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The Sheffer group and the Riordan group

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Abstract

We define the Sheffer group of all Sheffer-type polynomials and prove the isomorphism between the Sheffer group and the Riordan group. An equivalence of the Riordan array pair and generalized Stirling number pair is also presented. Finally, we discuss a higher dimensional extension of Riordan array pairs.

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1. Introduction

Riordan array is a special type of infinite lower-triangular matrix and the set of all Riordan matrices forms a group called the Riordan group, which was first defined in 1991 by Shapiro et al. [23]. Some of main results on the Riordan group and its applications to the combinatorial sums and identities can be found in [15,22,23,26]. In particular, in the work by Sprugnoli (cf. [24,25]). In this paper, we will define an operation on the set of all Sheffer polynomial sequences so it forms a group called as the Sheffer group, which gives a general pattern consisting of various special Sheffer-type polynomial sequences as elements. We will show that every element of the group and its inverse are the potential polynomials of a pair of generalized Stirling numbers (GSNs) (see 3.7), and the isomorphism between the Sheffer group and the Riordan group (see 2.2). Hence, the established results on the Sheffer group connect the Riordan group, GSN pairs, and Riordan arrays, which can lead a comprehensive study on all of the topics. For instance, the Sheffer group and the related GSN-pairs and their inverse relations can be used to derive combinatorial identities as well as algebraic identities containing the Sheffer-type polynomials.

As what mentioned in [15] (see also in [1]), "The concept of representing columns of infinite matrices by formal power series is not new and goes back to Schur's paper on Faber polynomials in 1945 (cf. [21])". A formal power series

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in auxiliary variable t of the form

$$b(t) = b_0 + b_1 t + b_2 t^2 + \cdots = \sum_{n \geqslant 0} b_n t^n$$

is called an ordinary generating function of the sequence $\{b_n\}$.

Definition 1.1. Let $A(t)$ and $g(t)$ be any given formal power series over the real number field $\mathbb{R}$ or complex number field $\mathbb{C}$ with $A(0)=1$, $g(0)=0$ and $g'(0) \neq 0$. We call the infinite matrix $D=[d_{n,k}]_{n,k \geqslant 0}$ with real entries or complex entries a generalized Riordan matrix (the originally defined Riordan matrices need $g'(0)=1$) if its kth column satisfies

$$\sum_{n \geqslant 0} d_{n,k} t^n = A(t)(g(t))^k; \tag{1.1}$$

that is,

$$d_{n,k} = [t^n]A(t)(g(t))^k.$$

The Riordan matrix is denoted by $[d_{n,k}]$ or $(A(t), f(t))$.

Example 1.1. Riordan matrices $(1, t)$ and $(1/(1-t), t/(1-t))$ are the identity matrix and Pascal's triangle, respectively.

If $(A(t), g(t)$ and $(B(t), f(t))$ are Riordan matrices, then

$$(A(t), g(t)) * (B(t), f(t)) := (A(t)B(g(t)), f(g(t))) \tag{1.2}$$

is called the matrix multiplication, i.e., for $(A(t), g(t)) = [d_{nk}]_{n \geqslant k \geqslant 0}$ and $(B(t), f(t)) = [c_{nk}]_{n \geqslant k \geqslant 0}$ we have

$$(A(t), g(t)) * (B(t), f(t)) := (A(t)B(g(t)) \cdot f(g(t))) = [d_{nk}][c_{nk}]. \tag{1.3}$$

The set of all Riordan matrices is a group under the matrix multiplication (cf. [23–25]).

Definition 1.2. Let $A(t)$ and $g(t)$ be defined as 1.1. Then the polynomials $p_n(x)$ $(n = 0, 1, 2, \ldots)$ defined by the generating function (GF)

$$A(t)e^{xg(t)} = \sum_{n \geqslant 0} p_n(x) t^n \tag{1.4}$$

are called Sheffer-type polynomials with $p_0(x) = 1$. Accordingly, $p_n(D)$ with $D \equiv \mathrm{d}/\mathrm{d}t$ is called Sheffer-type differential operator of degree n associated with $A(t)$ and $g(t)$. In particular, $p_0(D) \equiv I$ is the identity operator.

The set of all Sheffer-type polynomial sequences $\{p_n(x) = [t^n]A(t)e^{xg(t)}\}$ with an operation, "umbral composition" (cf. [18,19]), shown later forms a group called the Sheffer group. We will also show that the Riordan group and the Sheffer group are isomorphic.

In Roman's book [18], $\{S_n = n!p_n(x)\}$ is called Sheffer sequence (also cf. [19,20]). Certain recurrence relation of $p_n(x)$ can be found in Hsu–Shiue's paper [12]. There are two special kinds of weighted Stirling numbers defined by Carlitz [4] (see also [2,8]). We now give the following definition of the generalized Stirling numbers.

Definition 1.3. Let $A(t)$ and $g(t)$ be defined as 1.1, and let

$$\frac{1}{k!}A(t)(g(t))^k = \sum_{n \geqslant k} \sigma(n, k) \frac{t^n}{n!}. \tag{1.5}$$

Then $\sigma(n, k)$ is called the generalized Stirling number with respect to $A(t)$ and $g(t)$.

The special case of $A(t) \equiv 1$ was studied in [10].

As having been commonly employed in the calculus of finite differences as well as in combinatorial analysis, the operators E, Δ, D are defined by the following relations:

$$Ef(t) = f(t+1), \quad \Delta f(t) = f(t+1) - f(t), \quad Df(t) = \frac{\mathrm{d}}{\mathrm{d}t} f(t).$$

Powers of these operators are defined in the usual way. In particular, for any real numbers x, one may define $E^x f(t) = f(t+x)$. Also, the number 1 may be used as an identity operator, viz. $1f(t) \equiv f(t)$. It is easy to verify that these operators satisfy the formal relations (cf. [13])

$$E = 1 + \Delta = e^D, \quad \Delta = E - 1 = e^D - 1, \quad D = \log(1+\Delta).$$

From Definitions 1.1–1.3 we have

$$p_n(x) = [t^n]A(t)e^{xg(t)} = [t^n]\sum_{k\geqslant 0}\frac{1}{k!}A(t)(g(t))^k x^k = \sum_{k=0}^{n} d_{n,k}\frac{x^k}{k!} = \frac{1}{n!}\sum_{k=0}^{n}\sigma(n,k)x^k, \tag{1.6}$$

where we use $d_{n,k} = \sigma(n,k) = 0$ for all $k > n$. Therefore, with a constant multiple, $1/(k!)$, of the kth column, the rows of the Riordan array present the coefficients of the Sheffer-type polynomial sequences. As an example, the rows of the Riordan array $(1/(1-t), t/(t-1)) = [(-1)^k \binom{n}{k}]_{0\leqslant k\leqslant n}$ give the coefficients of the Laguerre polynomial sequences $\{p_n(x) = \sum_{k=0}^{n}(-1)^k \binom{n}{k} x^k/k!\}_{0\leqslant n}$.

This paper will be managed as follows. We shall define the Sheffer group and present its properties in Section 2. Then the Riordan array pairs and the generalized Stirling pairs will be shown in Section 3. Finally, we shall discuss their higher dimensional extension in Section 4.

2. Sheffer group

Let $\{p_n(x) = \sum_{k=0}^{n} p_{n,k}x^k\}$ and $\{q_n(x) = \sum_{k=0}^{n} q_{n,k}x^k\}$ be two Sheffer-type polynomial sequences. Then we define an operation, #, of $\{p_n(x)\}$ and $\{q_n(x)\}$, called the (polynomial) sequence multiplication (or the "umbral composition" see [18,19]), as

$$\left\{p_n(x)\}\#\{q_n(x)\} := \{r_n(x) = \sum_{k=0}^{n} r_{n,k}x^k\right\}, \tag{2.1}$$

where

$$r_{n,k} = \sum_{\ell=k}^{n} \ell! p_{n\ell} q_{\ell k}, \quad n \geqslant \ell \geqslant k. \tag{2.2}$$

It is clear that the defined operation is not commutative. Sheffer group under the "umbral composition" was defined with the $n!$-umbral calculus in Roman [18]. We now give a formulation with the matrix form, or the 1-umbral calculus (cf. [24]).

Theorem 2.1. *The set of all Sheffer-type polynomial sequences defined by Definition* (1.2) *with the operation # defined by* (2.2) *forms a group called the Sheffer group and denoted by* $(\{p_n(x)\}, \#)$. *The identity of the group is* $\{x^n/n!\}$. *The inverse of* $\{p_n(x)\}$ *in the group, denoted by* $\{p_n(x)\}^{(-1)}$, *is the Sheffer-type polynomial sequence generated by* $1/A(\bar{g}(t))\exp(x\bar{g}(t))$, *where* $\bar{g}$ *is the compositional inverse of* g; *i.e.*, $(g\circ\bar{g})(t) = (\bar{g}\circ g)(t) = t$.

Proof. We now give a sketch of the proof. Some obvious details are omitted. Let $\{p_n(x) = \sum_{k=0}^{n} p_{n,k}x^k\}$, $\{q_n(x) = \sum_{k=0}^{n} q_{n,k}x^k\}$, and $\{r_n(x) = \sum_{k=0}^{n} r_{n,k}x^k\}$ be three Sheffer-type polynomial sequences. It can also be found the operation of the sequence multiplication satisfies the associative law, namely,

$$\{p_n(x)\}\#(\{q_n(x)\}\#\{r_n(x)\}) = \left\{\sum_{k=0}^{n}\left(\sum_{u=k}^{n}\sum_{\ell=u}^{n}\ell!u!p_{n,\ell}q_{\ell,u}r_{u,k}\right)x^k\right\} = (\{p_n(x)\}\#\{q_n(x)\})\#\{r_n(x)\}.$$

It is clear that $\{p_n(x)\}\#\{x^n/n!\}=\{x^n/n!\}\#\{p_n(x)\}=\{p_n(x)\}$. Hence, the set of all Sheffer-type polynomial sequences forms a group. □

From (1.6) we can establish the mapping $\theta : [d_{n,k}] \mapsto \{p_n(x)\}$ or $\theta : (A(t), g(t)) \mapsto \{p_n(x)\}$ as follows:

$$\theta([d_{n,k}]_{n\geqslant k\geqslant 0}) := \sum_{j=0}^{n} d_{n,j}x^j/j! = [d_{n,k}]_{n\geqslant k\geqslant 0}X, \tag{2.3}$$

for fixed n, where $X=(1, x, x^2/2!, \ldots)^{\mathrm{T}}$, or equivalently,

$$\theta((A(t), g(t)) := [t^n]A(t)e^{xg(t)}. \tag{2.4}$$

It is clear that $(1, t)$, the identity Riordan array maps to the identity Sheffer-type polynomial sequence $\{p_n(x) \equiv x^n/(n!)\}_{n\geqslant 0}$. From the Definition 1.1 we immediately know that

$$p_n(x) = [t^n]A(t)e^{xg(t)} \quad \text{if and only if } d_{n,k} = [t^n]A(t)(g(t))^k. \tag{2.5}$$

Hence, the mapping θ is one-to-one and onto. From the mapping defined by (2.3), we understand that the operation # defined in the Sheffer group is equivalent to the matrix multiplication of two Riordan matrices in the Riordan group. In [16], the connection between usual matrix multiplication and Riordan matrix multiplication is given. Hence, a connection between usual matrix multiplication and the Sheffer-type sequence multiplication can be established similarly. Using symbolic calculus with operators D and E, we find via (2.4) or Definition 1.2:

$$A(t)f(g(t)) = A(t)E^{g(t)}f(0) = A(t)e^{g(t)D}f(0) = \sum_{k\geqslant 0} t^k p_k(D)f(0). \tag{2.6}$$

This is the desired expression given in [7] to expand the composite function $A(t)f(g(t))$. Hence, we have $p_n(D)f(0) = [t^n]A(t)f(g(t))$, which, from (2.5) and the multiplication in the Riordan group, is equivalent to

$$\sum_{k=0}^{n} d_{n,k}f_k = [t^n]A(t)f(g(t)), \tag{2.7}$$

where $(f_0, f_1, \ldots)$ has GF $f(t)$. The last expression was used to find the Riordan subgroup by Shapiro recently (cf. [22]). Eq. (2.7) can also be considered as a linear transform to $f(t)$ or $(f_0, f_1, \ldots)$ represented by Riordan matrix $(A(t), g(t))$. Thus, (2.3) or (2.4) is the linear transform of e^{xt}. With the aid of (2.7), we may transfer a property of the Riordan group to the Sheffer group.

Example 2.1. We now consider $(1/(1-t), t/(t-1))$, an involution in the Riordan group (cf. [3]), that possesses the matrix form:

$$[d_{n,k}]_{n\geqslant k\geqslant 0} = \begin{bmatrix} 1 & & & & & \\ 1 & -1 & & & & \\ 1 & -2 & 1 & & & \\ 1 & -3 & 3 & -1 & & \\ 1 & -4 & 6 & -4 & 1 & \\ & \cdots & & \cdots & & \ddots \end{bmatrix}. \tag{2.8}$$

It is easy to find that

$$d_{n,k} = (-1)^k\binom{n}{k}.$$

Consequently,

$$p_n(x) = \theta[d_{n,k}] = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{x^k}{k!}, \tag{2.9}$$

which is the Laguerre polynomial of order zero. Conversely, for the given polynomials (2.9), we obtain matrix (2.8), in which entries satisfy

$$-d_{n,k-1} + d_{n,k} = d_{n+1,k}.$$

Hence, its generating functions satisfy

$$-tA(t)(g(t))^{k-1} + tA(t)(g(t))^k = A(t)(g(t))^k.$$

It follows that $g(t) = t/(t-1)$. From the first column of matrix (2.8) we also obtain $A(t) = 1/(1-t)$.

If the sequences $\{p_n(x)\}$ and $\{q_n(x)\}$ are mapped from the Riordan arrays $[d_{n,k}]$ and $[c_{n,k}]$, respectively, then from the defined operation, (2.2), of the polynomial sequence multiplication, the coefficients of the polynomials $p_n(x)$ and $q_n(x)$ are, respectively, $p_{n,k} = d_{n,k}/(k!)$ and $q_{n,k} = c_{n,k}/(k!)$, and hence, we have

$$[k!r_{n,k}]_{n\geqslant k\geqslant 0} = [d_{n,k}][c_{n,k}],$$

where $r_{n,k}$ are obtained in (2.2). Consequently, the Sheffer-type polynomial sequence $\{r_n(x)\}$ is mapped from the Riordan array:

$$[e_{n,k}] := [d_{n,k}][c_{n,k}],$$

where $e_{n,k} = \sum_{\ell=0}^{n} d_{n,\ell}c_{\ell,k}$ $(n\geqslant \ell\geqslant k)$.

Similarly, $\{L_n^{(p-1)}(x)\}^{-1} = \{L_n^{(p-1)}(x)\}$ because of

$$\left(\frac{1}{(1-t)^p}, \frac{t}{t-1}\right)^{-1} = \left(\frac{1}{(1-t)^p}, \frac{t}{t-1}\right).$$

Theorem 2.2. *The Sheffer group and the Riordan group are isomorphic.*

Proof. Let $\{p_n(x) = \sum_{k=0}^{n} p_{n,k}x^k\}$ and $\{q_n(x) = \sum_{k=0}^{n} q_{n,k}x^k\}$ be two Sheffer-type polynomial sequences mapped by θ from $(A(t), g(t))$ and $(B(t), f(t))$, respectively, i.e., $\theta(A(t), g(t)) = \{p_n(x)\}$ and $\theta(B(t), f(t)) = \{q_n(x)\}$ we have

$$\{p_n(x)\}\#\{q_n(x)\} = \theta((A(t), g(t)))\#\theta((B(t), f(t)) = \theta((A(t), g(t)) * (B(t), f(t)). \tag{2.10}$$

Since mapping $\theta : [d_{n,k}] \mapsto \{p_n(x)\}$ or equivalently, $\theta : (A(t), g(t)) \mapsto \{p_n(x)\}$ is one-to-one and onto and satisfies (2.10), we obtain the theorem. □

It is clear that the identity Sheffer polynomial sequence, $\{x^n/n!\}$, is the mapping from the Riordan array $(1, t)$. Hence, the inverse of a Sheffer-type polynomial sequence $\{p_n(x)\}$, denoted by $\{p_n(x)\}^{-1}$, is defined as $\theta(1/(A(\bar{g}(t))), \bar{g}(t))$, where $\{p_n(x)\} = \theta(A(t), g(t))$ and $\bar{g}(t)$ is the compositional inverse of $g(t)$, i.e., $g(\bar{g}(t)) = \bar{g}(g(t)) = t$.

Example 2.2. We now consider an exponential Riordan array $(t/(e^t-1), t)$. Since $\theta(t/(e^t-1), t) = \{(1/n!)B_n(x)\}$ and $(t/(e^t-1), t)^{-1} = ((e^t-1)/t, t)$, we have

$$\left\{\frac{1}{n!}B_n(x)\right\}^{-1} = \left(\frac{e^t-1}{t}, t\right) = \left\{\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}\frac{x^{n-k}}{k+1}\right\}.$$

Similarly, $\{L_n^{(p-1)}(x)\}^{-1} = \{L_n^{(p-1)}(x)\}$ because of

$$\left(\frac{1}{(1-t)^p}, \frac{t}{t-1}\right)^{-1} = \left(\frac{1}{(1-t)^p}, \frac{t}{t-1}\right).$$

Example 2.3. We now consider the sequence multiplication of the $(p-1)$st order Laguerre polynomial sequences generated by involution $(1/(1-t)^p, t/(t-1))$ in the Riordan group (cf. [3]),

$$\{L_n^{p-1}(x)\}\#\{L_n^{p-1}(x)\} = \left(\frac{1}{(1-t)^p}, \frac{t}{t-1}\right) * \left(\frac{1}{(1-t)^p}, \frac{t}{t-1}\right) = (1, t),$$

or equivalently

$$\{L_n^{p-1}(x)\}\#\{L_n^{p-1}(x)\} = \left\{\frac{x^n}{n!}\right\},$$

which implies that $\{L_n^{p-1}(x)\}$ is the inverse of itself and the following identity:

$$\sum_{\ell=0}^{n}(-1)^{k+\ell}\frac{(n+p-1)!}{(p+k-1)!(n-\ell)!(\ell-k)!} = \delta_{n,k}, \tag{2.11}$$

where $\delta_{n,k}$ is the Kronecker symbol, which takes value 1 when $n=k$ and zero otherwise.

Example 2.4. Since the multiplication of two exponential Riordan arrays

$$\left(\frac{t}{e^t-1}, t\right) * \left(\frac{2}{e^t+1}, t\right) = \left(\frac{2t}{e^{2t}-1}, t\right),$$

we can present the result of the sequence operation of the Bernoulli polynomial sequence and the Euler polynomial sequence as

$$\left\{\frac{1}{n!}B_n(x)\right\} \# \left\{\frac{1}{n!}E_n(x)\right\} = \left\{\frac{1}{n!}B_n\left(\frac{x}{2}\right)\right\}.$$

Remark 2.1. Let $p_n(x) = a_0 + a_1 x + \cdots + a_n x^n$, $a_i \in \mathbb{R}$, $n = 0, 1, \ldots$, with the corresponding lower triangular array

$$A = \begin{bmatrix} d_{0,0} & & & & \\ d_{1,0} & d_{1,1} & & & \\ d_{2,0} & d_{2,1} & d_{2,2} & & \\ \vdots & \vdots & \vdots & \ddots & \\ d_{n,0} & d_{n,1} & d_{n,2} & \cdots & d_{n,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

Then the Sheffer polynomial sequence $\{p_n(x)\}$ can be regarded as the matrix transformation θ defined by

$$\theta : A \mapsto A \begin{bmatrix} 1 \\ x \\ \frac{x^2}{2!} \\ \vdots \\ \frac{x^n}{n!} \\ \vdots \end{bmatrix}. \tag{2.12}$$

By using (2.12), we may find subgroups of the Sheffer group if the corresponding subgroups of the Riordan group can be found. In addition, the above consideration can be extended to the higher dimensional setting.

Furthermore, the operation defined by (2.1)–(2.2), of $\theta_A X = \{p_n(x)\}$ and $\theta_B X = \{q_n(x)\}$, $X = [1, x, x^2/2!, \ldots, x^n/n!, \ldots]^T$, where $\{p_n(x)\}$ and $\{q_n(x)\}$ are two sequences in the Sheffer group with corresponding Riordan matrices

A and B, respectively, can be written as

$$\{p_n(x)\}\#\{q_n(x)\} := (\theta_{AB}) \begin{bmatrix} 1 \\ x \\ \frac{x^2}{2!} \\ \vdots \\ \frac{x^n}{n!} \\ \vdots \end{bmatrix} = (AB) \begin{bmatrix} 1 \\ x \\ \frac{x^2}{2!} \\ \vdots \\ \frac{x^n}{n!} \\ \vdots \end{bmatrix},$$

a regular matrix multiplication of A and B. Based on this point of view, $(\{p_n(x)\}, \#, +)$ can be considered as a ring, where $+$ is a used addition of matrices.

Definition 2.1. Let $\{p_n(x)\}$ and $\{q_n(x)\}$ be two Sheffer polynomial sequences. We say $\{p_n(x)\}$ and $\{q_n(x)\}$ are combinatorial orthogonal if they satisfy

$$\{p_n(x)\}\#\{q_n(x)\} = \{q_n(x)\}\#\{p_n(x)\} = \left\{\frac{x^n}{n!}\right\}, \tag{2.13}$$

and we denote $\{p_n(x)\}\perp_{\rm com}\{q_n(x)\}$.

Example 2.5. Laguerre polynomial sequence is combinatorial orthogonal, i.e., $\{L_n^{(p-1)}(x)\}\perp_{\rm com}\{L_n^{(p-1)}(x)\}$. Although Laguerre polynomials are also analytic orthogonal, i.e., orthogonal in an inner product sense, it is not necessary that the analytic orthogonality implies the combinatorial orthogonality or verse vise. For instance, $\theta(1/(1-t), t) = \{1, 1+x, 1+x+x^2/2, \ldots\}$ and $\theta(1-t, t) = \{1, -1+x, -x+x^2/2, \ldots\}$ are combinatoric orthogonal, but not analytic orthogonal.

At the end of this section, we give a list of some Sheffer polynomials for the interested readers to construct the inverses and the resulting polynomials under the sequence multiplication. In the table, we can see many array components are exponential Riordan array components.

$A(t)$	$g(t)$	$p_n(x)$	Name of polynomials
$t/(e^t-1)$	t	$\frac{1}{n!}B_n(x)$	Bernoulli
$2/(e^t+1)$	t	$\frac{1}{n!}E_n(x)$	Euler
e^t	$\log(1+t)$	$(PC)_n(x)$	Poisson–Charlier
$e^{-\alpha t}(\alpha \neq 0)$	$\log(1+t)$	$\hat{C}_n^{(\alpha)}(x)$	Charlier
1	$\log(1+t)/(1-t)$	$(ML)_n(x)$	Mittag–Leffler
$(1-t)^{-1}$	$\log(1+t)/(1-t)$	$p_n(x)$	Pidduck
$(1-t)^{(-p)}(p>0)$	$t/(t-1)$	$L_n^{(p-1)}(x)$	Laguerre
$e^{\lambda t}(\lambda \neq 0)$	$1-e^t$	$(Tos)_n^{(\lambda)}(x)$	Toscano
1	e^t-1	$\tau_n(x)$	Touchard
$1/(1+t)$	$t/(t-1)$	$A_n(x)$	Angelescu
$(1-t)/(1+t)^2$	$t/(t-1)$	$(De)_n(x)$	Denisyuk
$(1-t)^{-p}(p>0)$	e^t-1	$T_n^{(p)}(x)$	Weighted–Touchard

3. Riordan array pairs and generalized Stirling number pairs

We first define the Riordan pairs.

Definition 3.1. Let $A(t)$ and $g(t)$ be given as in Definition 1.1. Then we have a Riordan pair $\{d_{n,k}, \bar{d}_{n,k}\}$ as defined by

$$\begin{cases} A(t)(g(t))^k = \sum_{n=k}^{\infty} d_{n,k} t^n, \\ A(\bar{g}(t))^{-1}(\bar{g}(t))^k = \sum_{n=k}^{\infty} \bar{d}_{n,k} t^n, \end{cases} \tag{3.1}$$

where $\bar{g} \equiv g^{\langle -1 \rangle}$ is the compositional inverse of g with $\bar{g}(0) = 0$, $[t]\bar{g}(t) \neq 0$, and $d_{0,0} = \bar{d}_{0,0} = 1$.

We also need the following definition of generalized Stirling number pairs (cf. [10] for the case of $A \equiv 1$, and a special example has been also studied in [24]).

Definition 3.2. Let $A(t)$ and $g(t)$ be given as in Definition 1.1. Then we have a generalized Stirling number pair $\{\sigma(n,k), \bar{\sigma}(n,k)\}$ as defined by

$$\begin{cases} \frac{1}{k!} A(t)(g(t))^k = \sum_{n=k}^{\infty} \sigma(n,k) \frac{t^n}{n!}, \\ \frac{1}{k!} A(\bar{g}(t))^{-1}(\bar{g}(t))^k = \sum_{n=k}^{\infty} \bar{\sigma}(n,k) \frac{t^n}{n!}, \end{cases} \tag{3.2}$$

where $\bar{g} \equiv g^{\langle -1 \rangle}$ is the compositional inverse of g with $\bar{g}(0) = 0$, $[t]\bar{g}(t) \neq 0$, and $\sigma(0,0) = \bar{\sigma}(0,0) = 1$.

Remark 3.1. A closed connection between (3.1) and (3.2) is apparent. A special case of pair (3.2) for $A(t) = 1$ was established in [9], which were later applied to derive some combinatorial identities (cf. [26]).

Remark 3.2. If in (3.2) let $A(t)$ and $g(t)$ be defined by

$$A(t) = (1+\alpha t)^{\gamma/\alpha}, \quad g(t) = ((1+\alpha t)^{\beta/\alpha} - 1)/\beta,$$

where α, β, and γ are real or complex numbers with $\alpha\beta \neq 0$, then

$$A(\bar{g}(t)) = (1+\beta t)^{\gamma/\beta}, \quad \bar{g}(t) = ((1+\beta t)^{\alpha/\beta} - 1)/\alpha,$$

so that $\sigma(n,k) = S(n,k;\alpha,\beta,\gamma)$ and $\bar{\sigma}(n,k) = S(n,k;\beta,\alpha,-\gamma)$ just form a pair of GSNs with three parameters. Note that such a class of GSN-pairs includes various useful special number-pairs. A detailed investigation of GSNs was given in [11] in 1998. For a very recent development relating to this subject, see [17].

Note that (3.1)–(3.2) imply the orthogonality relations

$$\sum_{k \leqslant n \leqslant m} d_{m,n} \bar{d}_{n,k} = \sum_{k \leqslant n \leqslant m} \bar{d}_{m,n} d_{n,k} = \delta_{mk},$$

and

$$\sum_{k \leqslant n \leqslant m} \sigma(m,n)\bar{\sigma}(n,k) = \sum_{k \leqslant n \leqslant m} \bar{\sigma}(m,n)\sigma(n,k) = \delta_{mk},$$

with δ_{mk} denoting the Kronecker delta, and it follows that there hold the inverse relations:

$$\frac{f_n}{n!} = \sum_{k=0}^{n} d_{n,k} \frac{g_k}{k!} \iff \frac{g_n}{n!} = \sum_{k=0}^{n} \bar{d}_{n,k} \frac{f_k}{k!}, \tag{3.3}$$

and

$$f_n = \sum_{k=0}^{n} \sigma(n,k) g_k \iff g_n = \sum_{k=0}^{n} \bar{\sigma}(n,k) f_k. \tag{3.4}$$

For an element $\{p_n(x)\}$ in the Sheffer group $(\{p_n(x)\}, \#)$, it is easy to write its inverse $\{\bar{p}_n(x)\} = \{p_n(x)\}^{-1}$ as

$$\bar{p}_n(x) = \sum_{k=0}^{n} \bar{d}_{n,k} \frac{x^k}{k!} = \frac{1}{n!} \sum_{k=0}^{n} \bar{\sigma}(n,k) x^k,$$

which are generated by

$$A(\bar{g}(t))^{-1} e^{x\bar{g}(t)} = \sum_{n \geqslant 0} \bar{p}_n(x) t^n,$$

with $\bar{p}_0(x) = 1$.

We shall give an application of the inverse formulas (3.3)–(3.4) based on the following result.

Theorem 3.3. *The Sheffe-type operator $p_n(D)$ has an expression of the form*

$$p_n(D) = \frac{1}{n!} \sum_{k=0}^{n} \sigma(n,k) D^k, \tag{3.5}$$

where $\sigma(n,k)$ (associated with $A(t)$ and $g(t)$) may be written in the form

$$\sigma(n,k) = \sum_{r=k}^{n} \binom{n}{r} \alpha_{n-r} B_{rk}(a_1, a_2, \ldots), \tag{3.6}$$

provided that $A(t) = \sum_{m \geqslant 0} \alpha_m t^m / m!$ and $g(t) = \sum_{m \geqslant 1} a_m t^m / m!$ with $\alpha_0 = 1$, $a_1 \neq 0$.

Proof. Note that (3.6) follows from (1.6). Moreover, recall a known expression for potential polynomials (cf. e.g., Comtet [5, Section 3.5, Theorem B], etc.). We have

$$\frac{1}{k!} (g(t))^k = \sum_{r \geqslant k}^{\infty} \frac{t^r}{r!} B_{rk}(a_1, a_2, \ldots).$$

Substituting this into (3.2) and comparing the resulting expression with the RHS of (3.2), we see that (3.6) is true. □

Corollary 3.4. *Formula* (2.5) *may be rewritten in the form*

$$A(t) f(g(t)) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \left(\sum_{k=0}^{n} \sigma(n,k) f^{(k)}(0) \right) = \sum_{n=0}^{\infty} t^n \left(\sum_{k=0}^{n} \frac{d_{n,k}}{k!} f^{(k)}(0) \right), \tag{3.7}$$

where $\sigma(n,k)$'s are defined by (3.1) *and given by* (3.6).

Corollary 3.5. *For the case $A(t) = 1$,* (3.6) *gives*

$$\sigma(n,k) = B_{n,k}(a_1, a_2, \ldots),$$

and

$$d_{n,k} = \frac{k!}{n!} B_{n,k}(a_1, a_2, \ldots),$$

where the incomplete Bell polynomial $B_{n,k}(a_1, a_2, \ldots)$ *has an explicit expression* (*cf.* [5])

$$B_{n,k}(a_1, a_2, \ldots) = \sum_{(c)} \frac{n!}{c_1!c_2!\cdots} \left(\frac{a_1}{1!}\right) c_1 \left(\frac{a_2}{2!}\right) c_2 \cdots,$$

where the summation extends over all integers $c_1, c_2, \ldots \geqslant 0$, *such that* $c_1 + 2c_2 + 3c_3 + \cdots = n, c_1 + c_2 + \cdots = k$.

Corollary 3.6. *The generalized exponential polynomials related to the generalized Stirling numbers* $\sigma(n, k)$ *and* $\bar{\sigma}(n, k)$ *are given, respectively, by the following*:

$$n!p_n(x) = \sum_{k=0}^{n} \sigma(n, k)x^k, \tag{3.8}$$

and

$$n!\bar{p}_n(x) = \sum_{k=0}^{n} \bar{\sigma}(n, k)x^k, \tag{3.9}$$

where $p_n(x)$ *and* $\bar{p}_n(x)$ *are Sheffer-type polynomials associated with* $\{A(t),\ g(t)\}$ *and* $\{A(\bar{g}(t))^{-1}, \bar{g}(t)\}$, *respectively*

Applying the reciprocal relations (3.4)–(3.9) we get

Corollary 3.7. *There hold the relations*

$$\sum_{k=0}^{n} \bar{\sigma}(n, k)k!p_k(x) = x^n, \tag{3.10}$$

and

$$\sum_{k=0}^{n} \sigma(n, k)k!\bar{p}_k(x) = x^n. \tag{3.11}$$

These may be used as recurrence relations for $p_n(x)$ *and* $p_n^*(x)$ *respectively.*

Eqs. (3.10) and (3.11) are equivalently

$$\begin{cases} \sum_{k=0}^{n} \bar{d}_{n,k} p_k(x) = \dfrac{x^n}{n!}, \\ \sum_{k=0}^{n} d_{n,k} \bar{p}_k(x) = \dfrac{x^n}{n!}. \end{cases} \tag{3.12}$$

Evidently (3.4) and (3.7) imply a higher derivative formula for $A(t)f(g(t))$ at $t = 0$, namely

$$\left(\frac{\mathrm{d}^n}{\mathrm{d}t^n}\right)(A(t)f(g(t)))\bigg|_{t=0} = \sum_{k=0}^{n} \sigma(n, k)f^{(k)}(0) = n!p_n(D)f(0).$$

Certainly, this will reduce to the Faa di Bruno formula when $A(t) = 1$.

Example 3.1. As a simple instance take $\{\sigma(n, k), \bar{\sigma}(n, k)\}$ to be the ordinary Stirling numbers $\{s(n, k), \bar{s}(n, k)\}$ of the first and second kinds. Then (3.9) yields the Bell number $W(n)$ at $x = 1$, namely,

$$W(n) = n!\bar{p}_n(1) = \sum_{k=0}^{n} s(n, k).$$

Consequently, (3.11) gives the simple identity

$$\sum_{k=0}^{n} s(n,k)W(k)=1.$$

More examples could be constructed using Sheffer polynomials listed in the table at the end of Section 2.

4. Higher dimensional extension

We now extend the Riordan group to the higher dimensional setting. In what follows we shall adopt the multi-index notational system. Denote

$$\hat{t}\equiv(t_1,\ldots,t_r),\quad \hat{x}\equiv(x_1,\ldots,x_r),$$

$$\hat{t}+\hat{x}\equiv(t_1+x_1,\ldots,t_r+x_r),$$

$$\hat{0}\equiv(0,\ldots,0),\quad \widehat{g(t)}\equiv(g_1(t_1),\ldots,g_r(t_r)),$$

$$\hat{x}\cdot\widehat{g(t)}\equiv\sum_{i=1}^{r}x_i g_i(t_i).$$

Also, E_i means the shift operator acting on t_i, namely for $1\leqslant i\leqslant r$,

$$E_i f(\ldots,t_i,\ldots)=f(\ldots,t_i+1,\ldots),$$

$$E_i^{x_i} f(\ldots,t_i,\ldots)=f(\ldots,t_i+x_i,\ldots).$$

Formally we may denote $E_i=e^{D_i}=\exp(\partial/\partial t_i)$. Moreover, we write $t^\lambda\equiv t_1^{\lambda_1},\ldots,t_r^{\lambda_r}$ with $\lambda\equiv\hat{\lambda}\equiv(\lambda_1,\ldots,\lambda_r)$, r being positive integer. Also, $\lambda\geqslant\hat{0}$ means $\lambda_i\geqslant 0$ $(i=1,\ldots,r)$, and $\lambda\geqslant\mu$ means $\lambda_i\geqslant\mu_i$ for all $i=1,\ldots,r$.

We first give an analog of Definition 1.2.

Definition 4.1. Let $\hat{t}=(t_1,t_2,\ldots,t_r)$, $A(\hat{t})$, $\widehat{g(t)}=(g_1(t_1),g_2(t_2),\ldots,g_r(t_r))$ and $f(\hat{t})$ be any given formal power series over the complex number field $\mathbb{C}^r$ with $A(\hat{0})=1$, $g_i(0)=0$ and $g_i'(0)\neq 0$ $(i=1,2,\ldots,r)$. Then the polynomials $p_{\hat{n}}(\hat{x})$ $(\hat{n}\in\mathbb{N}^r\cup\hat{0})$ as defined by the GF

$$A(\hat{t})e^{\hat{x}\cdot\widehat{g(t)}}=\sum_{\hat{n}\geqslant\hat{0}}p_{\hat{n}}(\hat{x})t^{\hat{n}} \tag{4.1}$$

are called Sheffer-type polynomials with $p_{\hat{0}}(\hat{x})=1$. Accordingly, $p_{\hat{n}}(\hat{D})$ with $\hat{D}\equiv(D_1,D_2,\ldots,D_r)$ is called Sheffer-type differential operator of degree $\hat{n}$ associated with $A(\hat{t})$ and $\widehat{g(t)}$. In particular, $p_{\hat{0}}(\hat{D})\equiv I$ is the identity operator.

For formal power series $f(\hat{t})$, the coefficient of $t^\lambda=(t_1^{\lambda_1},t_2^{\lambda_2},\ldots,t_r^{\lambda_r})$ is usually denoted by $[t^\lambda]f(\hat{t})$. Accordingly, (4.1) is equivalent to the expression $p_\lambda(\hat{x})=[t^\lambda]A(\hat{t})e^{\hat{x}\cdot\widehat{g(t)}}$. Also, we shall frequently use the notation

$$p_\lambda(\hat{D})f(\hat{0})=[p_\lambda(\hat{D})f(\hat{t})]_{\hat{t}=\hat{0}}, \tag{4.2}$$

and $\lambda!\equiv\hat{\lambda}!=\lambda_1!\lambda_2!\cdots\lambda_r!$.

Definition 4.2. Let $A(\hat{t})$ and $\widehat{g(t)}$ be any formal power series defined on $\mathbb{C}^r$, with $A(\hat{0})=1$, $g_i(0)=0$ and $g_i'(0)\neq 0$ $(i=1,2,\ldots,r)$. Then we have a multivariate weighted Stirling-type pair $\{\sigma(\hat{n},\hat{k}),\sigma^*(\hat{n},\hat{k})\}$ as defined by

$$\frac{1}{\hat{k}!}A(\hat{t})\Pi_{i=1}^{r}(g_i(t_i))^{k_i}=\sum_{\hat{n}\geqslant\hat{k}}\sigma(\hat{n},\hat{k})\frac{t^{\hat{n}}}{\hat{n}!}, \tag{4.3}$$

$$\frac{1}{\hat{k}!}A(\widehat{g^*(t)})^{-1}\Pi_{i=1}^{r}(g_i^*(t_i))^{k_i} = \sum_{\hat{n}\geqslant\hat{k}} \sigma^*(\hat{n},\hat{k})\frac{t^{\hat{n}}}{\hat{n}!}, \tag{4.4}$$

where $\widehat{g^*(t)} = (g_1^*(t_1), g_2^*(t_2), \ldots, g_r^*(t_r))$, $g_i^* \equiv g_i^{\langle -1\rangle}$ is the compositional inverse of g_i $(i = 1, 2, \ldots, r)$ with $g_i^*(0) = 0$, $[t_i]g_i^*(t_i) \neq 0$, and $\sigma(\hat{0},\hat{0}) = \sigma^*(\hat{0},\hat{0}) = 1$. We call $\sigma(\hat{n},\hat{k})$ the dual of $\sigma^*(\hat{n},\hat{k})$ and vice verse. We will also call

$$d_{\hat{n},\hat{k}} := \frac{\hat{k}}{\hat{n}}\sigma(\hat{n},\hat{k}), \quad d^*_{\hat{n},\hat{k}} := \frac{\hat{k}}{\hat{n}}\sigma^*(\hat{n},\hat{k}),$$

the multivariate Riordan arrays and denote them by $(A(\hat{t}), \widehat{g(t)})$ and $(1/A(\widehat{g^*(t)}), \widehat{g^*(t)})$, respectively.

Example 4.1. As an example, considering $A(\hat{t}) = 1$ and $\widehat{g(t)} = \hat{t}$, we obtain the $p_\lambda(\hat{x})$ defined by (4.1), namely,

$$p_\lambda(\hat{x}) = \frac{x^\lambda}{\lambda!}.$$

Thus the multivariate weighted Stirling-type pair $\{\sigma(\hat{n},\hat{k}), \sigma^*(\hat{n},\hat{k})\}$ defined as (4.3)–(4.4) is $(\sigma(\hat{n},\hat{k}), \sigma^*(\hat{n},\hat{k}))$, where

$$\sigma(\hat{n},\hat{k}) = \sigma^*(\hat{n},\hat{k}) = \delta_{\hat{n},\hat{k}}.$$

The corresponding multivariate Riordan array pair is $(d_{\hat{n},\hat{k}}, d^*_{\hat{n},\hat{k}})$, where

$$d_{\hat{n},\hat{k}} = d^*_{\hat{n},\hat{k}} = \frac{\hat{k}}{\hat{n}}\delta_{\hat{n},\hat{k}}.$$

A similar argument as (2.3) and (2.4) can be established as follows.

Theorem 4.3. *Eqs.* (4.3) *and* (4.4) *imply the biorthogonality relations*

$$\sum_{\hat{m}\geqslant\hat{n}\geqslant\hat{k}} \sigma(\hat{m},\hat{n})\sigma^*(\hat{n},\hat{k}) = \sum_{\hat{m}\geqslant\hat{n}\geqslant\hat{k}} \sigma^*(\hat{m},\hat{n})\sigma(\hat{n},\hat{k}) = \delta_{\hat{m}\hat{k}}, \tag{4.5}$$

with $\delta_{\hat{m}\hat{k}}$ *denoting the Kronecker delta, i.e.,* $\delta_{\hat{m}\hat{k}} = 1$ *if* $\hat{m} = \hat{k}$ *and* 0 *otherwise, and it follows that there hold the inverse relations*:

$$f_{\hat{n}} = \sum_{\hat{n}\leqslant\hat{k}\leqslant\hat{0}} \sigma(\hat{n},\hat{k})g_{\hat{k}} \iff g_{\hat{n}} = \sum_{\hat{n}\leqslant\hat{k}\leqslant\hat{0}} \sigma^*(\hat{n},\hat{k})f_{\hat{k}}. \tag{4.6}$$

Proof. Transforming t_i by $g_i^*(t_i)$ in (4.3) and multiplying $A(\widehat{g^*(t)})^{-1}(\hat{k}!)$ on the both sides of the resulting equation yields

$$t^{\hat{k}} = \sum_{\hat{n}\geqslant\hat{k}} \sigma(\hat{n},\hat{k})\frac{\hat{k}!}{\hat{n}!}A(\widehat{g^*(t)})^{-1}\Pi_{i=1}^{r}(g_i^*(t_i))^{n_i}. \tag{4.7}$$

By substituting (4.4) into the above equation, we obtain

$$t^{\hat{k}} = \sum_{\hat{n}\geqslant\hat{k}} \sigma(\hat{n},\hat{k}) \sum_{\hat{m}\geqslant\hat{n}} \sigma^*(\hat{m},\hat{n})\frac{\hat{k}!}{\hat{m}!}t^{\hat{m}} = \sum_{\hat{m}\geqslant\hat{k}} \frac{\hat{k}!}{\hat{m}!}t^{\hat{m}} \sum_{\hat{m}\geqslant\hat{n}\geqslant\hat{k}} \sigma^*(\hat{m},\hat{n})\sigma(\hat{n},\hat{k}).$$

Equating the coefficients of the terms $t^{\hat{m}}$ on the leftmost side and the rightmost side of the above equation leads (4.5) and (4.6). This completes the proof. □

Remark 4.1. $[\sigma(\hat{m},\hat{n})]$ and $[\sigma^*(\hat{n},\hat{k})]$ are a pair of inverse r-dimensional matrices, which may be useful in the higher dimensional matrix theory.

From (4.5), we can see that the $2r$ dimensional infinite matrices $\sigma(\hat{n},\hat{k})$ and $\sigma^*(\hat{n},\hat{k})$ are invertible for each other, i.e., their product is the identity matrix $[\delta_{\hat{n},\hat{k}}]_{\hat{n}\geqslant\hat{k}\geqslant\hat{0}}$.

By introducing group multiplication

$$(A(\hat{t}),\widehat{g(t)}) * (B(\hat{t}),\widehat{h(t)}) = (A(\hat{t})B(\widehat{g(t)}),\widehat{h(g(t))}), \tag{4.8}$$

where $\widehat{h(g(t))} = (h_1(g_1(t_1)),\ldots,h_r(g_r(t_r)))$, from Theorem 4.3, we immediately see that the inverse of $(A(\hat{t}),\widehat{g(t)})$ is $(1/A(\widehat{\bar{g}(t)}),\widehat{\bar{g}(t)})$ because their multiplication result is the identity $I=(1,\hat{t})$. Hence, similar to [23], we obtain the following corollary.

Corollary 4.4. *Let $A(\hat{t})$ and $\widehat{g(t)}$ be any formal power series defined on $\mathbb{C}^r$, with $A(\hat{0})=1$, $g_i(0)=0$ and $g_i'(0)\neq 0$ $(i=1,2,\ldots,r)$. Then with respect to the multiplication defined by* (4.8), *$\{(A(\hat{t}),\widehat{g(t)})\}$ forms a group with the identity $I=(1,\hat{t})$ and for any element $(A(\hat{t}),\widehat{g(t)})$ in the group, its inverse is $(1/A(\widehat{g^*(t)}),\widehat{g^*(t)})$, where $\widehat{g^*(t)} = (g_1^*(t_1),g_2^*(t_2),\ldots,g_r^*(t_r))$, $g_i^*\equiv g_i^{\langle -1\rangle}$ is the compositional inverse of g_i $(i=1,2,\ldots,r)$ with $g_i^*(0)=0$, $[t_i]g_i^*(0)\neq 0$.*

Proof. This proof is an analog of the proof on the one variable Riordan group (cf. [23]). Indeed, from (4.8) we have

$$(A(\hat{t}),\widehat{g(t)}) * I = (A(\hat{t}),\widehat{g(t)}),$$

$$\begin{aligned}((A(\hat{t}),\widehat{g(t)}) * (B(\hat{t}),\widehat{h(t)})) * (C(\hat{t}),\widehat{f(t)}) &= (A(\hat{t})B(\widehat{g(t)})C(\widehat{h(g(t))}), f(\widehat{h(g(t))}))\\ &= (A(\hat{t}),\widehat{g(t)}) * ((B(\hat{t}),\widehat{h(t)}) * (C(\hat{t}),\widehat{f(t)})),\end{aligned}$$

and

$$(A(\hat{t}),\widehat{g(t)}) * \left(\frac{1}{A(\widehat{g^*(t)})},\widehat{g^*(t)}\right) = \left(A(\hat{t})\frac{1}{A(\widehat{g^*(g(t))})},\widehat{g^*(g(t))}\right) = (1,\hat{t}) = I.$$

This completes the proof of the corollary. □

From Definition 4.1, we have

$$p_\lambda(\hat{x}) = [t^\lambda]A(\hat{t})e^{\hat{x}\cdot\widehat{g(t)}} = [t^\lambda]\sum_{\hat{k}\geqslant\hat{0}}\frac{1}{\hat{k}!}A(\hat{t})\Pi_{i=1}^r(g_i(t_i))^{\lambda_i}x_i^{\lambda_i} = \sum_{\lambda\geqslant k\geqslant\hat{0}} d_{\lambda,\hat{k}}\frac{x^\lambda}{\hat{k}!}.$$

Therefore, we establish a one-to-one and onto mapping θ^r from $[d_{\hat{n},\hat{k}}]$ to $p_{\hat{n}}(\hat{x})$, where $\theta^1\equiv\theta$ shown as in (2.9). By defining the operation, denoted as #, to two higher dimensional Sheffer type polynomial sequences, $\{p_{\hat{n}}(\hat{x}) = \sum_{\hat{n}\geqslant\lambda\geqslant\hat{0}}p_{\hat{n},\lambda}x^\lambda\}$ and $\{q_{\hat{n}}(\hat{x}) = \sum_{\hat{n}\geqslant\lambda\geqslant\hat{0}}q_{\hat{n},\lambda}x^\lambda\}$, as follows, the set $\{\{p_{\hat{n}}\},\#\}$ forms a group, called the higher dimensional Sheffer group:

$$\{p_{\hat{n}}\}\#\{q_{\hat{n}}\} = \left\{\sum_{\hat{n}\geqslant\lambda\geqslant\hat{0}} r_{\hat{n},\lambda}x^\lambda\right\},$$

where

$$r_{\hat{n},\lambda} = \sum_{\hat{n}\geqslant\hat{\ell}\geqslant\lambda}\hat{\ell}!p_{\hat{n},\hat{\ell}}q_{\hat{\ell},\lambda}.$$

Similar to Theorem 2.2, we can establish the following result.

Theorem 4.5. *The set $\{\{p_{\hat{n}}\},\#\}$ with the operation # is a group, called the higher dimensional Sheffer group that is isomorphic to the higher dimensional Riordan group defined in Corollary* 4.4.

Example 4.2. As an example of (4.1), we set $A(\hat{t})=1$ and $\exp(\hat{x}\cdot\widehat{g(t)})=\exp(x_1(e^{t_1}-1)+x_2(e^{t_2}-1)+\cdots+x_r(e^{t_r}-1))$ in (4.1) and obtain

$$\exp(\hat{x}\cdot\widehat{g(t)})=\sum_{\lambda\geqslant\hat{0}}\hat{\tau}_\lambda(\hat{x})t^\lambda, \tag{4.9}$$

where

$$\hat{\tau}_\lambda(\hat{x})=\Pi_{j=1}^r\tau_{\lambda_j}(x_j),$$

and $\tau_u(t)$ is the Touchard polynomial of degree u. Hence, we may call $\hat{\tau}_\lambda(\hat{x})$ the higher dimensional Touchard polynomial of order λ.

Example 4.3. Sheffer-type expansion (4.1) also includes the following two special cases shown as in [14]. Let $A(\hat{t})=2^m/(\exp\sum_{i=1}^r t_i+1)^m$ and $\exp(\hat{x}\cdot\widehat{g(t)})=\exp(\sum_{i=1}^r x_i t_i)$. Then the corresponding Sheffer-type expansion of (4.1) shown as in [14] has the form

$$A(\hat{t})\exp(\hat{x}\cdot\widehat{g(t)})=\sum_{\lambda\geqslant\hat{0}}\frac{E_\lambda^{(m)}(\hat{x})}{\lambda!}t^\lambda,$$

where $E_\lambda^{(m)}(\hat{x})$ $(\lambda\geqslant\hat{0})$ is defined as the mth order r-variable Euler's polynomial in [14].

Similarly, substituting $A(\hat{t})=(\sum_{i=1}^r t_i)^m/(\exp\sum_{i=1}^r t_i-1)^m$ and $\exp(\hat{x}\cdot\widehat{g(t)})=\exp(\sum_{i=1}^r x_i t_i)$ into (4.1) yields

$$A(\hat{t})\exp(\hat{x}\cdot\widehat{g(t)})=\sum_{\lambda\geqslant\hat{0}}\frac{B_\lambda^{(m)}(\hat{x})}{\lambda!}t^\lambda,$$

where $B_\lambda^{(m)}(\hat{x})$ $(\lambda\geqslant\hat{0})$ is called in [14] the mth order r-variable Bernoulli polynomial. Some basic properties of $E_\lambda^{(m)}(\hat{x})$ and $B_\lambda^{(m)}(\hat{x})$ were studied in [14].

Since

$$\left(\frac{2^m}{(\exp\sum_{i=1}^r t_i+1)^m},\hat{t}\right)*\left(\frac{(\sum_{i=1}^r t_i)^m}{(\exp\sum_{i=1}^r t_i+1)^m},\hat{t}\right)=\left(\frac{(\sum_{i=1}^r 2t_i)^m}{(\exp\sum_{i=1}^r 2t_i+1)^m},\hat{t}\right),$$

we have

$$\{E_\lambda^{(m)}(\hat{x})\}\#\{B_\lambda^{(m)}(\hat{x})\}=\{B_\lambda^{(m)}(2\hat{x})\}.$$

Example 4.4. For the case $A(\hat{t})=1$ and if $\widehat{g(t)}=\sum_{\hat{m}\geqslant(1,\ldots,1)}a_m\,t^m/(\hat{m})!$, where $a_m=a_{m_1}^{(1)}\cdots a_{m_r}^{(r)}$, it follows that $e^{\hat{x}\cdot\widehat{g(t)}}$ may be written in the form:

$$\exp(\hat{x}\cdot\widehat{g(t)})=\Pi_{\ell=1}^r\exp\left\{x_\ell\sum_{m_\ell\geqslant1}a_{m_\ell}^{(\ell)}\frac{t_\ell^{m_\ell}}{m_\ell!}\right\}=\Pi_{\ell=1}^r\left(1+\sum_{k_\ell\geqslant1}\frac{t_\ell^{k_\ell}}{k_\ell!}\left\{\sum_{j_\ell=1}^{k_\ell}x_\ell^{j_\ell}B_{k_\ell j_\ell}(a_1^{(\ell)},a_2^{(\ell)},\ldots)\right\}\right), \tag{4.10}$$

so that

$$p_\lambda(\hat{x})=[t^\lambda]e^{\hat{x}\cdot\widehat{g(t)}}=\Pi_{\ell=1}^r\frac{1}{\lambda_\ell!}\sum_{j_\ell=1}^{\lambda_\ell}x_\ell^{j_\ell}B_{\lambda_\ell j_\ell}(a_1^{(\ell)},a_2^{(\ell)},\ldots).$$

Consequently, we have

$$[t^\lambda]f(\widehat{g(t)})=\Pi_{\ell=1}^r\frac{1}{\lambda_\ell!}\sum_{j_\ell=1}^{\lambda_\ell}B_{\lambda_\ell j_\ell}(a_1^{(\ell)},a_2^{(\ell)},\ldots)D_\ell^{j_\ell}f(\hat{0}).$$

This is precisely the multivariate extension of the univariate Faa di Bruno formula (cf. [6] for another type extension)

$$[(\mathrm{d}/\mathrm{d}t)^k f(g(t))]_{t=0} = \sum_{j=1}^{k} B_{kj}(g'(0), g''(0), \ldots) f^{(j)}(0). \tag{4.11}$$

In this paper, examples seems not so enough to illustrate the merit of the theoretical results obtained. Interested reader might do something including more applications subsequently.

Remark 4.2. The properties of the higher dimensional Sheffer group, such as construction of the subgroup with certain orders, application to the multivariate expansions, combinatorial identities, etc., remain much to be investigated while some application results in this topic can be referred to [7].

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