

An Explicit Formula for Bernoulli Numbers in Terms of Stirling Numbers of the Second Kind

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Abstract: In the paper, the authors recover an explicit formula for computing Bernoulli numbers in terms of Stirling numbers of the second kind.

Keywords: explicit formula; Bernoulli number; Stirling number of the second kind; Bell polynomial of the second kind

1 Introduction

It is well known that Bernoulli numbers B_k for $k \geq 0$ may be generated by

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!} = 1 - \frac{x}{2} + \sum_{k=1}^{\infty} B_{2k} \frac{x^{2k}}{(2k)!} \quad (1)$$

for $|x| < 2\pi$. See [1, p. 48]. In combinatorics, Stirling numbers of the second kind $S(n, k)$ for $n \geq k \geq 0$ may be computed by

$$S(n, k) = \frac{1}{k!} \sum_{\ell=0}^k (-1)^{k-\ell} \binom{k}{\ell} \ell^n \quad (2)$$

and may be generated by

$$\frac{(e^x - 1)^k}{k!} = \sum_{n=k}^{\infty} S(n, k) \frac{x^n}{n!}, \quad k \in \{0\} \cup \mathbb{N}. \quad (3)$$

See [1, p. 206]. Bell polynomials of the second kind $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ are defined by

$$B_{n,k}(x_1, \dots, x_{n-k+1}) = \sum_{\substack{1 \leq i \leq n, \ell_i \in \mathbb{N} \\ \sum_{i=1}^n i \ell_i = n \\ \sum_{i=1}^n \ell_i = k}} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n-k+1} \left(\frac{x_i}{i!}\right)^{\ell_i}$$

for $n \geq k \geq 1$, See [1, p. 134, Theorem A].

The aim of this paper is to recover an explicit formula for computing Bernoulli numbers B_n in terms of Stirling numbers of the second kind $S(n, k)$.

The main results may be summarized as the following theorem.

Theorem 1 For $n \geq k \geq 0$, we have

$$B_{n,k}(0, 1, \dots, 1) = \sum_{i=0}^k (-1)^i \binom{n}{i} S(n-i, k-i) \quad (4)$$

and

$$B_{n,k}\left(\frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n-k+2}\right) = \frac{n!}{(n+k)!} \sum_{i=0}^k (-1)^{k-i} \binom{n+k}{k-i} S(n+i, i). \quad (5)$$

For $n \geq 0$, we have

$$B_n = \sum_{i=0}^n (-1)^i \frac{\binom{n+1}{i+1}}{\binom{n+i}{i}} S(n+i, i). \quad (6)$$

2 Proof of Theorem 1

In combinatorics, Faà di Bruno formula may be described in terms of Bell polynomials of the second kind

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$B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ by

$$\begin{aligned} & \frac{d^n}{dx^n} f \circ g(x) \\ &= \sum_{k=1}^n f^{(k)}(g(x)) B_{n,k}(g'(x), g''(x), \dots, g^{(n-k+1)}(x)). \end{aligned} \quad (7)$$

See [1, p. 139, Theorem C]. It is easy to see that

$$\frac{x}{e^x - 1} = \frac{1}{\int_0^1 e^{xt} dt}.$$

Applying in (7) the functions $f(y) = \frac{1}{y}$ and $y = g(x) = \int_0^1 e^{xt} dt$ results in

$$\begin{aligned} & \frac{d^n}{dx^n} \left(\frac{x}{e^x - 1} \right) = \frac{d^n}{dx^n} \left(\frac{1}{\int_0^1 e^{xt} dt} \right) \\ &= \sum_{k=1}^n (-1)^k \frac{k!}{\left(\int_0^1 e^{xt} dt \right)^{k+1}} \\ & \quad \times B_{n,k} \left(\int_0^1 t e^{xt} dt, \int_0^1 t^2 e^{xt} dt, \dots, \int_0^1 t^{n-k+1} e^{xt} dt \right) \\ & \rightarrow \sum_{k=1}^n (-1)^k k! B_{n,k} \left(\int_0^1 t dt, \int_0^1 t^2 dt, \dots, \int_0^1 t^{n-k+1} dt \right) \\ &= \sum_{k=1}^n (-1)^k k! B_{n,k} \left(\frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n-k+2} \right) \end{aligned}$$

as $x \rightarrow 0$. On the other hand, differentiating n times on both sides of (1) leads to

$$\frac{d^n}{dx^n} \left(\frac{x}{e^x - 1} \right) = \sum_{k=n}^{\infty} B_k \frac{x^{k-n}}{(k-n)!} \rightarrow B_n, \quad x \rightarrow 0.$$

As a result, we obtain

$$B_n = \sum_{k=1}^n (-1)^k k! B_{n,k} \left(\frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n-k+2} \right). \quad (8)$$

In [1, p. 133], it was listed that

$$\frac{1}{k!} \left(\sum_{m=1}^{\infty} x_m \frac{t^m}{m!} \right)^k = \sum_{n=k}^{\infty} B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) \frac{t^n}{n!} \quad (9)$$

for $n \geq k \geq 0$. Letting $x_1 = 0$ and $x_m = 1$ for $m \geq 2$ in (9) and employing (3) give

$$\begin{aligned} \sum_{n=k}^{\infty} B_{n,k}(0, 1, \dots, 1) \frac{t^n}{n!} &= \frac{1}{k!} \left(\sum_{m=2}^{\infty} \frac{t^m}{m!} \right)^k = \frac{1}{k!} (e^t - 1 - t)^k \\ &= \frac{1}{k!} \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} (e^t - 1)^i t^{k-i} \\ &= \sum_{i=0}^k \frac{(-1)^{k-i}}{(k-i)!} \sum_{j=i}^{\infty} S(j, i) \frac{t^{k+j-i}}{j!}. \end{aligned}$$

This implies that

$$\begin{aligned} B_{n,k}(0, 1, \dots, 1) &= n! \sum_{i=0}^k \frac{(-1)^{k-i}}{(k-i)!} \frac{S(n-k+i, i)}{(n-k+i)!} \\ &= \sum_{i=0}^k (-1)^{k-i} \binom{n}{k-i} S(n-k+i, i) \\ &= \sum_{i=0}^k (-1)^i \binom{n}{i} S(n-i, k-i). \end{aligned}$$

The formula (4) follows.

By virtue of

$$\begin{aligned} B_{n,k} \left(\frac{x_2}{2}, \frac{x_3}{3}, \dots, \frac{x_{n-k+2}}{n-k+2} \right) \\ = \frac{n!}{(n+k)!} B_{n+k,k}(0, x_2, \dots, x_{n+1}), \end{aligned}$$

see [1, p. 136], and the formula (4), we obtain

$$\begin{aligned} B_{n,k} \left(\frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n-k+2} \right) \\ = \frac{n!}{(n+k)!} B_{n+k,k}(0, 1, \dots, 1) \\ = \frac{n!}{(n+k)!} \sum_{i=0}^k (-1)^i \binom{n+k}{i} S(n+k-i, k-i), \end{aligned}$$

from which, the formula (5) follows.

Substituting (5) into (8) leads to

$$\begin{aligned} B_n &= \sum_{k=1}^n \frac{k! n!}{(n+k)!} \sum_{i=0}^k (-1)^i \binom{n+k}{k-i} S(n+i, i) \\ &= \sum_{k=1}^n \sum_{i=0}^k (-1)^i \frac{\binom{k}{i}}{\binom{n+i}{i}} S(n+i, i) \\ &= \sum_{i=0}^n \frac{(-1)^i}{\binom{n+i}{i}} S(n+i, i) \sum_{k=i}^n \binom{k}{i} \\ &= \sum_{i=0}^n \frac{(-1)^i}{\binom{n+i}{i}} \binom{n+1}{i+1} S(n+i, i), \end{aligned}$$

which may be rewritten as the formula (6). The proof of Theorem 1 is complete.

3 Remarks

Finally we list several remarks on something to do with our main results.

Remark 1 The formula (5) may be alternatively proved as follows.

Taking $x_m = \frac{1}{m+1}$ for all $m \in \mathbb{N}$ in (9) and utilizing (3) yield

$$\begin{aligned} \sum_{n=k}^{\infty} B_{n,k} \left(\frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n-k+2} \right) \frac{t^n}{n!} &= \frac{1}{k!} \left[\sum_{m=1}^{\infty} \frac{t^m}{(m+1)!} \right]^k \\ &= \frac{1}{k!} \left(\frac{e^t - 1 - t}{t} \right)^k = \frac{1}{k!} \left(\frac{e^t - 1}{t} - 1 \right)^k \\ &= \frac{1}{k!} \sum_{\ell=0}^k (-1)^{k-\ell} \binom{k}{\ell} \left(\frac{e^t - 1}{t} \right)^\ell \\ &= \frac{1}{k!} \sum_{\ell=0}^k (-1)^{k-\ell} \binom{k}{\ell} \frac{\ell!}{t^\ell} \sum_{i=\ell}^{\infty} S(i, \ell) \frac{t^i}{i!} \\ &= \sum_{\ell=0}^k \frac{(-1)^{k-\ell}}{(k-\ell)!} \sum_{i=\ell}^{\infty} S(i, \ell) \frac{t^{i-\ell}}{i!}. \end{aligned}$$

This implies that

$$\begin{aligned} B_{n,k} \left(\frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n-k+2} \right) \\ = n! \sum_{\ell=0}^k \frac{(-1)^{k-\ell}}{(k-\ell)!(n+\ell)!} S(n+\ell, \ell). \end{aligned}$$

The formula (5) follows.

Remark 2 In [1, p. 220] and [3, pp. 559–560], the following explicit formula for computing Bernoulli numbers B_n in terms of Stirling numbers of the second kind $S(n, k)$ was presented: For $n \geq 0$, we have

$$B_n = \sum_{k=0}^n (-1)^k \frac{k!}{k+1} S(n, k). \quad (10)$$

Recently, four alternative proofs for the formula (10) were supplied in [4] and [11].

The first formula for Bernoulli numbers B_n listed in [2] is

$$B_n = \sum_{k=0}^n \frac{1}{k+1} \sum_{j=0}^k (-1)^j \binom{k}{j} j^n, \quad n \geq 0, \quad (11)$$

which is a special case of the general formula [8, (2.5)]. We observe that the formula (11) is equivalent to the one (10). In all, we may collect at least seven alternative proofs for the formula (10) or (11) in the references [2], [3], [4], [8], and [11].

Remark 3In [6, p. 1128, Corollary], among other things, it was found that, for $k \geq 1$,

$$B_{2k} = \frac{1}{2} - \frac{1}{2k+1} - 2k \sum_{i=1}^{k-1} \frac{A_{2(k-i)}}{2(k-i)+1}, \quad (12)$$

where A_m is defined by

$$\sum_{m=1}^n m^k = \sum_{m=0}^{k+1} A_m n^m.$$

In [5, Theorem 3.1], it was presented that Bernoulli numbers B_{2k} may be computed by

$$\begin{aligned} B_{2k} &= 1 + \sum_{m=1}^{2k-1} \frac{S(2k+1, m+1)S(2k, 2k-m)}{\binom{2k}{m}} \\ &\quad - \frac{2k}{2k+1} \sum_{m=1}^{2k} \frac{S(2k, m)S(2k+1, 2k-m+1)}{\binom{2k}{m-1}} \quad (13) \end{aligned}$$

for $k \in \mathbb{N}$. In [10, Theorem 1.4], among other things, it was discovered that

$$B_{2k} = \frac{(-1)^{k-1} k}{2^{2k-2} (2^{2k}-1)} \sum_{i=0}^{k-1} \sum_{\ell=0}^{k-i-1} (-1)^{i+\ell} \binom{2k}{\ell} (k-i-\ell)^{2k-1}.$$

for $n \in \mathbb{N}$.

Remark 4 The object of the paper [2], motivated by the paper [8], is to set matters straight by presenting a bibliography, including 33 references, on explicit formulas for Bernoulli numbers and to show how one can easily manufacture expressions for Bernoulli numbers.

In [2, p. 48, (11)], it was deduced that

$$B_n = \sum_{j=0}^n (-1)^j \binom{n+1}{j+1} \frac{n!}{(n+j)!} \sum_{k=0}^j (-1)^{j-k} \binom{j}{k} k^{n+j}$$

for $n \geq 0$. This may be rearranged as the form of the formula (6).

On 21 January 2014, the authors searched out that the formula (6) was ever derived in [7, p. 59] and [13, p. 140] by different tools from Faà di Bruno formula (7).

For more information on the history and literature of explicit formulas for computing Bernoulli numbers, please refer to [2], [7], [8], [12], and [13] and plenty references therein.

Remark 5 This paper is a revised version of the preprint [9].

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