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Some results for Carlitz's q-Bernoulli numbers and polynomials

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A further investigation for Carlitz's q-Bernoulli numbers and polynomials is performed, and several new formulae for these numbers and polynomials are established by applying some summation transform techniques. Special cases as well as immediate consequences of the main results are also presented.

1. INTRODUCTION

The classical Bernoulli polynomials $B_n(x)$ are usually defined by the following exponential generating function:

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}\quad (|t|<2\pi). \tag{1.1}$$

In particular, the rational numbers $B_n = B_n(0)$ are called the classical Bernoulli numbers. These numbers and polynomials play important roles in many different branches of mathematics including number theory, combinatorics, special function and analysis. Numerous interesting properties for them can be found in many books; see, for example, [**9, 23, 30**]).

In the present paper, we will be concerned with Carlitz's q-Bernoulli numbers $\beta_n(q)$ and q-Bernoulli polynomials $\beta_n(x,q)$, which are respectively given by means of (see, e.g., [**5, 6**])

$$\beta_0(q)=1,\quad q(q\beta(q)+1)^n-\beta_n(q)=\begin{cases}1, & \text{if } n=1,\\ 0, & \text{if } n>1,\end{cases} \tag{1.2}$$

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and

$$\beta_n(x,q) = (q^x\beta(q) + [x]_q)^n = \sum_{k=0}^{n} \binom{n}{k} q^{kx}\beta_k(q)[x]_q^{n-k} \quad (n \geq 0), \tag{1.3}$$

with the usual convention about replacing β_i by β^i. And the parameter q appearing in (1.2) and (1.3) satisfies that $q \in \mathbb{C}$ with $|q| < 1$ and $\mathbb{C}$ being complex number field, and the bracket notation $[x]_q$ appearing in (1.3) stands for the q-number defined by (see, e.g., [**3, 11**])

$$[x]_q = \frac{1-q^x}{1-q} = 1 + q + \cdots + q^{x-1}. \tag{1.4}$$

Obviously, $\beta_n(q) = \beta_n(0,q)$ and $\lim_{q\to 1}[x]_q = x$.

Since the above Carlitz's q-Bernoulli numbers and q-Bernoulli polynomials appeared, different properties for them have been well studied by many authors; see, for example, [**18, 19, 20, 31, 33**]. In fact, Carlitz's q-Bernoulli numbers and polynomials can be defined by the following exponential generating functions (see, e.g., [**24, 27**]):

$$\sum_{m=0}^{\infty} q^m e^{[m]_q t}(1 - q - q^m t) = \sum_{n=0}^{\infty} \beta_n(q)\frac{t^n}{n!} \quad (|t + \log q| < 2\pi), \tag{1.5}$$

and

$$\sum_{m=0}^{\infty} q^m e^{[x+m]_q t}(1 - q - q^{x+m} t) = \sum_{n=0}^{\infty} \beta_n(x,q)\frac{t^n}{n!} \quad (|t + \log q| < 2\pi). \tag{1.6}$$

From (1.5) and (1.6), one can easily get

$$\lim_{q\to 1}\beta_n(q) = B_n \quad \text{and} \quad \lim_{q\to 1}\beta_n(x,q) = B_n(x). \tag{1.7}$$

If the left-hand side of (1.6) is denoted by $F_q(t,x)$ then the Mellin transform gives

$$\frac{1}{\Gamma(s)}\int_0^{\infty} F_q(-t,x)t^{s-2}\mathrm{d}t = \sum_{n=0}^{\infty} \frac{q^{x+2n}}{[x+n]_q^s} + \frac{1-q}{s-1}\sum_{n=0}^{\infty}\frac{q^n}{[x+n]_q^{s-1}} \tag{1.8}$$

with $s \in \mathbb{C}$ and $x \neq 0, -1, -2, \ldots$. Based on the observation on (1.8), the q-Hurwitz zeta function can be defined by (see, e.g., [**24**])

$$\zeta_q(s,x) = \sum_{n=0}^{\infty}\frac{q^n}{[x+n]_q^s} + (1-q)\left(\frac{2-s}{s-1}\right)\sum_{n=0}^{\infty}\frac{q^n}{[x+n]_q^{s-1}}, \tag{1.9}$$

where $s \in \mathbb{C}$ with $Re(s) > 1$ and $x \notin \mathbb{Z}_0^- = \{0, -1, -2, \ldots\}$. Especially, the case $x = 1$ and $q \to 1$ in (1.9) respectively gives the q-zeta function given by Satoh [**27**] and the classical Hurwitz zeta function $\zeta(s,x)$:

$$\zeta(s,x) = \sum_{n=0}^{\infty}\frac{1}{(n+x)^s} \quad (s \in \mathbb{C}, Re(s) > 1; x \notin \mathbb{Z}_0^- = \{0, -1, -2, \ldots\}). \tag{1.10}$$

Recently, Choi, Anderson and Srivastava [**8**] systematically explore Carlitz's q-Bernoulli numbers and polynomials, and recover some interesting properties between Carlitz's q-Bernoulli numbers and polynomials and some related numbers and polynomials and functions. Inspired by their work, in this paper, we perform a further investigation for Carlitz's q-Bernoulli numbers and polynomials, and give some new formulae for these numbers and polynomials by applying some summation transform techniques. It turns out that various known results including the recent one presented in [**4**] are derived as special cases.

2. THE STATEMENT OF THE RESULTS

We begin by describing the falling factorial $(x)_k$ of order k and rising factorial $x^{(k)}$ of order k ($x \in \mathbb{C}$ and k non-negative integer):

$$(x)_k = x(x-1)(x-2)\dots(x-k+1) \quad (k \geq 1), \quad (x)_0 = 1, \tag{2.1}$$

and

$$x^{(k)} = x(x+1)(x+2)\dots(x+k-1) \quad (k \geq 1), \quad x^{(0)} = 1. \tag{2.2}$$

We now recall the following addition theorem of Carlitz's q-Bernoulli polynomials (see, e.g., [**8**]),

$$\beta_n(x+y,q) = \sum_{k=0}^{n} \binom{n}{k} q^{kx} \beta_k(y,q)[x]_q^{n-k} \quad (n \geq 0). \tag{2.3}$$

Clearly, $\beta_n(y,q) = \beta_n(-x+(x+y),q)$ for non-negative integer n, so from (2.3) we obtain that for non-negative integers m, n,

$$\begin{aligned} \sum_{k=0}^{m} \binom{m}{k} q^{(n+k)x} \frac{\beta_{n+k+r}(y,q)}{(n+k)_r} [x]_q^{m-k} &= \sum_{k=0}^{m} \binom{m}{k} q^{(n+k)x} \frac{[x]_q^{m-k}}{(n+k)_r} \\ &\quad \times \sum_{i=0}^{n+k+r} \binom{n+k+r}{i} q^{-ix} \beta_i(x+y,q)[-x]_q^{n+k+r-i}. \end{aligned} \tag{2.4}$$

Since $[x]_q = (-q^x)[-x]_q$ then from (2.4) we get

$$\begin{aligned} \sum_{k=0}^{m} \binom{m}{k} q^{(n+k)x} \frac{\beta_{n+k+r}(y,q)}{(n+k)_r} [x]_q^{m-k} &= \sum_{k=0}^{m} \binom{m}{k} q^{(m+n)x} \frac{(-1)^{m-k}}{(n+k)_r} \\ &\quad \times \sum_{i=0}^{n+k+r} \binom{n+k+r}{i} q^{-ix} \beta_i(x+y,q)[-x]_q^{m+n+r-i}. \end{aligned} \tag{2.5}$$

If we change the order of the summations in the right hand side of (2.5) then

$$\begin{aligned}(2.6)\quad \sum_{k=0}^{m}\binom{m}{k}q^{(n+k)x}\frac{\beta_{n+k+r}(y,q)}{(n+k)_r}[x]_q^{m-k} &= \sum_{i=0}^{m+n+r}q^{(m+n-i)x}\beta_i(x+y,q)[-x]_q^{m+n+r-i}\\ &\quad\times\sum_{k=0}^{m}\binom{m}{k}\binom{n+k+r}{i}\frac{(-1)^{m-k}}{(n+k)_r}.\end{aligned}$$

Observe that for non-negative integers n,k,r,

$$(2.7)\qquad (n+k)_r=(n+k)(n+k-1)\cdots(n+k-r+1)=r!\cdot\binom{n+k}{r}.$$

So from (2.6) and (2.7), we discover

$$\begin{aligned}(2.8)\quad r!\sum_{k=0}^{m}\binom{m}{k}\binom{n+k}{r}q^{(n+k)x}\frac{\beta_{n+k+r}(y,q)}{(n+k)_r}[x]_q^{m-k} &= \sum_{i=0}^{m+n+r}q^{(m+n-i)x}\beta_i(x+y,q)[-x]_q^{m+n+r-i}\\ &\quad\times\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\binom{n+k+r}{i}.\end{aligned}$$

Notice that for a complex number s and non-negative integers p,h (cf. the identity of Wu described in [**10, 14, 29**]),

$$(2.9)\qquad \sum_{k=0}^{p}(-1)^{p+k}\binom{p}{k}\binom{k+h+s}{h}=\mathrm{res}_x(1+x)^{s+h}x^{-h+p-1}=\binom{s+h}{h-p}.$$

Hence, by applying (2.9) to (2.8), we obtain

$$\begin{aligned}(2.10)\quad \sum_{k=0}^{m}\binom{m}{k}q^{(n+k)x}\beta_{n+k}(y,q)[x]_q^{m-k} &= \sum_{k=0}^{n}\binom{n}{k}q^{(n-k)x}\beta_{m+k}(x+y,q)[-x]_q^{n-k}.\end{aligned}$$

Since Carlitz's q-Bernoulli polynomials obey the symmetric distribution (see, e.g., [**8**])

$$(2.11)\qquad \beta_n(1-x,q^{-1})=(-q)^n\beta_n(x,q)\quad(n\geq 0),$$

so by setting $x+y+z=1$ in (2.10), in view of (2.11) and $[x]_q=(-q^x)[-x]_q$, we immediately get the following result.

Theorem 2.1. *Let m, n be non-negative integers. Then for $x + y + z = 1$,*

$$(2.12)\quad (-1)^m \sum_{k=0}^{m} \binom{m}{k} q^{(n+k)x} \beta_{n+k}(y, q)[x]_q^{m-k} = (-1)^n \sum_{k=0}^{n} \binom{n}{k} q^{-(m+k)} \beta_{m+k}(z, q^{-1})[x]_q^{n-k}.$$

It is worthy noticing that the case $n = 0$ in the formula (2.10) gives the formula (2.3) and the formula (2.10) can be also derived by applying the generating function methods, see [**16**] for a detail. And the theorem 2.1 above can be regarded as the corresponding q-analogue of a result of Sun [**32**], namely

$$(2.13)\quad (-1)^m \sum_{k=0}^{m} \binom{m}{k} x^{m-k} B_{n+k}(y) = (-1)^n \sum_{k=0}^{n} \binom{n}{k} x^{n-k} B_{m+k}(z).$$

If we set $x = 1$ and $y = z = 0$ in Theorem 2.1, we get that for non-negative integers m, n,

$$(2.14)\quad (-1)^m \sum_{k=0}^{m} \binom{m}{k} q^{n+k} \beta_{n+k}(q) = (-1)^n \sum_{k=0}^{n} \binom{n}{k} q^{-(m+k)} \beta_{m+k}(q^{-1}),$$

which is a q-analogue of the familiar formula described in [**13**]

$$(2.15)\quad (-1)^m \sum_{k=0}^{m} \binom{m}{k} B_{n+k} = (-1)^n \sum_{k=0}^{n} \binom{n}{k} B_{m+k} \quad (m, n \geq 0).$$

For some similar results on the q-Bernoulli numbers attached to formal group to (2.14), one is referred to [**27**].

We next give a more general form of Theorem 2.1. In a similar consideration to (2.6), we have

$$(2.16)\quad \sum_{k=0}^{m} \binom{m}{k} q^{(n+k)x} (n+k)_r \beta_{n+k-r}(y, q)[x]_q^{m-k} = \sum_{i=0}^{m+n-r} q^{(m+n-i)x} \beta_i(x+y, q)[-x]_q^{m+n-r-i} \times \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \binom{n+k-r}{i} (n+k)_r,$$

which together with (2.7) yields

$$\sum_{k=0}^{m}\binom{m}{k}\binom{n+k}{r}q^{(n+k)x}\beta_{n+k-r}(y,q)[x]_q^{m-k} = \sum_{i=0}^{m+n-r}q^{(m+n-i)x}\beta_i(x+y,q)[-x]_q^{m+n-r-i} \times\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\binom{n+k-r}{i}\binom{n+k}{r}. \tag{2.17}$$

Clearly, $(-m)^{(k)}=(-1)^k m(m-1)\cdots(m-k+1)$ and $(n+k)!=n!\cdot(n+1)^{(k)}$ for non-negative integers k,m, which follow that

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\binom{n+k-r}{i}\binom{n+k}{r} = \frac{(-1)^m n!}{i!\cdot r!\cdot(n-i-r)!}\sum_{k=0}^{m}\frac{(-m)^{(k)}(n+1)^{(k)}}{k!\cdot(n+1-i-r)^{(k)}}. \tag{2.18}$$

Note that for non-negative integer n and complex numbers a,b (cf. the Chu-Vandermonde summation formula stated in [**3, 11**]),

$$\sum_{k=0}^{n}\frac{(-n)^{(k)}\cdot a^{(k)}}{k!\cdot b^{(k)}}=\frac{(b-a)^{(n)}}{b^{(n)}}. \tag{2.19}$$

Hence, by applying (2.19) to (2.18), we get

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\binom{n+k-r}{i}\binom{n+k}{r} = \frac{n!\cdot(i+r)(i+r-1)\cdots(i+r-m+1)}{i!\cdot r!\cdot(m+n-i-r)!}=\binom{n}{i+r-m}\binom{i+r}{i}. \tag{2.20}$$

Combining (2.17) and (2.20) gives

$$\sum_{k=0}^{m}\binom{m}{k}\binom{n+k}{r}q^{(n+k)x}\beta_{n+k-r}(y,q)[x]_q^{m-k} = \sum_{k=0}^{n}\binom{n}{k}\binom{m+k}{r}q^{(n+r-k)x}\beta_{m+k-r}(x+y,q)[-x]_q^{n-k}. \tag{2.21}$$

If we set $x+y+z=1$ in (2.21), in light of (2.11) and $[x]_q=(-q^x)[-x]_q$, we get

$$\sum_{k=0}^{m}\binom{m}{k}\binom{n+k}{r}q^{(n+k-r)x}\beta_{n+k-r}(y,q)[x]_q^{m-k} = (-1)^{m+n-r}\sum_{k=0}^{n}\binom{n}{k}\binom{m+k}{r}q^{-(m+k-r)}\beta_{m+k-r}(z,q^{-1})[x]_q^{n-k}. \tag{2.22}$$

Thus, by substituting m for $m+r$ and n for $n+r$ in (2.22), we immediately obtain the following result.

Theorem 2.2. *Let m,n,r be non-negative integers. Then for $x+y+z=1$,*

$$\sum_{k=0}^{m+r}\binom{m+r}{k}\binom{n+k+r}{r}q^{(n+k)x}\beta_{n+k}(y,q)[x]_q^{m+r-k} = (-1)^{m+n+r}\sum_{k=0}^{n+r}\binom{n+r}{k}\binom{m+k+r}{r}q^{-(m+k)}\beta_{m+k}(z,q^{-1})[x]_q^{n+r-k}. \tag{2.23}$$

It follows that we show some special cases of Theorem 2.2. Setting $r=0$ in Theorem 2.2 gives Theorem 2.1. If we let $q\to 1$ in Theorem 2.2 then for non-negative integers m,n,r, (see, e.g., [**15**])

$$\sum_{k=0}^{m+r}\binom{m+r}{k}\binom{n+k+r}{r}B_{n+k}(y)x^{m+r-k} = (-1)^{m+n+r}\sum_{k=0}^{n+r}\binom{n+r}{k}\binom{m+k+r}{r}B_{m+k}(z)x^{n+r-k}. \tag{2.24}$$

If we set $x=1$ and $y=z=0$ in Theorem 2.2, we obtain that for non-negative integers m,n,r,

$$\sum_{k=0}^{m+r}\binom{m+r}{k}\binom{n+k+r}{r}q^{n+k}\beta_{n+k}(q) = (-1)^{m+n+r}\sum_{k=0}^{n+r}\binom{n+r}{k}\binom{m+k+r}{r}q^{-(m+k)}\beta_{m+k}(q^{-1}), \tag{2.25}$$

which is a q-analogue of a formula on the classical Bernoulli numbers due to Agoh (see, e.g., [**1, 25**])

$$\sum_{k=0}^{m+r}\binom{m+r}{k}\binom{n+k+r}{r}B_{n+k} = (-1)^{m+n+r}\sum_{k=0}^{n+r}\binom{n+r}{k}\binom{m+k+r}{r}B_{m+k} \quad (m,n,r\geq 0). \tag{2.26}$$

It is worthy mentioning that since $B_n = (-1)^n B_n$ for positive integer $n \geq 2$ then the case $r = 1$ in (2.26) gives that for non-negative integers m, n,

$$(2.27)\quad (-1)^m \sum_{k=0}^{m} \binom{m+1}{k} (n+k+r) B_{n+k} + (-1)^n \sum_{k=0}^{n} \binom{n+1}{k} (m+k+1) B_{m+k} = 0 \quad (m+n \geq 1),$$

which was obtained by Momiyama [**21**] who made use of p-adic integral over $\mathbb{Z}_p$ and used to give a brief proof of the famous Kummer congruence. And the case $m = n$ in (2.26) gives that for non-negative integer n and odd integer $r \geq 1$,

$$(2.28)\quad \sum_{k=0}^{n+r} \binom{n+r}{k} \binom{n+k+r}{r} B_{n+k} = 0,$$

which can be derived by applying the extended Zeilberger's algorithm (see, e.g., [**7**]). In particular, the case $r = 1$ in (2.28) was firstly discovered by Kaneko [**17**].

We are now in the position to give the corresponding q-analogue of Gessel's formula presented in [**4**] on the classical Bernoulli numbers. By setting $x = a, y = 0$ and $z = 1 - a$ in Theorem 2.2, we have

$$(2.29)\quad \sum_{k=0}^{m+r} \binom{m+r}{k} \binom{n+k+r}{r} q^{(n+k)a} \beta_{n+k}(q) [a]_q^{m+r-k} = (-1)^{m+n+r} \sum_{k=0}^{n+r} \binom{n+r}{k} \binom{m+k+r}{r} \times q^{-(m+k)} \beta_{m+k}(1-a, q^{-1}) [a]_q^{n+r-k}.$$

Since Carlitz's q-Bernoulli polynomials satisfy the difference equation (see, e.g., [**8**]):

$$(2.30)\quad q\beta_n(x+1) - \beta_n(x) = nq^x [x]_q^{n-1} + (q-1)[x]_q^n \quad (n \geq 0),$$

then for non-negative integers a, n,

$$(2.31)\quad \beta_n(1-a, q) = q^{a-1} \beta_n(q) - \sum_{i=1}^{a-1} q^{i-1} \big\{ nq^{i-a} [i-a]_q^{n-1} + (q-1)[i-a]_q^n \big\}.$$

Hence, in view of $[x]_q = (-q^x)[-x]_q$, the formula (2.31) can be rewritten as

$$(2.32)\quad \beta_n(1-a, q) = q^{a-1} \beta_n(q) + (-1)^n \sum_{i=1}^{a-1} q^{(i-a)n+i-1} \big\{ n[a-i]_q^{n-1} - (q-1)[a-i]_q^n \big\}.$$

If we apply (2.32) to (2.29) we get

$$\begin{aligned}(2.33)\quad &\sum_{k=0}^{m+r}\binom{m+r}{k}\binom{n+k+r}{r}q^{(n+k)a}\beta_{n+k}(q)[a]_q^{m+r-k}\\ &=(-1)^{m+n+r}\sum_{k=0}^{n+r}\binom{n+r}{k}\binom{m+k+r}{r}q^{1-(m+k+a)}\beta_{m+k}(q^{-1})[a]_q^{n+r-k}\\ &\quad+(-1)^{n+r}\sum_{k=0}^{n+r}\binom{n+r}{k}\binom{m+k+r}{r}(-1)^k q^{-(m+k)}\sum_{i=1}^{a-1}q^{(a-i)(m+k)-i+1}\\ &\qquad\times\left\{(m+k)[a-i]_{q^{-1}}^{m+k-1}-(q^{-1}-1)[a-i]_{q^{-1}}^{m+k}\right\}[a]_q^{n+r-k}.\end{aligned}$$

Note that $[x]_q=q^{x-1}[x]_{q^{-1}}$ and $[x+y]_q=[x]_q+q^x[y]_q$, then

$$\begin{aligned}[a]_q^{n+r-k} &= q^{(a-1)(n+r-k)}\left([i]_{q^{-1}}+q^{-i}[a-i]_{q^{-1}}\right)^{n+r-k}\\ &= q^{(a-1)(n+r-k)}\sum_{j=0}^{n+r-k}\binom{n+r-k}{j}[i]_{q^{-1}}^j\left(q^{-i}[a-i]_{q^{-1}}\right)^{n+r-k-j}\\ &= q^{(a-1)(n+r-k)}\sum_{j=1-n}^{r+1-k}\binom{n+r-k}{n+j-1}[i]_{q^{-1}}^{n+j-1}\\ (2.34)\qquad &\qquad\times\left(q^{-i}[a-i]_{q^{-1}}\right)^{r+1-k-j}.\end{aligned}$$

By applying (2.34) to the second summation of the right hand side of (2.33) and changing the order of the summation, we obtain

$$\begin{aligned}(2.35)\quad &\sum_{k=0}^{m+r}\binom{m+r}{k}\binom{n+k+r}{r}q^{(n+k)a}\beta_{n+k}(q)[a]_q^{m+r-k}\\ &=(-1)^{m+n+r}\sum_{k=0}^{n+r}\binom{n+r}{k}\binom{m+k+r}{r}q^{1-(m+k+a)}\beta_{m+k}(q^{-1})[a]_q^{n+r-k}\\ &\quad+(-1)^{n+r}\sum_{j=1-n}^{r+1}\sum_{k=0}^{n+r}\binom{n+r}{k}\binom{m+k+r}{r}\binom{n+r-k}{n+j-1}(-1)^k\\ &\quad\times\sum_{i=1}^{a-1}q^{(a-1)(m+n+r)-i(m+r+2-j)+1}\left\{(m+k)[i]_{q^{-1}}^{n+j-1}\cdot[a-i]_{q^{-1}}^{m+r-j}\right.\\ &\qquad\left.-(q^{-1}-1)[i]_{q^{-1}}^{n+j-1}\cdot[a-i]_{q^{-1}}^{m+r+1-j}\right\}.\end{aligned}$$

Observe that for $1-n\leq j\leq r+1$,

$$\sum_{k=0}^{n+r}\binom{n+r}{k}\binom{m+k+r}{r}\binom{n+r-k}{n+j-1}(-1)^k(m+k) \tag{2.36}$$
$$=\frac{(r+1)\cdot(n+r)!\cdot(m+r)!}{(m-1)!\cdot(r+1)!\cdot(n+j-1)!\cdot(r+1-j)!}$$
$$\times\sum_{k=0}^{n+r}\frac{(-(r+1-j))^{(k)}\cdot(m+r+1)^{(k)}}{k!\cdot m^{(k)}},$$

which together with (2.19) yields that for $1-n\leq j\leq r+1$,

$$\sum_{k=0}^{n+r}\binom{n+r}{k}\binom{m+k+r}{r}\binom{n+r-k}{n+j-1}(-1)^k(m+k) \tag{2.37}$$
$$=\frac{(r+1)\cdot(n+r)!\cdot(m+r)!}{(r+1)!\cdot(n+j-1)!\cdot(r+1-j)!}\cdot\frac{(-1)^{r+1-j}(r+1)r\cdots(j+1)}{(m+r-j)!}.$$

In the same way, for $1-n\leq j\leq r+1$, we have

$$\sum_{k=0}^{n+r}\binom{n+r}{k}\binom{m+k+r}{r}\binom{n+r-k}{n+j-1}(-1)^k \tag{2.38}$$
$$=\frac{(n+r)!\cdot(m+r)!}{r!\cdot(n+j-1)!\cdot(r+1-j)!}\cdot\frac{(-1)^{r+1-j}r(r-1)\cdots j}{(m+r+1-j)!}.$$

Thus, combining (2.35), (2.37) and (2.38) gives the following result.

Theorem 2.3. *Let m,n,r,a be non-negative integers. Then*

$$\sum_{k=0}^{m+r}\binom{m+r}{k}\binom{n+k+r}{r}q^{(n+k)a}\beta_{n+k}(q)[a]_q^{m+r-k} \tag{2.39}$$
$$=(-1)^{m+n+r}\sum_{k=0}^{n+r}\binom{n+r}{k}\binom{m+k+r}{r}q^{1-(m+k+a)}\beta_{m+k}(q^{-1})[a]_q^{n+r-k}$$
$$+\sum_{i=1}^{a-1}q^{(a-1)(m+n+r)-i(m+r+2)+1}$$
$$\times\Bigg\{(r+1)\sum_{j=0}^{r+1}\binom{m+r}{j}\binom{n+r}{r+1-j}q^{ij}[i]_{q^{-1}}^{n+j-1}\cdot[a-i]_{q^{-1}}^{m+r-j}$$
$$-(q^{-1}-1)\sum_{j=1}^{r+1}\binom{m+r}{j-1}\binom{n+r}{r+1-j}q^{ij}[i]_{q^{-1}}^{n+j-1}\cdot[a-i]_{q^{-1}}^{m+r+1-j}\Bigg\}.$$

It become obvious that the Theorem 2.3 can be regarded as a generalization of the formula (2.25). And the case $q \to 1$ in Theorem 2.3 gives that for non-negative integers m, n, r, a,

$$\begin{aligned} &\sum_{k=0}^{m+r} \binom{m+r}{k}\binom{n+k+r}{r} B_{n+k} a^{m+r-k} \\ &\quad + (-1)^{m+n+r-1} \sum_{k=0}^{n+r} \binom{n+r}{k}\binom{m+k+r}{r} B_{m+k} a^{n+r-k} \\ &\qquad = (r+1) \sum_{i=1}^{a-1} \sum_{j=0}^{r+1} \binom{m+r}{j}\binom{n+r}{r+1-j} i^{n+j-1} (a-i)^{m+r-j}, \end{aligned} \tag{2.40}$$

which was discovered by Gessel [**4**] who made use of the methods presented in [**13**].

We next give another type generalization of Theorem 2.1. In a similar consideration to (2.6), we have

$$\begin{aligned} &\sum_{k=0}^{m} \binom{m}{k} q^{(n+k)x} (n+k+1)^{(r)} \beta_{n+k+r}(y,q) [x]_q^{m-k} \\ &\quad = \sum_{i=0}^{m+n+r} q^{(m+n-i)x} \beta_i(x+y,q) [-x]_q^{m+n+r-i} \\ &\qquad \times \sum_{k=0}^{m} \binom{m}{k}\binom{n+k+r}{i} (-1)^{m-k} (n+k+1)^{(r)}. \end{aligned} \tag{2.41}$$

Observe that for non-negative integers n, k, r,

$$(n+k+1)^{(r)} = (n+k+1)\cdots(n+k+r) = \frac{1}{r! \cdot \binom{n+k+r}{r}}, \tag{2.42}$$

which together with (2.41) yields

$$\begin{aligned} &\sum_{k=0}^{m} \frac{\binom{m}{k}}{\binom{n+k+r}{r}} q^{(n+k)x} \beta_{n+k+r}(y,q) [x]_q^{m-k} \\ &\quad = \sum_{i=0}^{m+n+r} q^{(m+n-i)x} \beta_i(x+y,q) [-x]_q^{m+n+r-i} \\ &\qquad \times \sum_{k=0}^{m} \binom{m}{k}\binom{n+k+r}{i} \frac{(-1)^{m-k}}{\binom{n+k+r}{r}}. \end{aligned} \tag{2.43}$$

Hence, in light of (2.19), we obtain

$$\begin{aligned}
&\sum_{k=0}^{m}\binom{m}{k}\binom{n+k+r}{i}\frac{(-1)^{m-k}}{\binom{n+k+r}{r}}\\
&\quad=\frac{(-1)^m n!\cdot r!}{i!\cdot(n+r-i)!}\sum_{k=0}^{m}\frac{(-m)^{(k)}\cdot(n+1)^{(k)}}{(n+r+1-i)^{(k)}}\\
&\quad=\frac{(-1)^m n!\cdot r!\cdot(r-i)(r-i+1)\cdots(r-i+m-1)}{i!\cdot(m+n+r-i)!}. \qquad (2.44)
\end{aligned}$$

Combining (2.43) and (2.44) gives

$$\begin{aligned}
&\sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{n+k+r}{r}}q^{(n+k)x}\beta_{n+k+r}(y,q)[x]_q^{m-k} \qquad (2.45)\\
&=\sum_{k=0}^{n}\frac{\binom{n}{k}}{\binom{m+k+r}{r}}q^{(n-r-k)x}\beta_{m+k+r}(x+y,q)[-x]_q^{n-k}+(-1)^m r q^{(m+n+1-r)x}\\
&\qquad\times[-x]_q^{m+n+1}\sum_{i=0}^{r-1}\frac{\binom{r-1}{i}}{\binom{m+n+i}{n}}q^{ix}\beta_{r-1-i}(x+y,q)\frac{[-x]_q^i}{m+n+i+1}.
\end{aligned}$$

Thus, by setting $x+y+z=1$ in (2.45), in light of (2.11) and $[x]_q=(-q^x)[-x]_q$, we state the following result.

Theorem 2.4. *Let m,n,r be non-negative integers. Then for $x+y+z=1$,*

$$\begin{aligned}
&(-1)^m\sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{n+k+r}{r}}q^{(n+k+r)x}\beta_{n+k+r}(y,q)[x]_q^{m-k} \qquad (2.46)\\
&=(-1)^{n+r}\sum_{k=0}^{n}\frac{\binom{n}{k}}{\binom{m+k+r}{r}}q^{-(m+k+r)}\beta_{m+k+r}(z,q^{-1})[x]_q^{n-k}\\
&\qquad+(-1)^{m+n+r}r[x]_q^{m+n+1}\sum_{i=0}^{r-1}\frac{\binom{r-1}{i}}{\binom{m+n+r-1-i}{n}}q^{-i}\beta_i(z,q^{-1})\frac{[x]_q^{r-1-i}}{m+n+r-i}.
\end{aligned}$$

We next discuss some special cases of Theorem 2.4. Clearly, the case $r=0$ in Theorem 2.4 gives the Theorem 2.1. If we set $r=1$ in Theorem 2.4, we obtain that for non-negative integers m,n,

$$\begin{aligned}
&(-1)^m\sum_{k=0}^{m}\binom{m}{k}q^{(n+k+1)x}\frac{\beta_{n+k+1}(y,q)}{n+k+1}[x]_q^{m-k} \qquad (2.47)\\
&\qquad+(-1)^n\sum_{k=0}^{n}\binom{n}{k}q^{-(m+k+1)}\frac{\beta_{m+k+1}(z,q^{-1})}{m+k+1}[x]_q^{n-k}=\frac{(-[x]_q)^{m+n+1}}{(m+n+1)\binom{m+n}{m}},
\end{aligned}$$

which is a q-analogue of Sun's formula on the classical Bernoulli polynomials (see, e.g., [**7, 15, 32**])

$$(-1)^m \sum_{k=0}^{m} \binom{m}{k} x^{m-k} \frac{B_{n+k+1}(y)}{n+k+1} + (-1)^n \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \frac{B_{m+k+1}(z)}{m+k+1} = \frac{(-x)^{m+n+1}}{(m+n+1)\binom{m+n}{m}} \quad (m,n \geq 0). \tag{2.48}$$

In fact, the formula (2.47) has other applications. For example, since Carlitz's q-Bernoulli polynomials can be expressed by the closed formula (see, e.g., [**8**]):

$$\beta_n(x,q) = \frac{1}{(1-q)^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k q^{kx} \frac{k+1}{[k+1]_q} \quad (n \geq 0), \tag{2.49}$$

by applying the derivative operation $\partial/\partial x$ to both sides of (2.49), with the help of

$$k\binom{n}{k} = n\binom{n-1}{k-1} = n\left\{\binom{n}{k} - \binom{n-1}{k}\right\} \quad (k,n \geq 0), \tag{2.50}$$

one can easily derive that for non-negative integer n,

$$\frac{\partial}{\partial x}\beta_n(x,q) = \ln q\left(n\beta_n(x,q) - \frac{n}{1-q}\beta_{n-1}(x,q)\right). \tag{2.51}$$

Hence, replacing z with $1-x-y$ and applying the derivative operation $\partial/\partial y$ to both sides of (2.47), in view of (2.51), we obtain that for non-negative integers m, n,

$$\begin{aligned}(-1)^m \sum_{k=0}^{m} \binom{m}{k} q^{(n+k+1)x} [x]_q^{m-k} \{\beta_{n+k}(y,q) + (q-1)\beta_{n+k+1}(y,q)\} \\ = (-1)^n \sum_{k=0}^{n} \binom{n}{k} q^{-(m+k+1)} [x]_q^{n-k} \{-q\beta_{m+k}(z,q^{-1}) \\ + (q-1)\beta_{m+k+1}(z,q^{-1})\} \quad (x+y+z=1),\end{aligned} \tag{2.52}$$

which is another q-analogue of Sun's formula (2.13). On the other hand, if we set

$x=a, y=0, z=1-a$ in (2.47), by (2.32) and (2.34) we get

$$
\begin{aligned}
(2.53)\quad & (-1)^m \sum_{k=0}^{m} \binom{m}{k} q^{(n+k+1)a} \frac{\beta_{n+k+1}(q)}{n+k+1} [a]_q^{m-k} \\
& + (-1)^n \sum_{k=0}^{n} \binom{n}{k} q^{-(m+k+a)} \frac{\beta_{m+k+1}(q^{-1})}{m+k+1} [a]_q^{n-k} \\
& + (-1)^{m+n+1} \sum_{j=1-n}^{1} \sum_{k=0}^{n} \binom{n}{k} \binom{n-k}{n+j-1} \frac{(-1)^k}{m+k+1} \\
& \times \sum_{i=1}^{a-1} q^{(a-1)(m+n+1)-i(m+3-j)+1} \big\{ (m+k+1)[i]_{q^{-1}}^{n+j-1} \cdot [a-i]_{q^{-1}}^{m+1-j} \\
& \qquad - (q^{-1}-1)[i]_{q^{-1}}^{n+j-1} \cdot [a-i]_{q^{-1}}^{m+2-j} \big\} = \frac{(-1)^{m+n+1} \cdot m! \cdot n!}{(m+n+1)!} [a]_q^{m+n+1}.
\end{aligned}
$$

Note that from (2.38) we have

$$
(2.54)\qquad \sum_{k=0}^{n} \binom{n}{k} \binom{n-k}{n+j-1} (-1)^k = \begin{cases} 1, & j=1, \\ 0, & 1-n \leq j \leq 0, \end{cases}
$$

and from (2.19), we obtain that for $1-n \leq j \leq 1$,

$$
\begin{aligned}
& \sum_{k=0}^{n} \binom{n}{k} \binom{n-k}{n+j-1} \frac{(-1)^k}{m+k+1} \\
& = \frac{m! \cdot n!}{(m+1)! \cdot (n+j-1)! \cdot (1-j)!} \sum_{k=0}^{n} \frac{(-(1-j))^{(k)} \cdot (m+1)^{(k)}}{k! \cdot (m+2)^{(k)}} \\
(2.55)\quad & = \frac{m! \cdot n!}{(n+j-1)! \cdot (m+2-j)!}.
\end{aligned}
$$

Hence, combining (2.53), (2.54) and (2.55) gives the following result.

Theorem 2.5. *Let m, n, a be non-negative integers. Then*

$$\begin{aligned}(2.56)\quad &(-1)^{n+1}\sum_{k=0}^{m}\binom{m}{k}q^{(n+k+1)a}\frac{\beta_{n+k+1}(q)}{n+k+1}[a]_q^{m-k}\\ &+(-1)^{m+1}\sum_{k=0}^{n}\binom{n}{k}q^{-(m+k+a)}\frac{\beta_{m+k+1}(q^{-1})}{m+k+1}[a]_q^{n-k}\\ &=\frac{m!\cdot n!}{(m+n+1)!}[a]_q^{m+n+1}-\sum_{i=1}^{a-1}q^{(a-1)(m+n+1)-i(m+2)+1}[i]_{q^{-1}}^{n}\cdot[a-i]_{q^{-1}}^{m}\\ &\quad+(q^{-1}-1)\cdot m!\cdot n!\sum_{i=1}^{a-1}q^{(a-1)(m+n+1)-i(m+3)+1}\\ &\quad\times\sum_{j=1-n}^{1}\frac{q^{ij}}{(n+j-1)!\cdot(m+2-j)!}[i]_{q^{-1}}^{n+j-1}\cdot[a-i]_{q^{-1}}^{m+2-j}.\end{aligned}$$

It is obvious that the case $a = 1$ in Theorem 2.5 gives that for non-negative integers m, n,

$$\begin{aligned}(2.57)\quad &(-1)^{n+1}\sum_{k=0}^{m}\binom{m}{k}q^{n+k+1}\frac{\beta_{n+k+1}(q)}{n+k+1}\\ &+(-1)^{m+1}\sum_{k=0}^{n}\binom{n}{k}q^{-(m+k+1)}\frac{\beta_{m+k+1}(q^{-1})}{m+k+1}=\frac{m!\cdot n!}{(m+n+1)!},\end{aligned}$$

which is a q-analogue of a formula of Saalschütz [**26**], later rediscovered by Gelfand [**12**], namely

$$\begin{aligned}(2.58)\quad &(-1)^{n+1}\sum_{k=0}^{m}\binom{m}{k}\frac{B_{n+k+1}}{n+k+1}+(-1)^{m+1}\sum_{k=0}^{n}\binom{n}{k}\frac{B_{m+k+1}}{m+k+1}\\ &=\frac{m!\cdot n!}{(m+n+1)!}\quad (m,n\geq 0).\end{aligned}$$

And the case $q \to 1$ in Theorem 2.5 gives that for non-negative integers m, n, a,

$$\begin{aligned}(2.59)\quad &(-1)^{n+1}\sum_{k=0}^{m}\binom{m}{k}\frac{B_{n+k+1}}{n+k+1}a^{m-k}+(-1)^{m+1}\sum_{k=0}^{n}\binom{n}{k}\frac{B_{m+k+1}}{m+k+1}a^{n-k}\\ &=\frac{m!\cdot n!}{(m+n+1)!}a^{m+n+1}-\sum_{i=1}^{a-1}i^{n}(a-i)^{m}.\end{aligned}$$

which was considered by Neuman and Schonbach [**22**] from the point of view of numerical analysis. See also [**2**] for a different proof and detail introduction for the formula (2.59).

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