Arab Journal of Mathematical Sciences (2011) 17, 31–44

King Saud University
Arab Journal of Mathematical Sciences
www.ksu.edu.sa
www.sciencedirect.com

ORIGINAL ARTICLE

On the recursive sequence $x_{n+1} = \frac{a + bx_n}{A + Bx_{n-1}^k}$

Alaa E. Hamza [a,*], A.M. Ahmed [b], A.M. Youssef [b]

[a] *Department of Mathematics, Faculty of Science, Cairo University, Giza 12211, Egypt*
[b] *Department of Natural Science, Community College, King Saud University, Malaz, P.O. Box 28095, Riyadh 11437, Saudi Arabia*

Received 29 September 2010; accepted 23 October 2010
Available online 17 December 2010

KEYWORDS
Difference equations;
Recursive sequences;
Global asymptotic stability

Abstract In this paper, we investigate the global behavior and boundedness of the difference equation

$$x_{n+1} = \frac{a + bx_n}{A + Bx_{n-1}^k}, \quad n = 0, 1, \ldots,$$

with positive coefficients and non-negative initial conditions.
© 2011 King Saud University. Production and hosting by Elsevier B.V. All rights reserved.

1. Introduction and preliminaries

Recently there has been great interest in studying the behavior of rational and non-rational nonlinear difference equations. We believe that the results about

* Corresponding author.
E-mail addresses: hamzaaeg2003@yahoo.com (A.E. Hamza), ahmedelkb@yahoo.com (A.M. Ahmed), amr_youssef@hotmail.com (A.M. Youssef).

1319-5166 © 2011 King Saud University. Production and hosting by Elsevier B.V. All rights reserved.

Peer review under responsibility of King Saud University.
doi:10.1016/j.ajmsc.2010.12.002

Production and hosting by Elsevier

second order rational difference equations are of paramount importance in their own right, and furthermore we believe that these results offer a prototype towards the development of the basic theory of the global behavior of solutions of non-linear difference equations of order greater than one. Many authors studied the global behavior of the recursive sequence

$$x_{n+1} = \frac{a + bx_n}{A + Bx_{n-1}}, \quad n = 0, 1, \ldots, \tag{1.1}$$

where $a,\ b,\ A$, and B are nonnegative real numbers (see Jaroma et al., 1995; Kocic and Ladas, 1993; Kocic et al., 1993). Eq. (1.1) is a very simple looking equation for which it has long been conjectured that its equilibrium is globally asymptotically stable. To this day, the conjecture has not been proven or refuted. Also, Hamza and El-Sayed (1998) studied the stability of the recursive sequence

$$x_{n+1} = \frac{\alpha + \beta x_n^2}{1 + \gamma x_{n-1}}, \quad n = 0, 1, \ldots, \tag{1.2}$$

where $\alpha \geqslant 0$ and $\beta, \gamma > 0$. For related results see Berg (2002), El-Owaidy et al. (2005a,b), Gibbsons et al. (2000), Jaroma et al. (1995), Aboutaleb et al. (2001), Kelly and Peterson (1991), Kocic and Ladas (1993), Kocic et al. (1993), Kulenović and Ladas (2002) and Stević (2001, 2002a,b,c, 2003).

In this paper we generalize the results due to Eq. (1.1) to the rational difference equation

$$x_{n+1} = \frac{a + bx_n}{A + Bx_{n-1}^k}, \quad n = 0, 1, \ldots,$$

where $a,\ b,\ A,\ B$ and k are positive real numbers with non-negative initial conditions such that

$$A + Bx_{n-1}^k > 0, \quad \forall n \geqslant 0.$$

Let I be some interval of real numbers and let

$$f : I \times I \to I$$

be a continuously differentiable function. Then for every set of initial conditions $\{x_0, x_{-1}\} \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \ldots, \tag{1.3}$$

has a unique solution $\{x_n\}_{n=-1}^{\infty}$.

Definition 1.1. A point $\overline{x} \in I$ is called an equilibrium point of Eq. (1.3) if

$$\overline{x} = f(\overline{x}, \overline{x}),$$

or equivalently, $\overline{x}$ is a fixed point of $g(x) = f(x, x)$.

Definition 1.2. Let $\bar{x}$ be an equilibrium point of Eq. (1.3), then we have:

(i) The equilibrium point $\bar{x}$ of Eq. (1.3) is called locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that $x_{-1}, x_0 \in I$ with

$$|x_0 - \bar{x}| + |x_{-1} - \bar{x}| < \delta,$$

then we have

$$|x_n - \bar{x}| < \epsilon \quad \text{for all } n \geqslant -1.$$

(ii) The equilibrium point $\bar{x}$ of Eq. (1.3) is called locally asymptotically stable if it is locally stable, and if there exists $\gamma > 0$ such that $x_{-1}, x_0 \in I$ with

$$|x_0 - \bar{x}| + |x_{-1} - \bar{x}| < \gamma,$$

then we have

$$\lim_{n\to\infty} x_n = \bar{x}.$$

(iii) The equilibrium point $\bar{x}$ of Eq. (1.3) is called a global attractor if for every $x_{-1}, x_0 \in I$ we have

$$\lim_{n\to\infty} x_n = \bar{x}.$$

(iv) The equilibrium point $\bar{x}$ of Eq. (1.3) is called globally asymptotically stable if it is locally asymptotically stable and a global attractor.

(v) The equilibrium point $\bar{x}$ of Eq. (1.3) is unstable if $\bar{x}$ is not locally stable.

Let

$$p = \frac{\partial f}{\partial u}(\bar{x}, \bar{x}) \quad \text{and} \quad q = \frac{\partial f}{\partial v}(\bar{x}, \bar{x})$$

denote the partial derivatives of $f(u, v)$ evaluated at the equilibrium point $\bar{x}$ of Eq. (1.3), i.e. $\bar{x} = f(\bar{x}, \bar{x})$. Then the equation

$$y_{n+1} = py_n + qy_{n-1}, \quad n = 0, 1, \ldots \tag{1.4}$$

is called the *linearized equation associated with Eq. (1.3), about the equilibrium point* $\bar{x}$. Then its characteristic equation is

$$\lambda^2 - p\lambda - q = 0. \tag{1.5}$$

We need the following theorems.

Theorem 1.1 (Linearized Stability (Kulenović and Ladas, 2002)).

(*a*) *If both roots of the quadratic Eq.* (1.5) *lie in the open unit disk* $|\lambda| < 1$, *then the equilibrium point* $\bar{x}$ *of Eq.* (1.3) *is locally asymptotically stable.*

(*b*) *If at least one of the roots of Eq.* (1.5) *has absolute value greater than one, then the equilibrium point* $\bar{x}$ *of Eq.* (1.3) *is unstable.*

(*c*) *A necessary and sufficient condition for both roots of Eq.* (1.5) *to lie in the open unit disk* $|\lambda| < 1$, *is*

$|p| < 1 - q < 2.$

In this case the locally asymptotically stable equilibrium $\overline{x}$ is also called a sink.

(*d*) *A necessary and sufficient condition for both roots of Eq. (1.5) to have absolute value greater than one is*

$|q| > 1 \quad \text{and} \quad |p| < |1 - q|.$

In this case $\overline{x}$is a repeller.

(*e*) *A necessary and sufficient condition for one root of Eq. (1.5) to have absolute value greater than one and for the other to have absolute value less than one is*

$p^2 + 4q > 0 \quad \text{and} \quad |p| > |1 - q|.$

In this case the unstable equilibrium point $\overline{x}$ is called a saddle point.

(*f*) *A necessary and sufficient condition for a root of Eq. (1.5) to have absolute value equal to one is*

$|p| = |1 - q|,$

or

$q = -1 \quad \textit{and} \quad |p| \leqslant 2.$

In this case the equilibrium point $\overline{x}$ is called a non-hyperbolic point.

For this issue, we refer the reader to Elaydi (1999), Kelly and Peterson (1991), Kocic and Ladas (1993) and Kulenović and Ladas (2002).

2. Stability analysis

In this paper we consider the following recursive sequence

$$x_{n+1} = \frac{a + bx_n}{A + Bx_{n-1}^k}, \quad n = 0, 1, \ldots, \tag{2.1}$$

where a, b, A, B and k are positive real numbers.

By changing the variables $x_n = \sqrt[k]{b/B}y_n$, Eq. (2.1) is reduced to

$$y_{n+1} = \frac{s + y_n}{r + y_{n-1}^k}, \quad n = 0, 1, \ldots, \tag{2.2}$$

where $s = a/b\sqrt[k]{B/p}$ and $r = A/b$.

We summarize the results of this section in the following three theorems.

Theorem 2.1. *The following statements are true:*

(*1*) *Assume that $r > 1$. Then Eq. (2.2) has a unique equilibrium point in $(0, \frac{s}{r-1})$.*

(*2*) *Assume that $r < 1$. Then we have:*

  (*i*) *If $r \geqslant s$, then Eq. (2.2) has a unique equilibrium point in $(s, 1]$;*

  (*ii*) *If $r < s$, then Eq. (2.2) has a unique equilibrium point in $(1, \frac{s}{r})$.*

(*3*) *Assume that $r = 1$. Then $\overline{y} = \sqrt[k+1]{s}$ is an equilibrium of Eq. (2.2).*

Proof. A point $\overline{y}$ is an equilibrium point of Eq. (2.2) if and only if $\overline{y}$ is a root of the function

$$f(x) = x^{k+1} + (r-1)x - s. \tag{2.3}$$

(1) Let $r > 1$, then $f(0) = -s$ and $f(\frac{s}{r-1}) = (\frac{s}{r-1})^{k+1} > 0$, whence $f(x)$ has a root in $(0, \frac{s}{r-1})$.
(2) Let $r < 1$.
 (i) Assume that $r \geqslant s$. Then $f(s) < 0$ and $f(1) = r - s \geqslant 0$, whence $f(x)$ has a root in $[s, 1]$.
 (ii) Assume that $r < s$, then $f(1) < 0$ and $f(\frac{s}{r}) = \frac{s^{k+1} - sr^k}{r^{k+1}} > 0$, whence $f(x)$ has a root in $(1, \frac{s}{r})$.
 The uniqueness of the equilibrium point in cases (1) and (2) is obvious.
(3) Let $r = 1$, it is obvious that $\overline{y} = \sqrt[k+1]{s}$ is the unique equilibrium point of Eq. (2.2). □

In the sequel $\overline{y}$ denotes the unique equilibrium point of Eq. (2.2). In the following Theorem we determine the conditions under which $\overline{y}$ is locally asymptotically stable and unstable.

Lemma 2.1. *The following statements are true:*

(*1*) *If* $r > (k-1)\overline{y}^k$, *then the equilibrium point* $\overline{y}$ *is locally asymptotically stable.*
(*2*) *If* $r < (k-1)\overline{y}^k$, *then the equilibrium point* $\overline{y}$ *is unstable, in fact a repeller.*
(*3*) *If* $r = (k-1)\overline{y}^k$, *then the equilibrium point* $\overline{y}$ *is a non-hyperbolic point.*

Proof. The characteristic equation of the associated linearized Eq. (2.2) is

$$\lambda^2 = p\lambda + q,$$

where

$$p = \frac{1}{r + \overline{y}^k} \quad \text{and} \quad q = \frac{-k\overline{y}^k}{(r + \overline{y}^k)}.$$

The results follow directly by applying the Linearized Stability Theorem 1.1. □

In the following Theorem we determine more precisely necessary conditions (on parameters) for $\overline{y}$ to be locally asymptotically stable and for $\overline{y}$ to be unstable.

Theorem 2.2. *The following statements are true:*

(*1*) *Assume that* $k \leqslant 1$. *Then the equilibrium point* $\overline{y}$ *is locally asymptotically stable.*
(*2*) *Assume that* $k > 1$. *Then we have:*

(a) *If* $k(r-1)+1 \leqslant 0$, *then* $\overline{y}$ *is unstable;*
(b) *If* $k(r-1)+1 > 0$, *then*
  (i) $s < r^{\frac{1}{k}}[k(r-1)+1]/(k-1)^{\frac{k+1}{k}} \Rightarrow \overline{y}$ *is locally asymptotically stable;*
  (ii) $s > r^{\frac{1}{k}}[k(r-1)+1]/(k-1)^{\frac{k+1}{k}} \Rightarrow \overline{y}$ *is unstable, in fact a repeller;*
  (iii) $s = r^{\frac{1}{k}}[k(r-1)+1]/(k-1)^{\frac{k+1}{k}} \Rightarrow \overline{y}$ *is a non-hyperbolic point.*

Proof

(1) It is clear that for $k \leqslant 1$, we have $(k-1)\overline{y}^k < r$, so, by Lemma (2.1) $\overline{y}$ is locally asymptotically stable.
(2) Now assume that $k > 1$.
(a) Assume that $k(r-1)+1 = 0$, so $k-1 = \frac{r}{1-r}$. Hence, $(k-1)\overline{y}^k > r$, and $\overline{y}$ is unstable, in fact a repeller. Now assume that $k(r-1)+1 < 0$. Then

$$\frac{s}{\overline{y}} > \frac{k(r-1)+1}{k-1}.$$

Therefore, $(k-1)\overline{y}^k > r$.
(b) Assume that $k(r-1)+1 > 0$. It is easy to show that

$$(k-1)\overline{y}^k < r \Longleftrightarrow \overline{y} > \frac{s(k-1)}{k(r-1)+1};$$

$$(k-1)\overline{y}^k > r \Longleftrightarrow \overline{y} < \frac{s(k-1)}{k(r-1)+1};$$

$$(k-1)\overline{y}^k = r \Longleftrightarrow \overline{y} = \frac{s(k-1)}{k(r-1)+1}.$$

We have,

$$f\left(\frac{s(k-1)}{k(r-1)+1}\right) = \frac{s}{[k(r-1)+1]}\left[\frac{s^k(k-1)^{k+1}}{[k(r-1)+1]^k} - r\right],$$

where $f(x)$ is defined in (2.3).
(i) If $s < r^{\frac{1}{k}}[k(r-1)+1]/(k-1)^{\frac{k+1}{k}}$, then $\frac{s(k-1)}{k(r-1)+1} < \overline{y}$, consequently $(k-1)\overline{y}^k < r$. Hence $\overline{y}$ is locally asymptotically stable.
(ii) If $s > r^{\frac{1}{k}}[k(r-1)+1]/(k-1)^{\frac{k+1}{k}}$, then $\frac{s(k-1)}{k(r-1)+1} > \overline{y}$ and $(k-1)\overline{y}^k > r$, hence $\overline{y}$ is unstable, in fact $\overline{y}$ is a repeller.
(iii) If $s = r^{\frac{1}{k}}[k(r-1)+1]/(k-1)^{\frac{k+1}{k}}$, then $\frac{s(k-1)}{k(r-1)+1} = \overline{y}$, and $\overline{y}$ is a non-hyperbolic point of Eq. (2.2). □

The following result is very useful in studying the global attractivity. By an invariant interval I of a real function $G(x,y)$ we mean that $G(x,y) \in I,\ \forall x,y \in I$.

Theorem 2.3 (Cunningham et al., 2001; Devault et al., 2001). *Assume that $G(x,y)$ is a continuous function which is non-decreasing (non-increasing) in x for each y and non-increasing (non-decreasing) in y for each x. Assume that every solution of the equation*

$$y_{n+1} = G(y_n, y_{n-k}), \quad n = 0, 1, \ldots \tag{2.4}$$

has an inferior limit λ and superior limit Λ such that λ and Λ belong to an invariant interval $I = [a,b]$ under G. Let $\overline{y}$ be a unique equilibrium point in I. If the system

$$x = G(x,y) \quad \text{and} \quad y = G(y,x) \tag{2.5}$$

$$(x = G(y,x) \quad \text{and} \quad y = G(x,y)) \tag{2.6}$$

has exactly one solution in I^2, then $\overline{y}$ is a global attractor.

Proof. Let $\{y_n\}_{n=-1}^{\infty}$ be a solution of (2.4) with initial conditions $y_{-k}, y_{-k+1}, \ldots, y_0 \in I$, $\lambda = \lim_{n\to\infty} \inf y_n$ and $\Lambda = \lim_{n\to\infty} \sup y_n$. Assume that $G(x,y)$ is non-decreasing (non-increasing) in x for each y and non-increasing (non-decreasing) in y for each x. Take $U_1 = G(\Lambda, \lambda)$ $(U_1 = G(\lambda, \Lambda))$ and $L_1 = G(\lambda, \Lambda)$ $(L_1 = G(\Lambda, \lambda))$. For every $\epsilon \in (0, \lambda - a)$, $\exists n_0 \in \mathbb{N}$ such that

$$\lambda - \epsilon < y_n < \Lambda + \in, \quad \forall n \geqslant n_0.$$

Then

$$L_1 \leqslant \lambda \leqslant \Lambda \leqslant U_1.$$

Set $U_{n+1} = G(U_n, L_n)(U_{n+1} = G(L_n, U_n))$ and $L_{n+1} = G(L_n, U_n)(L_{n+1} = G(U_n, L_n))$, $n = 1, 2, \ldots$ One can see that

$$a \leqslant \cdots \leqslant L_2 \leqslant L_1 \leqslant \lambda \leqslant \Lambda \leqslant U_1 \leqslant U_2 \leqslant \cdots \leqslant b.$$

Hence $\{U_n\}$ is monotonically increasing to a number, say $U \in I$, and $\{L_n\}$ is monotonically decreasing to a number, say $L \in I$. This implies that $(U, L) \in I^2$ is a solution of the system (2.5) and (2.6). Therefore, $U = L = \overline{y} = \lambda = \Lambda$. □

Corollary 2.1. *Assume that $G(x,y)$ is a continuous function which is non- decreasing (non-increasing) in x for each y and non-increasing (non-decreasing) in y for each x. Let $I = [a,b]$ be an invariant interval under $G(x,y)$. Assume that $\overline{y} \in I$ is a unique equilibrium point of Eq. (2.4). Assume that J is a closed interval such that $G(x,y) \in I$, $\forall x, y \in J$. If the system*

$$x = G(x,y) \quad \text{and} \quad y = G(y,x)$$
$$(x = G(y,x) \quad \text{and} \quad y = G(x,y))$$

has exactly one solution in I^2, then $\overline{y}$ is a global attractor with basin I^{k+1}.

3. Boundedness of solutions

In this section we show that every solution of Eq. (2.2) is bounded and persists. Moreover, if $r > 1$ we can determine a lower and upper bound for every solution $\{y_n\}_{n=-1}^{\infty}$ that depends on the coefficients and the initial value y_0.

Theorem 3.1. *Every solution of Eq.* (2.2) *is bounded from above and from below by positive constants.*

Proof. Let $\{y_n\}$ be a solution of Eq. (2.2). Clearly, if the solution is bounded from above by a positive constant M, then

$$y_{n+1} \geqslant \frac{s}{r + M^k},$$

and so it is also bounded from below. Now assume for the sake of contradiction that the solution is not bounded from above. Then there exists a subsequence $\{y_{1+n_m}\}_{m=0}^{\infty}$ such that $\lim_{n\to\infty} n_m = \infty$, $\lim_{m\to\infty} y_{1+n_m} = \infty$, and $y_{1+n_m} = \max\{y_n : n \leqslant 1 + n_m\}$ for $m \geqslant 0$. From Eq. (2.2) we see that

$$y_{n+1} < \frac{s}{r} + \frac{1}{r} y_n \quad \text{for } n \geqslant 0,$$

and so,

$$\lim_{n\to\infty} y_{n_m} = \lim_{n\to\infty} y_{n_m-1} = \infty.$$

Hence, for sufficiently large m,

$$0 \leqslant y_{1+n_m} - y_{n_m} = \frac{s + [(1-r) - y_{n_m-1}^k] y_{n_m}}{r + y_{n_m-1}^k} < 0,$$

which is a contradiction and the proof is complete. □

Theorem 3.2. *Assume that* $r \neq 1$ *and* $\{y_n\}_{n=-1}^{\infty}$ *is a solution of Eq.* (2.2). *Then*

$$\frac{\alpha}{1 + \beta\left[\frac{\alpha(1-\beta^{n-2})}{1-\beta} + y_0\beta^{n-2}\right]^k} \leqslant y_n \leqslant \alpha\left(\frac{1-\beta^n}{1-\beta}\right) + y_0\beta^n, \quad n \geqslant 2,$$

where $\alpha = \frac{s}{r}$ *and* $\beta = \frac{1}{r}$.

Proof. By Eq. (2.2) we have

$$\frac{s}{r + y_{n-1}^k} \leqslant y_{n+1} < \frac{s}{r} + \frac{y_n}{r}.$$

Set $\alpha = \frac{s}{r}$ and $\beta = \frac{1}{r}$. By induction on n, we can get

$$\frac{\alpha}{1 + \beta\left[\frac{\alpha(1-\beta^{n-2})}{1-\beta} + y_0\beta^{n-2}\right]^k} \leqslant y_n \leqslant \alpha\left(\frac{1-\beta^n}{1-\beta}\right) + y_0\beta^n, \quad n \geqslant 2. \qquad \square$$

It is obvious that when $r > 1$, then every solution $\{y_n\}_{n=-1}^{\infty}$ satisfies the following inequality

$$\frac{\alpha}{1+\beta[\frac{\alpha}{1-\beta}+y_0]^k} < y_n < \frac{\alpha}{1-\beta}+y_0, \quad n \geqslant 2.$$

4. Global behavior of Eq. (2.2) when $r > 1$

In this section we investigate the global asymptotic stability of Eq. (2.2) when $r > 1$.

Lemma 4.1. *Let $\{y_n\}_{n=-1}^{\infty}$ be a solution of Eq. (2.2), $\Lambda = \lim_{n\to\infty}\sup y_n$ and $\lambda = \lim_{n\to\infty}\inf y_n$, then Λ and λ satisfy the following two inequalities,*

$$\frac{s}{r+(\frac{s}{r-1})^k} \leqslant \lambda \leqslant \Lambda \leqslant \frac{s}{r-1} \tag{4.1}$$

and

$$\frac{s+\lambda}{r+\Lambda^k} \leqslant \lambda \leqslant \Lambda \leqslant \frac{s+\Lambda}{r+\lambda^k}. \tag{4.2}$$

Proof. Inequality (4.1) is a direct consequence of Theorem (2.2), since $r > 1$. For every $\epsilon \in (0, \lambda)$, $\exists n_0 \in \mathbb{N}$ such that

$$\lambda - \epsilon \leqslant y_n \leqslant \Lambda + \epsilon \quad \text{for every } n \geqslant n_0,$$

so,

$$\frac{s+\lambda-\epsilon}{r+(\Lambda+\epsilon)^k} \leqslant y_n \leqslant \frac{s+\Lambda+\epsilon}{r+(\lambda-\epsilon)^k} \quad \forall n \geqslant n_0+1.$$

Therefore,

$$\frac{s+\lambda}{r+\Lambda^k} \leqslant \lambda \leqslant \Lambda \leqslant \frac{s+\Lambda}{r+\lambda^k}. \qquad \square$$

In the following we define

$$I_0 = \left[0, \frac{s}{r-1}\right].$$

Lemma 4.2. *The interval I_0 is invariant under the function*

$$G(x,y) = \frac{s+x}{r+y^k}. \tag{4.3}$$

Proof. Let $x, y \in I_0$. Then

$$0 < \frac{s}{r + (\frac{s}{r-1})^k} < G(x, y) < \frac{s + x}{r} < \frac{s}{r-1}. \qquad \square$$

Theorem 4.1. *If the system*

$$y = \frac{s + y}{r + x^k} \quad \textit{and} \quad x = \frac{s + x}{r + y^k} \tag{4.4}$$

has exactly one solution in I_0^2*, then the equilibrium point* $\bar{y}$ *is a global attractor.*

Proof. Let $\{y_n\}_{n=-1}^{\infty}$ be a solution of Eq. (2.2), $\Lambda = \lim_{n\to\infty} \sup y_n$ and $\lambda = \lim_{n\to\infty} \inf y_n$. By Lemma (4.1), we have $\Lambda, \lambda \in I_0$ which is invariant under $G(x, y)$. By Theorem (2.3), $\bar{y}$ is a global attractor. $\square$

The following Theorem determines conditions under which system (4.4) has exactly one solution.

Theorem 4.2. *Assume that* $k > 1$. *If* $s \leqslant \left[\frac{(r-1)^{k+1}}{k}\right]^{1/k}$, *then system* (4.4) *has exactly one solution in* I_0^2.

Proof. Assume that (x, y) is a solution of system (4.4) in I^2. Then we have

$$r + x^k = \frac{s}{y} + 1 \quad \text{and} \quad r + y^k = \frac{s}{x} + 1.$$

Hence,

$$x^k - y^k = \frac{s}{y} - \frac{s}{x} = \frac{s(x - y)}{xy}.$$

Assume towards a contradiction that $x \neq y$, (say $y < x$). Hence,

$$\frac{x^k - y^k}{x - y} = \frac{s}{xy}.$$

By the Mean Value Theorem, there exists $c \in (y, x)$ such that $\frac{s}{xy} = kc^{k-1}$. Since $k > 1$, we have $\frac{s}{xy} < kx^{k-1}$ and $\frac{s}{\frac{s}{r-1}} < \frac{s}{y} < kx^k < k(\frac{s}{r-1})^k$. This implies that

$$r - 1 < k\left(\frac{s}{r-1}\right)^k,$$

which is a contradiction. Then system (4.4) has exactly one solution $(x, y) = (\bar{y}, \bar{y})$. $\square$

Now, we are ready to prove the main result of this section.

Theorem 4.3

(*1*) *Assume that* $k \leqslant 1$. *Then the equilibrium point* $\bar{y}$ *is globally asymptotically stable.*

(2) *Assume that* $k > 1$. *If* $s \leqslant \left[\frac{(r-1)^{k+1}}{k}\right]^{1/k}$, *then the equilibrium point* $\bar{y}$ *is globally asymptotically stable.*

Proof. (1) Assume that $k \leqslant 1$. In view of Theorem (2.2), it remains to show that every solution $\{y_n\}$ of Eq. (2.2) tends to $\bar{y}$ as $n \to \infty$. Let $\{y_n\}_{n=-1}^{\infty}$ be a solution of Eq. (2.2). By Theorem (3.1) it is bounded by two positive numbers. Let

$$\lambda = \lim_{n\to\infty} \inf y_n \quad \text{and} \quad \Lambda = \lim_{n\to\infty} \sup y_n$$

by inequality(4.2), we have

$$\frac{s+\lambda}{r+\Lambda^k} \leqslant \lambda \leqslant \Lambda \leqslant \frac{s+\Lambda}{r+\lambda^k},$$

from which we see that

$$\lambda^k(1-r) + s\lambda^{k-1} \leqslant s\Lambda^{k-1} + \Lambda^k(1-r).$$

If $\lambda < \Lambda$, then

$$\lambda^k(1-r) + s\lambda^{k-1} > s\Lambda^{k-1} + \Lambda^k(1-r),$$

which is a contradiction, whence $\lambda = \Lambda$, from which the result follows.

(2) Assume that $k > 1$ and $s \leqslant \left[\frac{(r-1)^{k+1}}{k}\right]^{1/k}$. We have $\frac{(k-1)^{k+1}s^k}{[k(r-1)+1]^k} < r$, since

$$\frac{(k-1)^{k+1}s^k}{r[k(r-1)+1]^k} < \frac{(k-1)^{k+1}(r-1)^{k+1}}{kr[k(r-1)+1]^k} = \left(\frac{k-1}{k}\right)\left(\frac{r-1}{r}\right)\left[\frac{(k-1)(r-1)}{k(r-1)+1}\right]^k < 1.$$

In view of Theorem (2.2), we get $\bar{y}$ is locally asymptotically stable. By combining Theorems (4.1), and (4.2), we see that $\bar{y}$ is globally asymptotically stable. □

Open Problem (1): Investigate the global behavior of the solution of Eq. (2.2) when $r > 1$ under the condition $s > \left[\frac{(r-1)^{k+1}}{k}\right]^{1/k}$.

5. Global behavior of Eq. (2.2) when $r = 1$

In this section, we investigate the global behavior of Eq. (2.2) when $r = 1$, so Eq. (2.2) yields

$$y_{n+1} = \frac{s+y_n}{1+y_{n-1}^k}, \quad n = 0, 1, \ldots \tag{5.1}$$

Eq. (5.1) has a unique equilibrium point $\bar{y} = \sqrt[k+1]{s}$. Theorem (2.2) can be restated as follows:

Theorem 5.1. *The following statements are true for Eq. (5.1).*

(1) Assume that $k \leqslant 1$. Then the equilibrium point $\bar{y} = \sqrt[k+1]{s}$ is locally asymptotically stable.
(2) Assume that $k > 1$. Then we have:
 (*i*) *If* $s < 1/(k-1)^{\frac{k+1}{k}}$, *then* $\bar{y}$ *is locally asymptotically stable.*
 (*ii*) *If* $s > 1/(k-1)^{\frac{k+1}{k}}$, *then* $\bar{y}$ *is unstable, in fact a repeller.*
 (*iii*) *If* $s = 1/(k-1)^{\frac{k+1}{k}}$, *then* $\bar{y}$ *is a non-hyperbolic point.*

Theorem 5.2. *Assume that* $k < 1$. *Then the equilibrium point* $\bar{y} = \sqrt[k+1]{s}$ *is globally asymptotically stable.*

Proof. Assume that $k < 1$. In view of Theorem (5.1), it remains to show that every solution $\{y_n\}$ of Eq. (5.1) tends to $\bar{y}$ as $n \to \infty$. Let

$$\lambda = \lim_{n\to\infty} \inf y_n \quad \text{and} \quad \Lambda = \lim_{n\to\infty} \sup y_n,$$

then we have

$$\lambda \geqslant \frac{s+\lambda}{1+\Lambda^k} \quad \text{and} \quad \Lambda \leqslant \frac{s+\Lambda}{1+\lambda^k}.$$

Hence, $\Lambda\lambda^k \leqslant s \leqslant \lambda\Lambda^k$ and since $k < 1$, then $\lambda \geqslant \Lambda$ whence $\Lambda = \lambda$. □

Remark 5.1. The case where $r = 1$ and $k = 1$, was studied in Kulenović and Ladas (2002).

Computer observations show that when $k > 1$ and $s < 1/(k-1)^{\frac{k+1}{k}}$, then $\bar{y} = \sqrt[k+1]{s}$ of Eq. (5.1) is globally asymptotically stable. Also under the condition $s > 1/(k-1)^{\frac{k+1}{k}}$, the following properties hold

(1) When $s > 1$, then every solution converges to a five-period solution.
(2) When $s = 1$, then every solution converges to a twenty-period solution.

6. Global behavior of Eq. (2.2) when $r < 1$

In this section we show that the equilibrium point $\bar{y}$ of the equation

$$y_{n+1} = \frac{s+y_n}{r+y_{n-1}^k}, \quad n = 0, 1, \ldots, \tag{6.1}$$

where $r < 1$, is a global attractor with some basin that depends on the coefficients. Let $\bar{y}$ be a unique equilibrium point of Eq. (6.1). In the sequel define

$$G(x,y) = \frac{s+x}{r+y^k}.$$

The following lemma determines invariant intervals dependent on r, s, and k.

Lemma 6.1. *Assume that* $k < 1$.

(*1*) *If* $r \geqslant s^{\frac{k-1}{k}}$, *then* $I = [1, \frac{s}{r}]$ *is invariant under G and I contains* $\bar{y}$.
(*2*) *If* $r \geqslant 1 + s - s^k$, *then* $I = [s, 1]$ *is invariant under G and I contains* $\bar{y}$.

Proof. (1) The condition $r \geqslant s^{\frac{k-1}{k}}$, implies that $s > 1$. Hence by Theorem (2.1), $\bar{y} \in (1, \frac{s}{r})$. Let $x, y \in [1, \frac{s}{r}]$, then we have

$$1 \leqslant \frac{s+1}{1+(\frac{s}{r})^k} \leqslant G(x,y) = \frac{s+x}{r+y^k} \leqslant \frac{s+\frac{s}{r}}{r+1} = \frac{s}{r}.$$

(2) The condition $r \geqslant 1 + s - s^k$, implies that $s < 1$. Also we can see that $s \leqslant r$. Hence by Theorem (2.1) $\bar{y} \in [s, 1]$. Let $x, y \in [s, 1]$, then we have

$$s = \frac{s+s}{1+1} \leqslant G(x,y) = \frac{s+x}{r+y^k} \leqslant \frac{s+1}{r+s^k} \leqslant 1. \qquad \square$$

Consider the following system

$$y = \frac{s+y}{r+x^k} \quad \text{and} \quad x = \frac{s+x}{r+y^k}. \tag{6.2}$$

In the next theorem we determine some conditions under which system (6.2) has exactly one solution.

Theorem 6.1. *Assume that* $k < 1$.

(*i*) *If* $r^{\frac{k}{k-1}} < s < (\frac{1}{k} r^{k+1})^{\frac{1}{k}}$, *then system* (6.2) *has exactly one solution* $(x, y) \in [1, \frac{s}{r}]^2$.
(*ii*) *If* $k < s \leqslant r$, *then system* (6.2) *has exactly one solution* $(x, y) \in [s, 1]^2$.

Proof. Assume that $(x, y) \in I^2$ is a solution of system (6.2), and $y < x$, where $I = [1, \frac{s}{r}]$ in statement (i) and $I = [s, 1]$ in statement (ii). Then as before

$$\frac{x^k - y^k}{x - y} = \frac{s}{xy}.$$

There exists $c \in (y, x)$ such that $\frac{s}{xy} = kc^{k-1} < ky^{k-1}$. Hence $ky^k \geqslant \frac{s}{x}$.

(i) Since the condition $r^{\frac{k}{k-1}} < s < (\frac{1}{k} r^{k+1})^{\frac{1}{k}}$ implies $s^{k-1} < r^k$, then $[1, \frac{s}{r}]$ is an invariant under G by Lemma (6.1). Since $1 \leqslant x, y \leqslant \frac{s}{r}$, then $k(\frac{s}{r})^k \geqslant ky^k \geqslant \frac{s}{x} \geqslant \frac{rs}{s} = r$, whence $s^k \geqslant \frac{r^{k+1}}{k}$, which is a contradiction. Therefore, $x = y = \bar{y}$.
(ii) Since $s \leqslant x,\ y \leqslant 1$, then $k \geqslant ky^k \geqslant \frac{s}{x} \geqslant s$, which is a contradiction. Therefore, $x = y = \bar{y}$. $\square$

Now, we are ready to prove the main result of this section.

Theorem 6.2. *Assume that* $k < 1$.

(*i*) *If* $r^{\frac{k}{k-1}} < s < (\frac{1}{k}r^{k+1})^{\frac{1}{k}}$, *then* $\overline{y}$ *is globally asymptotically stable with basin* $[1, \frac{s}{r}]^2$.
(*ii*) *If* $1 - r \leqslant s^k - s$ *and* $k < s$, *then* $\overline{y}$ *is globally asymptotically stable with basin* $[s, 1]^2$.

Proof. (i) By Theorem (2.2) $\overline{y}$ is locally asymptotically stable and by Lemma (6.1) the interval $[1, \frac{s}{r}]$ is invariant under G and contains $\overline{y}$. By Theorem (6.1) the condition $r^{\frac{k}{k-1}} < s < (\frac{1}{k}r^{k+1})^{\frac{1}{k}}$, implies that system (6.2) has a unique solution in $[1, \frac{s}{r}]^2$. By Corollary (2.1) $\overline{y}$ is a global attractor with basin $[1, \frac{s}{r}]^2$. (ii) By the same argument of (i) we can prove (ii). □

Open Problem (2): Investigate the global behavior of the solution of Eq. (2.2) when $r < 1$ and $k > 1$.

References

Aboutaleb Mona T, El-Sayed MA, Hamza Alaa E. Stability of the recursive sequence $x_{n+1} = \frac{(\alpha - \beta x_n)}{(\gamma + x_{n-1})}$. J Math Anal Appl 2001;261:126–33.

Berg L. On the asymptotic of nonlinear difference equations. Z Anal Anwend 2002;21(4):1061–74.

Cunningham KC, Kulenović MRS, Ladas G, Valicenti SV. On the recursive sequence $x_{n+1} = (\alpha + \alpha x_n)/(Bx_n + Cx_{n-1})$. Nonlinear Anal TMA 2001;47:4603–14.

Devault R, Kosmala W, Ladas G, Schultz SW. Nonlinear Anal TMA 2001;47:4743–51.

Elaydi S. An introduction to difference equations. 2nd ed. New York: Springer-Verlag; 1999.

El-Owaidy HM, Ahmed AM, Youssef AM. On the dynamics of the recursive sequence $x_{n+1} = (\alpha x_{n-1})/(\beta + \gamma x_{n-2}^p)$. Appl Math Lett 2005a;18(9):1013–8.

El-Owaidy HM, Youssef AM, Ahmed AM. On the dynamics of $x_{n+1} = (bx_{n-1}^2)(A + Bx_{n-2})^{-1}$. Rostock Math Kolloq 2005b;59:11–8.

Gibbsons C, Kulenović M, Ladas G. On the recursive sequence $y_{n+1} = (\alpha + \beta y_{n-1})/(\gamma + y_n)$. Math Sci Res Hot-Line 2000;4(2):1–11.

Hamza Alaa E, El-Sayed MA. Stability problem of some nonlinear difference equations. Int J Math Math Sci 1998;21(2):331–40.

Jaroma JH. On the global asymptotic stability of $x_{n+1} = \frac{a + bx_n}{A + x_{n-1}}$. In: Proceeding of the first international conference on difference equations and applications, San Antonio, Texas, May 25–28, 1994. Gordon and Breach Science Publishers; 1995. p. 281–94.

Kelly WG, Peterson AC. Difference equations. New York: Academic Press; 1991.

Kocic VL, Ladas G. Global behavior of nonlinear difference equations of higher order with applications. Dordyecht: Kluwer Academic Publisher; 1993.

Kocic VL, Ladas G, Rodrigues IW. On the rational recursive sequences. J Math Anal Appl 1993;173:127–57.

Kulenović MRS, Ladas G. Dynamics of second order rational difference equations. Chapman & Hall/CRC; 2002.

Stević S. A generalization of the Copson's theorem concerning sequences which satisfy a linear inequality. Indian J Math 2001;43(3):277–82.

Stević S. On the recursive sequence $x_{n+1} = g(x_n, x_{n-1})/(A + x_n)$. Appl Math Lett 2002a;15:305–8.

Stević S. On the recursive sequence $x_{n+1} = x_{n-1}/g(x_n)$. Taiwanese J Math 2002b;6(3):405–14.

Stević S. On the recursive sequence $x_{n+1} = \frac{\alpha + \beta x_{n-1}}{1 + g(x_n)}$. Indian J Pure Appl Math 2002c;33(12):1767–74.

Stević S. Asymptotic behavior of a nonlinear difference equation. Indian J Pure Appl Math 2003;34(12):1681–9.