International Mathematical Forum, Vol. 8, 2013, no. 15, 701 - 706
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A Generalization of Hermite Polynomials

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Abstract

The intended objective of this paper is to extend the Hermite polynomials based on hypergeometric functions and to prove basic properties of the extended Hermite polynomials.

Mathematics Subject Classification: 33C45, 05A15, 11B37.

Keywords: Orthogonal polynomials, Hermite polynomials, Hypergeometric functions**,** Generalized hypergeometric functions.

1. Introduction

Hermite Polynomials and its applications have been studied for long and still attract attention. One can refer a long list of books and journals for advanced knowledge of Hermite polynomials and its extensions, for example [7] and [6], for books and [2], [4] , [5], [7], [8], [9], [10] and [11] for journals. Based on a generalized hypergeometric function, we introduce here a generalization of the Hermite polynomials that provide natural extensions of basic results involving the Hermite polynomials as a study of the Laguerre polynomials in [3].

For a positive integer p, the set $\{S_{p,n}(x)\}$ generated by the function $G(x,t)=\exp\left(pxt-t^{p}\right)$ is to be known as the generalized Hermite polynomial set. Note that for $p=2$, it reduces to the known generating function for the Hermite polynomials. We first deduce an explicit expression for this generalized Hermite polynomials.

Theorem 1:

For a non-negative integer n and a positive integer p, we have

$$S_{p,n}(x)=\sum_{k=0}^{\left[\frac{n}{p}\right]}\frac{(-1)^{k}\,n!(px)^{n-pk}}{k!(n-pk)!}. \tag{1.1}$$

Proof:

By considering

$$\sum_{n=0}^{\infty}\frac{S_{p,n}(x)t^{n}}{n!}=e^{\left(pxt-t^{p}\right)},$$

note that

$$e^{\left(pxt-t^{p}\right)}=\left(\sum_{n=0}^{\infty}\frac{(pxt)^{n}}{n!}\right)\left(\sum_{k=0}^{\infty}\frac{\left(-t^{p}\right)^{k}}{k!}\right)$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{k}}{k!}\frac{(px)^{n}\,t^{n+pk}}{n!}.$$

A use of a variation of Lemma 11 pp. 57 of [6] with $\frac{n}{p}$ in place of $\frac{n}{2}$ leads to

$$\sum_{n=0}^{\infty}\frac{S_{p,n}(x)t^{n}}{n!}=\sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{p}\right]}\frac{(-1)^{k}\,(px)^{n-pk}\,t^{n}}{k!(n-pk)!},$$

which implies that

$$S_{p,n}(x)=\sum_{k=0}^{\left[\frac{n}{p}\right]}\frac{(-1)^{k}\,n!(px)^{n-pk}}{k!(n-pk)!}.$$

We now determine a few recurrence relations for the generalized Hermite polynomials.

Theorem 2:

For all finite x, t, a positive integer p and a non-negative integer n,

(i) $xS'_{p,n}(x) - nS_{p,n}(x) = n(n-1)(n-2)\cdots(n-p+2)S'_{p,n-p+1}(x)$, (1.2)

(ii) $np(n-p+2)S_{p,n}(x)$

$$= px(n-p+2)S'_{p,n}(x) - n(n-1)(n-2)\cdots(n-p+1)S''_{p,n-p+2}(x) \cdot \tag{1.3}$$

Proof:

Consider $F = G\left(pxt - t^p\right) = \sum_{n=0}^{\infty} f_n(x)t^n$.

Taking partial derivatives of F w.r.t x and t, we have

$$\frac{\partial F}{\partial x} = ptG', \ \frac{\partial F}{\partial t} = \left(px - pt^{p-1}\right)G'.$$

Also, $\sum_{n=0}^{\infty} xf'_n(x)t^n - \sum_{n=0}^{\infty} f'_n(x)t^{p+n-1} - \sum_{n=0}^{\infty} nf_n(x)t^n = 0$.

These relations give rise to

$$\left(x - t^{p-1}\right)\frac{\partial F}{\partial x} - t\frac{\partial F}{\partial t} = 0 \ , \tag{1.4}$$

and consequently

$$xf'_n(x) - nf_n(x) = f'_{n-p+1}(x).$$

Since by taking $G = e^{pxt - t^p}$, $f_n(x) = \dfrac{S_{p,n}(x)}{n!}$, we finally get

$$xS'_{p,n}(x) - nS_{p,n}(x) = n(n-1)(n-2)\cdots(n-p+2)S'_{p,n-p+1}(x).$$

Similarly, we can prove (ii).

Theorem 3:

For any real number c and a positive integer p, we have

$$(i)\sum_{n=0}^{\infty}\frac{(c)_n S_{p,n}(x)t^n}{n!}=(1-pxt)^{-c}\times$$

$${}_pF_0\left(\frac{c}{p},\frac{c+1}{p},\frac{c+2}{p},\ldots,\frac{c+p-1}{p};-\;;(-1)\left(\frac{pt}{1-pxt}\right)^p\right),\quad (1.5)$$

$$(ii)S_{p,n}(x)=(px)^n\,{}_pF_0\left(\frac{-n}{p},\frac{-n+1}{p},\frac{-n+2}{p},\ldots,\frac{-n+p-1}{p};-;\frac{-1}{x^p}\right),\quad (1.6)$$

$$(iii)x^n=\sum_{k=0}^{\left[\frac{n}{p}\right]}\frac{n!S_{p,n-pk}(x)}{p^n(n-pk)!k!}.\qquad (1.7)$$

Proof:

Note that

$$\sum_{n=0}^{\infty}\frac{(c)_n S_{p,n}(x)t^n}{n!}=\sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{p}\right]}\frac{(-1)^k(c)_n(px)^{n-pk}}{k!(n-pk)!}t^n$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k(c)_{n+pk}(px)^n t^{n+pk}}{k!n!}$$

$$=\sum_{k=0}^{\infty}\left(\sum_{n=0}^{\infty}\frac{(c+pk)_n(pxt)^n}{n!}\right)\left(\frac{(-1)^k(c)_{pk}t^{pk}}{k!}\right),$$

so that

$$\sum_{n=0}^{\infty}\frac{(c)_n S_{p,n}(x)t^n}{n!}=\sum_{k=0}^{\infty}\frac{(-1)^k(c)_{pk}t^{pk}}{k!(1-pxt)^{c+pk}},$$

which implies that

$$\sum_{n=0}^{\infty}\frac{(c)_n S_{p,n}(x)t^n}{n!}=(1-pxt)^{-c}\times$$

$${}_pF_0\left(\frac{c}{p},\frac{c+1}{p},\frac{c+2}{p},\ldots,\frac{c+p-1}{p};-\;;(-1)\left(\frac{pt}{1-pxt}\right)^p\right).$$

Starting with $S_{p,n}(x) = (px)^n \sum_{k=0}^{\left[\frac{n}{p}\right]} \frac{n!(-1)^k}{(n-pk)!} \frac{(p)^{-pk}(x)^{-pk}}{k!}$, and

$e^{pxt} = e^{t^p} \sum_{n=0}^{\infty} \frac{S_{p,n}(x)t^n}{n!}$, we can prove (ii) and (iii).

Following traditional theory, we can prove orthogonality, integrals and expansions involving the Hermite polynomials and its relations with other polynomials. We can also consider $q-$Hermite polynomials and prove corresponding results.

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Received: January 11, 2013