frac{r}{m}\right).$$

(iv) The Euler-Maclaurin summation formula for polynomials ([**8**], [**13**]). If

$$f(x) = \sum_{n=0}^{k} f^{(n)}(0)\frac{x^n}{n!}$$

is a polynomial of degree k in x, then

$$f(B+1)-f(B) \equiv \sum_{n=0}^{k} \frac{f^{(n)}(0)}{n!}\{(B+1)^n - B^n\} \equiv f'(0), \tag{8}$$

by the recurrence formula (2). Unit translations of this result give, successively,

$$\begin{aligned} &f(B+2)-f(B+1) \equiv f'(1),\\ &f(B+3)-f(B+2) \equiv f'(2),\\ &\cdots\cdots\cdots\cdots\cdots\cdots\\ &f(B+n)-f(B+n-1) \equiv f'(n-1). \end{aligned}$$

Adding these with (8) we get the umbral form of the Euler–Maclaurin summation formula

$$f'(0)+f'(1)+f'(2)+\cdots+f'(n-1) \equiv f(B+n)-f(B). \tag{9}$$

In particular, if p is a positive integer and $f(x) = x^{p+1}/(p+1)$,

$$1^p+2^p+3^p+\cdots+(n-1)^p \equiv \frac{(B+n)^{p+1}-B^p}{p+1}.$$

Also if we put $f(x) = (x+z)^p$ in (8) we get $B_p(z+1)-B_p(z) = pz^{p-1}$, which is the difference formula for the Bernoulli polynomials.

If we expand the right-hand side of (9) in powers of B by Taylor's theorem, it is to be interpreted as

$$\sum_{r=0}^{k} \frac{B^r}{r!}\{f^{(r)}(n)-f^{(r)}(0)\}.$$

If we put $\phi(x) = f'(x)$ and recall that $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_3 = B_5 = \cdots = 0$, then (9) gives

$$\tfrac{1}{2}\phi(0)+\phi(1)+\phi(2)+\cdots+\phi(n-1)+\tfrac{1}{2}\phi(n) = \int_0^n \phi(x)\,dx + \sum_{r=1}^{[\frac{1}{2}p+\frac{1}{2}]} \frac{B_{2r}}{2r!}\{\phi^{(2r-1)}(n)-\phi^{(2r-1)}(0)\}.$$

This is the usual non-umbral form of the Euler–Maclaurin summation formula.

4. The Euler numbers. The Euler numbers E_n in the expansion

$$\operatorname{sech} x = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!} = 1-\frac{x^2}{2!}+5\frac{x^4}{4!}-61\frac{x^6}{6!}+\cdots, \qquad (|x|<\tfrac{1}{2}\pi)$$

can be associated with an umbra E. That is $E^n \equiv E_n$. Since $\operatorname{sech} x$ is an even function of x, it follows that $E_1 = E_3 = E_5 = \cdots = 0$. Then from

$$\frac{2}{e^x+e^{-x}} = \operatorname{sech} x \equiv e^{Ex}$$

we get

$$e^{(E+1)x}+e^{(E-1)x}\equiv 2.$$

and on equating coefficients, the recurrence formula

$$(E+1)^n+(E-1)^n\equiv\begin{cases}2, & (n=0),\\ 0, & (n=1,2,3,\ldots).\end{cases}$$

If $f(x)$ is a polynomial in x of degree k, then the recurrence implies that

$$f(E-1)+f(E+1)\equiv 2f(0).$$

Hence

$$\begin{aligned}f(1)-f(3)+f(5)-\cdots-f(4n-1)&\\ \equiv\tfrac{1}{2}\{f(E)+f(E+2)-f(E+2)-f(E+4)+\cdots-f(E+4n-2)-f(E+4n)\}&\\ \equiv\tfrac{1}{2}\{f(E)-f(E+4n)\}.&\qquad(10)\end{aligned}$$

This is the umbral form of an analogue for alternating series of the Euler–Maclaurin summation formula. In particular

$$1^p-3^p+5^p-7^p+\cdots-(4n-1)^p\equiv\tfrac{1}{2}\{E^p-(E+4n)^p\}$$

for positive integral p. In non-umbral form (10) becomes

$$f(1)-f(3)+f(5)-f(7)+\cdots-f(4n-1)=\tfrac{1}{2}\sum_{r=0}^{\left[\frac{1}{2}k+\frac{1}{2}\right]}\frac{E_{2r}}{2r!}\{f^{(2r)}(0)-f^{(2r)}(4n)\}.\qquad(11)$$

5. Asymptotic remainder formulas. The Euler–Maclaurin summation formula and its analogue (11) are, of course, valid for functions other than polynomials, and many useful asymptotic formulas for remainders in summing series can be derived from them ([**8**]). For example, if we put $f(x)=1/x$ in (11), the result is the formula ([**5**])

$$\tfrac{1}{4}\pi=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots-\frac{1}{4n-1}+\tfrac{1}{2}\sum_{r=0}^{k}\frac{E_{2r}}{(4n)^{2r+1}}+O(n^{-2k-3})\qquad(12)$$

for fixed k and $n\to\infty$. This result cannot be deduced from (10) since $f(E)=1/E$ has no umbral interpretation, but another approach can give a short umbral proof.

For any algebraic quantity, by summing geometric series,

$$\sum_{s=0}^{m-1}\left\{\frac{(E+1)^s}{x^{s+1}}-\frac{E^s}{(x+1)^{s+1}}\right\}=\frac{\{E/(x+1)\}^m-\{(E+1)/x\}^m}{x-1-E}=O(x^{-m-1})\qquad(13)$$

for fixed m and $x\to\infty$. Similarly

$$\sum_{s=0}^{m-1}\left\{\frac{(E-1)^s}{x^{s+1}}-\frac{E^s}{(x-1)^{s+1}}\right\}=O(x^{-m-1}).\qquad(14)$$

If we now regard E as the umbra for the Euler numbers, add (13) and (14), and use the recurrence formula, we get

$$\frac{2}{x}-\sum_{r=0}^{m-1}E^s\left\{\frac{1}{(x-1)^{s+1}}+\frac{1}{(x+1)^{s+1}}\right\}\equiv O(x^{-m-1}).\qquad(15)$$

If we put $m=2k+2$, recall that odd order Euler numbers vanish, and divide by 2, then (15), in non-umbral form, becomes

$$u(x)=\frac{1}{x}-\sum_{r=0}^{k}E_{2r}\left\{\frac{1}{(x-1)^{2r+1}}+\frac{1}{(x+1)^{2r+1}}\right\}=O(x^{-2k-3}). \tag{16}$$

Now $u(x)$ is a rational algebraic function of x, so for sufficiently large x it tends to zero monotonically as x increases. Setting $x=4n+1,4n+3,4n+5,\ldots$ it follows that

$$u(4n+1)-u(4n+3)+u(4n+5)-\cdots=O(n^{-2k-3})$$

as $n\to\infty$. That is

$$\frac{1}{4n+1}-\frac{1}{4n+3}+\frac{1}{4n+5}-\frac{1}{4n+7}+\cdots$$

$$-\sum_{r=0}^{k}E_{2r}\left\{\frac{1}{(4n)^{2r+1}}+\frac{1}{(4n+2)^{2r+1}}-\frac{1}{(4n+2)^{2r+1}}-\cdots\right\}$$

$$=O(n^{-2k-3}). \tag{17}$$

Since all terms after the first cancel out in the second part of (17), and by Leibniz's series

$$\tfrac{1}{4}\pi=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots, \tag{18}$$

the required result (12) follows from (17).

The Leibniz series (18) is usually dismissed as useless for any practical calculation of π, though it is certainly the simplest series which could be so used. Using (12) with $n=2$, $k=1$ gives an estimate of π as 3.14133, an error of some 2.6×10^{-4}. Direct use of the Leibniz series (18) would require about 8000 terms to reach this order of accuracy.

If we set $x=4n+2,4n+4,4n+6,\ldots$ in (16), then similar arguments show that

$$\log 2=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots-\frac{1}{2n}+\sum_{r=0}^{k}\frac{E_{2r}}{(4n+1)^{2r+1}}+O(n^{-2k-3}).$$

6. Reciprocal sequences and related identities. Two sequences $\{a_n\},\{b_n\},(n=0,1,2,\ldots)$ are said to be reciprocal if [**1**]

$$b_n=\sum_{r=0}^{n}(-)^r\binom{n}{r}a_r, \tag{19}$$

since it is then also true that

$$a_n=\sum_{r=0}^{n}(-)^r\binom{n}{r}b_r. \tag{20}$$

To prove this, designate a and b as umbrae. Then (19) becomes $b^n\equiv(1-a)^n$. Hence $e^{bx}\equiv e^{(1-a)x}$.

Replacing x by $-x$, we have

$$e^{ax}\equiv e^{(1-b)x}, \tag{21}$$

whence (20) follows on reversing the argument. Note that in this case a_0 and b_0 are not necessarily equal to one, unlike B_0 and E_0.

If $\{\alpha_n\},\{\beta_n\}$ is another pair of reciprocal sequences, then

$$e^{(a-\alpha)x}\equiv e^{ax}\times e^{-\alpha x}\equiv e^{(1-b)x}\times e^{-(1-\beta)x}\equiv e^{(\beta-b)x}.$$

Hence we get the identity

$$\sum_{r=0}^{n}(-)^r\binom{n}{r}a_r\alpha_{n-r}=\sum_{r=0}^{n}(-)^r\binom{n}{r}\beta_r b_{n-r}.$$

Identities involving Bernoulli numbers and reciprocal sequences occur in certain relations between angle-sums in n-dimensional simplexes ([7]). An extension of these results, in umbral form, is

$$(B+a)^n \equiv (-B-b)^n \equiv \tfrac{1}{2}\{(2B+a)^n+(-2B-b)^n\}.$$

That is, in non-umbral form,

$$\sum_{r=0}^{n}\binom{n}{r}B_r a_{n-r} = (-)^n\sum_{r=0}^{n}\binom{n}{r}B_r b_{n-r} = \sum_{r=0}^{n}\binom{n}{r}2^{r-1}B_r\{a_{n-r}+(-)^n b_{n-r}\}.$$

To prove this, it suffices to show that

$$e^{(B+a)x}\equiv e^{-(B+b)x}\equiv \tfrac{1}{2}\{e^{(2B+a)x}+e^{-(2B+b)x}\}.$$

From (3) and (21)

$$e^{(B+a)x}\equiv \frac{x}{e^x-1}e^{(1-b)x}\equiv \frac{-x}{e^{-x}-1}e^{-bx}\equiv e^{-Bx}\times e^{-bx}\equiv e^{-(B+b)x}.$$

Also

$$\tfrac{1}{2}\{e^{(2B+a)x}+e^{-(2B+b)x}\}\equiv \tfrac{1}{2}\left\{\frac{2xe^{ax}}{e^{2x}-1}+\frac{-2xe^{-(1-a)x}}{e^{-2x}-1}\right\}$$

$$\equiv \frac{xe^{ax}}{e^x-1}\equiv e^{(B+a)x},$$

as required.

7. Sums of products and dual umbral notation. Many identities involving sums of products of Bernoulli numbers have long been known. In particular, an identity of Euler, written in the even suffix notation, takes the form [12]

$$\sum_{r=1}^{n-1}\binom{2n}{2r}B_{2r}B_{2n-2r} = -(2n+1)B_{2n}. \tag{22}$$

Mordell [10] has commented that proofs of (22) in the literature are involved. Since $B_1 = -\frac{1}{2}$ and other odd order Bernoulli numbers vanish, (22) can be written for even m as

$$\sum_{s=0}^{m}(-)^s\binom{m}{s}B_sB_{m-s} = (1-m)B_m. \tag{23}$$

For odd m, (23) is trivially true in that both sides vanish. The form of (23) suggests the use of a dual umbral notation, with two umbrae B and B', both for Bernoulli numbers, subject to the interpretation that terms $(B)^p(B')^q \equiv B_pB_q$. Then (23) becomes

$$(B-B')^m \equiv (1-m)B^m. \tag{24}$$

Such a notation has been used previously to simplify the expression of similar sums of products ([6], [17]). It can also suggest simple proofs; for Euler's identity in the form (24), we have

$$\sum_{m=0}^{\infty}(B-B')^m\frac{x^m}{m!}\equiv e^{(B-B')x}\equiv e^{Bx}\times e^{-B'x}$$

$$\equiv \frac{x}{e^x-1}\times\frac{-x}{e^{-x}-1} = \frac{x^2e^x}{(e^x-1)^2}$$

$$= -x^2\frac{d}{dx}\left\{\frac{e^{Bx}}{x}\right\}\equiv(1-Bx)e^{Bx}$$

$$\equiv \sum_{m=0}^{\infty}B^m\frac{x^m}{m!}-\sum_{m=0}^{\infty}B^{m+1}\frac{x^{m+1}}{m!}$$

$$\equiv \sum_{m=0}^{\infty} (1-m)B^m \frac{x^m}{m!}.$$

Equating coefficients of $x^m/m!$ we obtain (24).

8. Multiple umbral methods and other extensions. Endless similar results can be found by the methods of the previous sections. With a multiple umbral notation, $(B)^p(B')^q(B'')^r \cdots \equiv B_pB_qB_r\cdots$, we have

$$(B+B'+B'')^n \equiv 3\binom{n}{3}\left\{\frac{B^n}{n}+3\frac{B^{n-1}}{n-1}+2\frac{B^{n-2}}{n-2}\right\},$$

$$(B+B'-B'')^n \equiv 3\binom{n}{3}\left\{\frac{B^n}{n}+\frac{B^{n-1}}{n-1}\right\}, \tag{25}$$

$$(B+B'+B''+B'''+B'''')^n \equiv 5\binom{n}{5}\left\{\frac{B^n}{n}+10\frac{B^{n-1}}{n-1}+35\frac{B^{n-2}}{n-2}+50\frac{B^{n-3}}{n-3}+24\frac{B^{n-4}}{n-4}\right\}. \tag{26}$$

The pattern of coefficients in (25) and (26) follows that of

$$(x+1)(x+2)=x^2+3x+2 \quad \text{and} \quad (x+1)(x+2)(x+3)(x+4)=x^4+10x^3+35x^2+50x+24.$$

Another series of identities starts with the Euler identity (24), and continues with

$$(2B-B')^{2n} \equiv (1-2n)B^{2n},$$

$$(4B-B')^{2n} \equiv (1-2n)B^{2n}+n(2n-1)E^{2n-2},$$

$$(8B-B')^{2n} \equiv (1-2n)B^{2n}+n(2n-1)E^{2n-2}$$
$$+2n(2n-1)\{(2E-1)^{2n-2}+(2E)^{2n-2}+(2E+1)^{2n-2}\}.$$

Other identities with Euler numbers are

$$(2B+E)^n \equiv (4B+1)^n, \qquad (n+2)(E+E')^n \equiv 2^{n+2}(2^{n+2}-1)B^{n+2},$$
$$2(E+E'+E'')^n \equiv E^n - E^{n+2}.$$

The properties of the Euler polynomials can be derived in the same way as was used for the Bernoulli polynomials in Section 3 (iii). It is found that their umbral form is ([2]) $E_n(x) \equiv (\frac{1}{2}E+x-\frac{1}{2})^n$.

Proofs of the results of this section follow the same lines as preceding sections.

9. Remarks. The use of a special symbol, such as $\equiv$ or $\doteqdot$, is not essential; some writers use one ([**9**], [**15**]), others do not ([**13**]). I have used the former in this paper to lessen possible ambiguities.

The ban on multiplication of expressions with a common umbra, mentioned in Rule II, does not appear to have been stated explicitly in the literature. Presumably this is because it is usual for rules to tell what is permitted rather than what is forbidden. Riordan [**15**] does briefly imply that there are operations to be avoided; in so doing he also uses a form of dual umbral notation without any distinguishing marks for twin umbrae. For an expression such as $(a+a)^n$ he says, "Like sequences are treated exactly as unlike sequences." This becomes confusing if applied to the more involved expressions of Section 8, so I have preferred to make definite distinctions by using B, B', B'', and so on.

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THE SEVENTH U.S.A. MATHEMATICAL OLYMPIAD: A REPORT

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The Seventh U.S.A. Mathematical Olympiad was held on May 2, 1978. From the Honor Roll of the Annual High School Mathematics Examination, 108 students, who had scored 118 points or better, were invited to take part, and 106 finally did participate. The papers were graded, first by Professors Michael Aissen and John Bender of Rutgers University, and then by Professor Murray Klamkin, of the University of Alberta, and this writer. A copy of the Olympiad problems appears below.

The table below compares the students' scores on the Olympiad with those on the Annual High School Mathematics Examination. As has been the case on all previous Olympiads, there is little or no correlation, which is not surprising, considering the differing content and goals of the contests.

Olympiad / H.S. Exam	0–10	11–20	21–30	31–40	41–50	51–60	61–70	71–80	81–90	91–100
148–150					1	1				2
145–147	1		1	1					1	1
142–144										
139–141	1			1	1	1		1		
136–138		1								
133–135	2	1				1		1		1
130–132	3	2	3	3	1		2			
127–129	2	4	3	4	1		1			
124–126	2	1	6	3	2	1	1			
121–123	2	3	5	1	3	1	1			
118–120	5	4	5	7	3					