Generalization of Bernoulli polynomials

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The Bernoulli polynomials are generalized and some properties of the resulting generalizations are presented.

1. Introduction

It is well known that the Bernoulli numbers B_n can be defined [1–3] as

$$\phi(x)\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} \times x^n \qquad |x| < 2\pi \tag{1}$$

The Bernoulli polynomials $B_n(x)$ can be defined [1–3] by

$$\phi(z, x)\frac{z\,e^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} \times z^n \qquad |z| < 2\pi \tag{2}$$

where we write $B_n = B_n(0)$ for the Bernoulli numbers.

The usual definition of the generalized Bernoulli polynomials is

$$\frac{t^\sigma e^{ut}}{(e^t - 1)^\sigma} = \sum_{n=0}^{\infty} B_n^\sigma(u) \times \frac{t^n}{n!} \qquad |t| < 2\pi \tag{3}$$

For more information about Bernoulli numbers and Bernoulli polynomials, see [4, 5]. Many approaches to calculating Bernoulli numbers are presented in [1–3, 6].

Now we introduce a new function $B_n(a, b)$ for $b > a > 0$ by

$$\phi(x; a, b)\frac{x}{b^x - a^x} = \sum_{n=0}^{\infty} B_n(a, b) \times \frac{x^n}{n!} \qquad |x| < \frac{2\pi}{\ln b - \ln a} \tag{4}$$

In this note, we give some relations between B_n, $B_n(x)$ and $B_n(a, b)$, and some properties of the function $B_n(a, b)$.

2. Relationships between B_n, $B_n x$ and $B_n ab$

It is clear that

$$B_0(a, b) = \frac{1}{\ln b - \ln a} \quad \text{and} \quad B_n(1, e) = B_n \tag{5}$$

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Since

$$\frac{x}{b^x - a^x} = \frac{1}{a^x} \times \frac{x}{e^{x(\ln b - \ln a)} - 1}$$
$$= \left(\sum_{n=0}^{\infty} \frac{(\ln b - \ln a)^{n-1}}{n!} B_n x^n\right)\left(\sum_{k=0}^{\infty} \frac{(\ln a)^k}{k!}(-1)^k x^k\right)$$
$$= \sum_{j=0}^{\infty}\left(\sum_{i=0}^{j}(-1)^{j-1} B_i \times \frac{(\ln b - \ln a)^{i-1}(\ln\ a)^{j-i}}{i!(j-i)!}\right) x_j$$

hence

$$B_j(a,b) = \sum_{i=0}^{j}(-1)^{j-i}(\ln b - \ln a)^{i-1}(\ln a)^{j-i}\binom{j}{i} B_i \tag{6}$$

Further, because

$$\frac{x}{b^x - a^x} = \frac{x\,e^{-x\ln a}}{e^{x(\ln b - \ln a)} - 1}$$
$$= \frac{1}{\ln b - \ln a}\sum_{n=0}^{\infty}\frac{(\ln b - \ln a)^n}{n!} \times B_n\left(\frac{\ln a}{\ln a - \ln b}\right) \times x^n$$
$$= \sum_{n=0}^{\infty}\frac{(\ln b - \ln a)^{n-1}}{n!} \times B_n\left(\frac{\ln a}{\ln a - \ln b}\right) \times x^n$$

then we have

$$B_n(a,b) = (\ln b - \ln a)^{n-1} \times B_n\left(\frac{\ln a}{\ln a - \ln b}\right) \tag{7}$$

Moreover, since

$$\frac{x\,e^{tx}}{e^x - 1} = \frac{x}{(e^{1-t})^x - (e^{-t})^x}$$

thus

$$B_n(t) = B_n(e^{-t}, e^{1-t}) \tag{8}$$

3. Some properties of the generalization of Bernoulli polynomials

For real numbers $b > a > 0$ and $x \in \mathbb{R}$, define

$$g(x) = g(x; a, b) = \begin{cases} \dfrac{b^x - a^x}{x}, & x \neq 0 \\ \ln b - \ln a, & x = 0 \end{cases} \tag{9}$$

Since $(b^x - a^x)\phi(x; a, b) = x$ and $g(x; a, b) \times \phi(x; a, b) = 1$, using the series expansions of a^x and b^x at $x = 0$ and formula (4), by standard arguments, we have

$$\sum_{i=0}^{k}[(\ln b)^i - (\ln a)^i]\binom{k+1}{i} B_{k-i+1}(a,b) = 0, \qquad k \geqslant 1 \tag{10}$$

$$\sum_{i=0}^{k}[(\ln b)^i - (\ln a)^i]\binom{k}{i} B_{k-i}(a,b) = 0, \qquad k \geqslant 2 \tag{11}$$

Since $B_n(1-y) = (-1)^n B_n(y)$, from formula (7), we have

$$B_n(a,b) = -B_n(b,a) \tag{12}$$

From formula (7), it is easy to see that

$$B_n(a^\alpha, b^\alpha) = \alpha^{n-1} B_n(a,b), \qquad \alpha \in \mathbb{R} \tag{13}$$

Using $\mathrm{d}B_n(y)/\mathrm{d}y = nB_{n-1}(y)$ and by direct calculation, formula (7) leads to

$$\begin{aligned}\frac{\partial B_n(a,b)}{\partial a} = \frac{(\ln b - \ln a)^{n-3}}{a}\bigg[&(n-1)(\ln a - \ln b)B_n\left(\frac{\ln a}{\ln a - \ln b}\right)\\ &-n(\ln b)B_{n-1}\left(\frac{\ln a}{\ln a - \ln b}\right)\bigg]\end{aligned} \tag{14}$$

Further, differentiating formula (8) with respect to t on both sides gives

$$B_n(t) = -\frac{1}{(n+1)\mathrm{e}^t}\left(\frac{\partial B_{n+1}(x,y)}{\partial x} + e\frac{\partial B_{n+1}(x,y)}{\partial y}\right)\bigg|_{\substack{x=\mathrm{e}^{-t}\\ y=\mathrm{e}^{1-t}}} \tag{15}$$

It is noted that many inequalities and properties of $g(x;a,b)$ have been established and researched by the authors and others in [7]–[12].

The function g can be expressed in integral form as

$$g(x;a,b) = \int_a^b t^{x-1}\,\mathrm{d}t \tag{16}$$

Mathieu's series defined in [13] can be expressed as

$$S(r) = \frac{1}{r^2}\int_0^\infty \frac{\sin t}{g(t/r;1,e)}\,\mathrm{d}t = \frac{1}{r}\int_0^\infty \phi(x)\sin(rt)\,\mathrm{d}t \tag{17}$$

Recently, some new results of Mathieu's series have appeared in [14].

By mathematical induction on $n \in \mathbb{N}$, we obtain a recursion formula for derivatives with respect to x of g as follows

$$(n+1)g^{(n)}(x) + xg^{(n+1)}(x) = (\ln b)^{n+1}b^x - (\ln a)^{n+1}a^x \tag{18}$$

Put $b = e$ and $a = 1$, then

$$(n+1)g^{(n)}(x;1,e) + xg^{(n+1)}(x;1,e) = \mathrm{e}^x \tag{19}$$

Note that the function $g(x;1,e)$ is absolutely monotonic increasing, see [7]–[10].

Since $[g'(x;1,e)]^2 \geqslant g(x;1,e) \times g''(x;1,e)$, by standard argument, we deduce that $\phi(x)$ is convex and $3(\phi'(x))^2 \leqslant \phi(x)\phi''(x)$.

Using the expression (16) for the function g, many new Steffensen pairs are established in [8, 9, 15].

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The Moore–Penrose inverse and the vector product

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In this note it is shown that the Moore–Penrose inverse of real 3×3 matrices can be expressed in terms of the vector product of their columns. Moreover, a simple formula of a generalized inverse is presented, which also involves the vector product.

1. Introduction

Given the 3×3 real matrix $\mathbf{A} = (\mathbf{a}, \mathbf{b}, \mathbf{c})$, where $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are its columns, it is easily seen that the adjoint matrix of $\mathbf{A}$ is given by

$$\mathbf{A}^{\#} = (\mathbf{b} \times \mathbf{c}, \mathbf{c} \times \mathbf{a}, \mathbf{a} \times \mathbf{b})' \tag{1.1}$$

Here '$\times$' denotes the vector product in $\mathbb{R}^3$ and prime means transpose. For a nonsingular matrix $\mathbf{A}$ we have

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \mathbf{A}^{\#} \tag{1.2}$$