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Generalized Fibonacci-Like Polynomial and its Determinantal Identities

V. K. Gupta [1], **Yashwant K. Panwar** [2] **and Omprakash Sikhwal** [3]

[1] Department of Mathematics, Govt. Madhav Science College, Ujjain, India
dr_vkg61@yahoo.com

[2] Department of Mathematics
Mandsaur Institute of Technology,Mandsaur, India
yashwantpanwar@gmail.com

[3] Department of Mathematics
Mandsaur Institute of Technology,Mandsaur, India
opbhsikhwal@rediffmail.com, opsikhwal@gmail.com

Abstract

It is well known that the Fibonacci polynomials are of great importance in the study of many subjects such as Algebra, geometry, combinatorics and number theory itself. Fibonacci polynomials defined by the recurrence relation $f_n(x) = x\, f_{n-1}(x) + f_{n-2}(x)$, $n \geq 2$ with $f_0(x) = 0$, $f_1(x) = 1$. In this paper we introduce Generalized Fibonacci-Like Polynomials. Further we present its generalized determinantal identities with classical polynomials like Fibonacci Polynomial, Lucas Polynomials, Pell Polynomials and Pell-Lucas Polynomials.

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1. INTRODUCTION

Fibonacci polynomials defined by the recurrence relation $f_n(x) = x\, f_{n-1}(x) + f_{n-2}(x)$, $n \geq 2$ with $f_0(x) = 0$, $f_1(x) = 1$. It is well known that the Fibonacci polynomials are of great importance in the study of many subjects such as Algebra, geometry, combinatorics and number theory itself.

Many authors have studied Fibonacci polynomials and Generalized Fibonacci polynomials identities. They applied concept of Matrix and Determinants to establish some identities. Spivey [8] describe sum property for determinants and presented new proof identities like Cassini identity, d'Ocagne identity and Catalan identity. Koken and Bozkurt [6] define Jacobsthal M-matrix and Jacobsthal Q-matrix similar to Fibonacci Q-matrix and using these matrix representations to found the Binet like formula for jacobsthal numbers. A.J.Macfarlane [4] use the property for determinants and give new identities involving Fibonacci and related numbers. Some determinantal identities involving Fibonacci polynomials, Lucas polynomials, Chebyshev Polynomials, Pell polynomials, Pell-Lucas polynomials, Vieta-Lucas Polynomials are described [5]. In this paper, we introduce Generalized Fibonacci-Like Polynomials and its determinantal identities. Also we establish result in terms of Generalized Pell Polynomials and Generalized Pell-Lucas Polynomials.

2. GENERALIZED FIBONACCI-LIKE POLYNOMIALS

We define Generalized Fibonacci-Like Polynomials by recurrence relation,

$$V_n(x) = xV_{n-1}(x) + V_{n-2}(x)\ ;\ n \geq 3 \quad with \quad V_1(x) = a\,,\, V_2(x) = bx \qquad [2.1]$$

First few polynomials are

$V_3(x) = bx^2 + a$

$V_4(x) = \left(x^3 + x\right)b + ax$

$V_5(x) = \left(x^4 + 2x^2\right)b + \left(x^2 + 1\right)a$

$V_6(x) = \left(x^5 + 3x^3 + x\right)b + \left(x^3 + 2x\right)a$

…

If $a = b = 1$, then

$f_n(x) = x\, f_{n-1}(x) + f_{n-2}(x)\ ;\ with\ f_1(x) = 1,\ f_2(x) = x$ (Fibonacci polynomials)

If $a = 2$, $b = 1$, then

$l_{n-1}(x) = x\, l_{n-2}(x) + l_{n-3}(x)\ ;\ with\ l_0(x) = 2,\ l_1(x) = x$ (Lucas polynomials)

If $a = 1$, $b = 2$, then

$P_n(x) = 2x\, P_{n-1}(x) + P_{n-2}(x)\ ;\ with\ P_1(x) = 1,\ P_2(x) = 2x$ (Pell polynomials)

If $a = b = 2$, then

$Q_{n-1}(x) = 2xQ_{n-2}(x) + Q_{n-3}(x); with\, Q_0(x) = 2\,, Q_1(x) = 2x$ (Pell-Lucas polynomials)

Now we define a family of Fibonacci-Like polynomial as

$$V=\{V_{n+k}(x),V_{n+i}(x),V_{n+m}(x),V_{n+m+j}(x),V_{n+m+i}(x)\},$$

Where n, i, j, k, m, are positive integers with $0<k<i, i+1<m, j=1$.
Then Generalized Fibonacci-Like polynomials are

$$V_{n+i+j}(x)=xV_{n+i}(x)+V_{n+k}(x) \qquad [2.2]$$

$$V_{n+m+j}(x)=xV_{n+m}(x)+V_{n+i}(x) \qquad [2.3]$$

$$V_{n+m+i}(x)=xV_{n+m+j}(x)+V_{n+m}(x) \qquad [2.4]$$

If $(a,b)=(1,1)$, *then* $V_{n+i+j}(x)=f_{n+i+j}(x)$, the Generalized Fibonacci Polynomials.
If $(a,b)=(2,1)$, *then* $V_{n+i+j}(x)=l_{n+i+j}(x)$, the Generalized Lucas Polynomials.
If $(a,b)=(1,2)$, *then* $V_{n+i+j}(x)=P_{n+i+j}(x)$, the Generalized Pell Polynomials.
If $(a,b)=(2,2)$, *then* $V_{n+i+j}(x)=Q_{n+i+j}(x)$, the Generalized Pell-Lucas Polynomials.
If $(x,a,b)=(1,1,1)$, *then* $V_{n+i+j}(1)=F_{n+i+j}$, the Generalized Fibonacci numbers.
If $(x,a,b)=(1,2,1)$, *then* $V_{n+i+j}(1)=L_{n+i+j}$, the Generalized Lucas numbers.
If $(x,a,b)=(1,1,2)$, *then* $V_{n+i+j}(1)=P_{n+i+j}$, the Generalized Pell numbers.
If $(x,a,b)=(1,2,2)$, *then* $V_{n+i+j}(1)=Q_{n+i+j}$, the Generalized Pell-Lucas numbers.

3. DETERMINANTAL IDENTITIES

Now we present determinantal identities
Theorem 1: If n, i, j, k, m, are positive integers with $0<k<i, i+1<m, j=1$, then

$$\begin{vmatrix} \dfrac{b^2V_{n+k}^2(x)+a^2V_{n+i}^2(x)}{V_{n+i+j}(x)} & V_{n+i+j}(x) & V_{n+i+j}(x) \\ bV_{n+k}(x) & \dfrac{a^2V_{n+i}^2(x)+V_{n+i+j}^2(x)}{bV_{n+K}(x)} & bV_{n+k}(x) \\ aV_{n+i}(x) & aV_{n+i}(x) & \dfrac{V_{n+i+j}^2(x)+b^2V_{n+k}^2(x)}{aV_{n+i}(x)} \end{vmatrix} = 4abV_{n+i}(x)V_{n+k}(x)V_{n+i+j}(x)$$

Proof: Let

$$\Delta=\begin{vmatrix} \dfrac{b^2V_{n+k}^2(x)+a^2V_{n+i}^2(x)}{V_{n+i+j}(x)} & V_{n+i+j}(x) & V_{n+i+j}(x) \\ bV_{n+k}(x) & \dfrac{a^2V_{n+i}^2(x)+V_{n+i+j}^2(x)}{bV_{n+K}(x)} & bV_{n+k}(x) \\ aV_{n+i}(x) & aV_{n+i}(x) & \dfrac{V_{n+i+j}^2(x)+b^2V_{n+k}^2(x)}{aV_{n+i}(x)} \end{vmatrix} \qquad [3.1]$$

Assume $bV_{n+k}(x) = \alpha$, $aV_{n+i}(x) = \beta$, then by [2.1] $V_{n+i+j}(x) = \alpha + x\beta$, Now

$$\Delta = \begin{vmatrix} \frac{\alpha^2+\beta^2}{\alpha+x\beta} & \alpha + x\beta & \alpha + x\beta \\ \alpha & \frac{\beta^2+(\alpha+x\beta)^2}{\alpha} & \alpha \\ \beta & \beta & \frac{\alpha^2+(\alpha+x\beta)^2}{\beta} \end{vmatrix} \quad [3.2]$$

Multiplying and divided R_1 by $(\alpha + x\beta), R_2$ by α, R_3 by β by

$$\Delta = \frac{1}{\alpha\beta(\alpha+x\beta)} \begin{vmatrix} \alpha^2+\beta^2 & (\alpha+x\beta)^2 & (\alpha+x\beta)^2 \\ \alpha^2 & \beta^2+(\alpha+x\beta)^2 & \alpha^2 \\ \beta^2 & \beta^2 & \alpha^2+(\alpha+x\beta)^2 \end{vmatrix} \quad [3.3]$$

Applying $R_1 \to R_1 + R_2 + R_3$ & $R_2 - R_1 \to R_2$, $R_3 - R_1 \to R_3$

$$\Delta = \frac{2}{\alpha\beta(\alpha+x\beta)} \begin{vmatrix} \alpha^2+\beta^2 & \beta^2+(\alpha+x\beta)^2 & \alpha^2+(\alpha+x\beta)^2 \\ -\beta^2 & 0 & -(\alpha+x\beta)^2 \\ -\alpha^2 & -(\alpha+x\beta)^2 & 0 \end{vmatrix} \quad [3.4]$$

Expand along first row, we get

$$\Delta = 4abV_{n+i}(x)V_{n+k}(x)V_{n+i+j}(x) \quad [3.5]$$

Put $bV_{n+k}(x) = \alpha$, $aV_{n+i}(x) = \beta$, $V_{n+i+j}(x) = \alpha + x\beta$

Theorem 2: If n, i, j, k, m, are positive integers with $0 < k < i, i+1 < m, j = 1$, then

$$\begin{vmatrix} b^2V_{n+k}^2(x) & aV_{n+i}(x)V_{n+i+j}(x) & bV_{n+k}(x)V_{n+i+j}(x)+V_{n+i+j}(x) \\ b^2V_{n+k}^2(x)+abV_{n+i}(x)V_{n+k}(x) & a^2V_{n+i}^2(x) & bV_{n+k}(x)V_{n+i+j}(x) \\ abV_{n+i}(x)V_{n+k}(x) & a^2V_{n+i}^2(x)+aV_{n+i}(x)V_{n+i+j}(x) & V_{n+i+j}^2(x) \end{vmatrix} = \left\{2abV_{n+i}(x)V_{n+k}(x)V_{n+i+j}(x)\right\}^2$$

Proof: Let

$$\Delta = \begin{vmatrix} b^2V_{n+k}^2(x) & aV_{n+i}(x)V_{n+i+j}(x) & bV_{n+k}(x)V_{n+i+j}(x)+V_{n+i+j}(x) \\ b^2V_{n+k}^2(x)+abV_{n+i}(x)V_{n+k}(x) & a^2V_{n+i}^2(x) & bV_{n+k}(x)V_{n+i+j}(x) \\ abV_{n+i}(x)V_{n+k}(x) & a^2V_{n+i}^2(x)+aV_{n+i}(x)V_{n+i+j}(x) & V_{n+i+j}^2(x) \end{vmatrix} \quad [3.6]$$

Assume $bV_{n+k}(x) = \alpha$, $aV_{n+i}(x) = \beta$, then by [1.1] $V_{n+i+j}(x) = \alpha + x\beta$, Now

$$\Delta = \begin{vmatrix} \alpha^2 & \beta(\alpha+x\beta) & \alpha(\alpha+x\beta)+(\alpha+x\beta)^2 \\ \alpha^2+\alpha\beta & \beta^2 & \alpha(\alpha+x\beta) \\ \alpha\beta & \beta^2+\beta(\alpha+x\beta) & (\alpha+x\beta)^2 \end{vmatrix} \quad [3.7]$$

Taking $\alpha, \beta, (\alpha + x\beta)$ common from C_1, C_2, C_3 & Applying $R_2 \to R_2 - (R_1 + R_3)$

$$\Delta = \alpha\beta(\alpha + x\beta)\begin{vmatrix} \alpha & (\alpha + x\beta) & \alpha + (\alpha + x\beta) \\ 0 & -2(\alpha + x\beta) & -2(\alpha + x\beta) \\ \beta & \beta + (\alpha + x\beta) & (\alpha + x\beta) \end{vmatrix} \qquad [3.8]$$

Applying $C_2 \to C_2 - C_3$ & Expansion by R_2

$$\Delta = (2\alpha\beta(\alpha + x\beta))^2 \qquad [3.9]$$

Put $bV_{n+k}(x) = \alpha$, $aV_{n+i}(x) = \beta$, $V_{n+i+j}(x) = \alpha + x\beta$, we get

$$\Delta = \left\{2abV_{n+i}(x)V_{n+k}(x)V_{n+i+j}(x)\right\}^2 \qquad [3.10]$$

Theorem 3: If n, i, j, k, m, are positive integers with $0 < k < i, i+1 < m, j = 1$, then

$$\begin{vmatrix} -b^2V_{n+k}^2(x) & abV_{n+i}(x)V_{n+k}(x) & bV_{n+k}(x)V_{n+i+j}(x) \\ abV_{n+i}(x)V_{n+k}(x) & -a^2V_{n+i}^2(x) & aV_{n+i}(x)V_{n+i+j}(x) \\ bV_{n+k}(x)V_{n+i+j}(x) & aV_{n+i}(x)V_{n+i+j}(x) & -V_{n+i+j}^2(x) \end{vmatrix} = \left\{2abV_{n+i}(x)V_{n+k}(x)V_{n+i+j}(x)\right\}^2 \qquad [3.11]$$

Theorem 4: If n, i, j, k, m, are positive integers with $0 < k < i, i+1 < m, j = 1$, then

$$\begin{vmatrix} a^2V_{n+i}^2(x)+V_{n+i+j}^2(x) & abV_{n+i}(x)V_{n+k}(x) & bV_{n+k}(x)V_{n+i+j}(x) \\ abV_{n+i}(x)V_{n+k}(x) & b^2V_{n+k}^2(x) & aV_{n+i}(x)V_{n+i+j}(x) \\ bV_{n+k}(x)V_{n+i+j}(x) & aV_{n+i}(x)V_{n+i+j}(x) & a^2V_{n+i}^2(x)+b^2V_{n+k}^2(x) \end{vmatrix} = \left\{2abV_{n+i}(x)V_{n+k}(x)V_{n+i+j}(x)\right\}^2 \qquad [3.12]$$

Theorem 5: If n, i, j, k, m, are positive integers with $0 < k < i, i+1 < m, j = 1$, then

$$\begin{vmatrix} 2V_{n+i+j}(x)+aV_{n+i}(x)+bV_{n+k}(x) & bV_{n+k}(x) & aV_{n+i}(x) \\ V_{n+i+j}(x) & 2bV_{n+k}(x)+aV_{n+i}(x)+V_{n+i+j}(x) & aV_{n+i}(x) \\ V_{n+i+j}(x) & bV_{n+k}(x) & 2aV_{n+i}(x)+bV_{n+k}(x)+V_{n+i+j}(x) \end{vmatrix}$$

$$= 2\left\{aV_{n+i}(x) + bV_{n+k}(x) + V_{n+i+j}(x)\right\}^3 \qquad [3.13]$$

Theorem 6: If n, i, j, k, m, are positive integers with $0 < k < i, i+1 < m, j = 1$, then

$$\begin{vmatrix} 1+bV_{n+k}(x) & 1 & 1 \\ 1 & 1+aV_{n+i}(x) & 1 \\ 1 & 1 & 1+V_{n+i+j}(x) \end{vmatrix} = \left\{abV_{n+i}(x)V_{n+k}(x)V_{n+i+j}(x)\right\}\left\{\frac{1}{bV_{n+k}(x)} + \frac{1}{aV_{n+i}(x)} + \frac{1}{V_{n+i+j}(x)} + 1\right\}$$

$$= \left\{abV_{n+i}(x)V_{n+k}(x)V_{n+i+j}(x) + aV_{n+i}(x)V_{n+i+j}(x) + bV_{n+k}(x)V_{n+i+j}(x) + abV_{n+i}(x)V_{n+k}(x)\right\} \qquad [3.14]$$

Above Theorems 3 to 6 can be solved same as Theorem: 1.

4. CONCLUSION

This paper describes Generalized Fibonacci-Like polynomials and its determinantal identities. Also results derived in terms of classical polynomials like Fibonacci Polynomial, Lucas Polynomials, Pell Polynomials and Pell-Lucas Polynomials.

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References

[1] Alexandru Lupas, A Guide of Fibonacci and Lucas *Polynomial*, Octagon Mathematics Magazine, Vol. 7, No.1 (1999), 2-12.

[2] A. Benjamin, J. Quinn and N. Cameron, Fibonacci Determinants- A Combinatorial Approach, Fibonacci Quarterly, 45, No. 1 (2007), 39-55.

[3] A. F. Horadam and Bro. J. M. Mahon, Pell and Pell-Lucas Polynomials, Fibonacci Quarterly, 23, No. 1 (1985), 17-20.

[4] A. J. Macfarlane, Use of Determinants to Present Identities Involving Fiboncci and Related Numbers, Fibonacci Quarterly, 48, No. 1 (2010), 68-76.

[5] B. Singh, O. Sikhwal, Y.K.Panwar, Generalized Determinantal Identities Involving Lucas Polynomials, Applied Mathematical Sciences, Vol. 3 (2009), No. 8, 377-388.

[6] F. Koken and D. Bozkurt, On the Jacobsthaal Numbers by Matrix Methods, Int. J. Contemp. Math. Sciences, 3(2008), 605-614.

[7] N. Cahill and D. Narayan, Fibonacci and Lucas numbers s Tridigonal Matrix Determinant, Fibonacci Quarterly, 42, (2004), 216-221.

[8] Z. Spivey, Fibonacci Identities via the Determinant sum property, College Mathematics Journal, 37(2006), 286-289.

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