A Combinatorial Approach to r-Fibonacci Numbers

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Abstract

In this paper we explore generalized "r-Fibonacci Numbers" using a combinatorial "tiling" interpretation. This approach allows us to provide simple, intuitive proofs to several identities involving r-Fibonacci Numbers presented by F.T. Howard and Curtis Cooper in the August, 2011, issue of the *Fibonacci Quarterly*. We also explore a connection between the generalized Fibonacci numbers and a generalized form of binomial coefficients.

Contents

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Chapter 1

Introduction

Recall the combinatorial Fibonacci numbers $\{f_n\}$, defined by

Definition 1.1.

$$f_n = \begin{cases} 1, & \text{if } n = 0; \\ 1, & \text{if } n = 1; \\ f_{n-1} + f_{n-2}, & \text{if } n > 1. \end{cases}$$

f_n counts the number of ways to tile an n-board, a $1 \times n$ grid with cells labeled $1, 2, \ldots, n$, with 1×1 squares and 1×2 dominoes. Length zero and length one boards can each be tiled in exactly one way (via the empty set and a single square, respectively), whereas for a board of length n, every tiling can be formed by either adding a domino to a tiled $(n-2)$-board (in f_{n-2} ways) or adding a square to a tiled $(n-1)$-board (f_{n-1} ways). We will sometimes refer to a tiled n-board as an n-tiling. R.T. Howard and Curtis Cooper (2011) have defined a generalization of this sequence.

Definition 1.2. *Let $r \geq 1$ be an integer. The r-generalized Fibonacci sequence $\{G_n\}$ is defined as*

$$G_n = \begin{cases} 0, & \text{if } 0 \leq n < r - 1; \\ 1, & \text{if } n = r - 1; \\ G_{n-1} + G_{n-2} + ... + G_{n-r}, & \text{if } n \geq r. \end{cases}$$

Howard and Cooper remark upon (but do not make use of) the fact that G_{n+r-1} counts the number of tilings of an n-board with tiles of length at most r. To simplify notation, we define a new sequence $\{g_n\}$ which corresponds more naturally to tilings.

Definition 1.3. *Let $r \geq 1$ be an integer. The r-generalized combinatorial Fibonacci sequence $\{g_n\}$ is defined as*

$$g_n = \begin{cases} 0, & \text{if } n < 0; \\ 1, & \text{if } n = 0; \\ g_{n-1} + g_{n-2} + ... + g_{n-r}, & \text{if } n \geq 0. \end{cases}$$

Note that $g_n = G_{n+r-1}$, so that g_n counts the number of n-tilings with tiles of length at most r. (The same reasoning from the $r = 2$ case applies: to form a tiling of length n, for $1 \leq k \leq r$ we can start with a tiled $(n-k)$-board and add a length k tile.) The initial conditions specified in the definition also make sense from a combinatorial perspective: they correspond to adopting the convention that a negative-length board cannot be tiled (as the existence of such a board does not make sense), but that a board of length zero can be tiled in exactly one way (by the empty set of tiles).

Howard and Cooper have provided algebraic proofs of a number of interesting original identities involving the r-Fibonacci sequences $\{G_n\}$. We provide combinatorial proofs for g_n versions of all of their identities.

Chapter 2

Results

Theorem 2.1. *For $n \geq r + 1$,*

$$2g_n = g_{n+1} + g_{n-r}.$$

Proof. We prove the identity by finding a 2:1 correspondence between the set of n-tilings with tiles of length at most r (counted by g_n) and the set of such tilings of length $n + 1$ or $n - r$ (counted by $g_{n+1} + g_{n-r}$.

First, given an n-tiling, we can produce an $n + 1$-tiling simply by adding a square. This maps every n-tiling to an $n + 1$-tiling that ends with a square.

Second, for each n-tiling, we can do one of two things. If the last tile has length k and $k < r$, then we can replace it with a tile of length $k + 1$ to get another unique length $n + 1$ tiling, this time ending with a tile of length $k + 1$, where $2 \leq k + 1 \leq r$. Otherwise, $k = r$ and we can simply remove the last tile to get a tiling of length $n - r$.

Thus each length n tiling maps to two unique tilings on the right-hand side of the equation, which may have length either $n + 1$ or $n - r$, proving the theorem. Conversely, each $(n + 1)$-tiling and $(n - r)$-tiling is achieved by this construction, and the theorem is proved. □

Theorem 2.2. *For $1 \leq n \leq r$,*

$$g_n = 2^{n-1}.$$

Proof. Begin by tiling the board with n squares. There are then $n - 1$ dividing lines between squares. Since $n \leq r$, we can obtain any length n tiling by selectively removing or leaving these dividing lines.

Thus g_n is equal to the number of ways to remove dividing lines. There are $n - 1$ such lines, and two choices for each: keep or remove. This gives 2^{n-1} possibilities in all. □

Note that, in the proof above, it was essential that we be able to remove dividing lines without restriction. This was possible because the length of the board was at most r, so that there was no risk of removing dividing lines in a way that created an "illegal" tile of length $r+1$ or greater. The next theorem considers boards that are long enough to contain exactly one illegal tile.

Theorem 2.3. *For* $r+1 \leq n \leq 2r+1$,

$$g_n = 2^{n-1} - (n-r+1)2^{n-r-2}.$$

Proof. The left-hand side counts the number of length n tilings with tiles of length at most r.

We obtain the right-hand side by counting the total number of possible length n tilings (using tiles of any length), then subtracting away the tilings which contain a tile of length $r+1$ or greater to get the number of length n tilings with tiles of length at most r.

We obtain the total number of tilings by starting with the all-squares tiling and counting the number of ways to remove dividing lines. There are again $n-1$ such lines, so the total number of tilings is 2^{n-1}.

To count the number of illegal tilings, we consider the position of the illegal tile's left edge. If its left edge coincides with the left edge of the board, then there are r dividing lines internal to the illegal tile which must remain absent for it to have length at least $r+1$. This leaves $n-1-r$ dividing lines elsewhere on the board, each of which can be either removed or kept, so that there are a total of 2^{n-r-1} illegal tilings with the illegal tile flush with the left edge of the board.

Otherwise, there are $n-r-1$ places to place the illegal tile's left edge (and leave room to the right for it to have length at least $r+1$). Once the left edge has been established, the status of that dividing line is fixed, as is the status of the first r "lines" internal to the illegal tile, so that there are in this case $n-2-r$ dividing lines which we can choose to either keep or remove. There are thus $(n-r-1)2^{n-r-2}$ such tilings.

The total number of illegal tilings is thus $(n-r-1)2^{n-r-2} + 2^{n-r-1} = (n-r+1)2^{n-r-2}$. Subtracting this from the total number of tilings, we obtain $g_n = 2^{n-1} - (n-r+1)2^{n-r-2}$. □

We now consider tilings where the number of illegal tiles is at most k.

Theorem 2.4. *For* $k(r+1) \leq n < (k+1)(r+1)$,

$$g_n = 2^{n-1} + \sum_{i=1}^{k} (-1)^i a_{n,i} 2^{(n-1)-i(r+1)},$$

where $a_{n,0} = 1$, $a_{n,1} = 0$ *for* $n < (r+1)$, $a_{n,r+1} = 2$, *and* $a_{n,i} = a_{n-1,i} + a_{n-(r+1),i-1}$.

Proof. Consider a board of length n. The number of ways to tile this board with tiles of length at most r is of course g_n.

We can also count tilings by counting all those with tiles of any length, then subtracting away the "illegal" tilings in which a tile of length $r+1$ or greater appears.

The total number of tilings is 2^{n-1}. We first subtract, for each cell j, the number of tilings with an illegal tile starting at cell j. However, this over-subtracts tilings with more than one illegal tile, so by the principle of inclusion/exclusion we have to add back in tilings with two illegal starting points, then subtract tilings with three illegal starting points, and so on.

Next we count the number of tilings with i designated starting points (where these starting points are at least $r+1$ apart). For such a tiling, i of the board's $n-1$ dividers are fixed, as are the r dividers that must be removed for each illegal tile to ensure it is sufficiently long. Thus there are $n-1-i(r+1)$ choices to make about the remaining dividers, so there are $2^{(n-1)-i(r+1)}$ ways to tile the rest of the board.

Note that when we designate the left end of the board as the edge of an illegal tile, we are not fixing an internal dividing line, so as in the previous theorem we have an additional factor of two in that case. We get around this by counting as normal, then weighting the special tilings by two.

All that remains to consider is the number of ways to designate the left edges. Let $a_{n,i}$ denote the number of ways to designate left edges for i illegal tiles on a length n board, where we give weight 2 to a designation where an illegal tile is flush with the left end of the board.

Then inclusion/exclusion gives us

$$\sum_{i=1}^{k}(-1)^i a_{n,i} 2^{(n-1)-i(r+1)},$$

so the total number of legal tilings is

$$g_n = 2^{n-1} + \sum_{i=1}^{k}(-1)^i a_{n,i} 2^{(n-1)-i(r+1)}.$$

We must show that $a_{n,i}$ behaves as we claim. There is of course exactly one way to designate left edges for zero such tiles, so $a_{n,0} = 1$. For $n < r+1$ there's no way to designate the left edge of an illegal tile, since the board isn't long enough to accomodate one, so $a_{n,1} = 0$ for $n < r+1$. When

$n = r+1$, there is exactly one way to designate the left edge of an illegal tile (since such a tile fills the board), but since this puts the illegal tile flush with the left edge of the board, we give it weight 2 so that $a_{r+1,1} = 2$. Finally, we can obtain a recurrence by conditioning on whether or not the last illegal tile we designate must occupy the last $r+1$ spaces of the board. (That is, if the left edge is placed such that to be sufficiently long, the tile must be flush with the right edge of the board.) If it doesn't, the number of ways to designate the left edges is $a_{n-1,i}$, since we can remove the rightmost tile of the board without changing anything. If it does, we already know the position of one tile, so the number of ways to designate the rest is $a_{n-1-r,i-1}$. Summing over the two cases, we have $a_{n,i} = a_{n-1,i} + a_{n-(r+1),i-1}$. This completes the proof.

□

Note that the expression for g_n found above could have been written a little more compactly. Specifically,

$$g_n = \sum_{i=0}^{k} (-1)^i a_{n,i} 2^{(n-1)-i(r+1)}.$$

Theorem 2.5. *For $k(r+1) \le n < (k+1)(r+1)$,*

$$g_n = \sum_{i=0}^{k} (-1)^i \left[\binom{n-ri}{i} + \binom{n-ri-1}{i-1} \right] 2^{(n-1)-i(r+1)}.$$

Proof. It suffices to show that in the previous theorem,

$$a_{n,i} = \binom{n-ri}{i} + \binom{n-ri-1}{i-1}.$$

That is, we must count the number of ways to designate the left edges of i illegal tiles on a board of length n, where we give weight 2 to designations that place an illegal tile flush with the left edge of the board.

First we count the number of ways to choose left endpoints, ignoring the weighting condition. There are n cells on the board, $n-r$ of which can be designated the leftmost cell of an illegal tile (cells $n-r+1$ through n are too close to the right edge of the board to permit a sufficiently long tile to begin at them). Thus we wish to choose i cells $x_1, ..., x_i$from the set $\{1, ..., n-r\}$ to serve as edge cells for our illegal tiles. The cells must be spaced far enough apart for the illegal tiles to "fit", so we require that $x_j - x_{j-1} \ge r+1$ for all j. To do this, we first choose $y_1, ..., y_i$ from the set $\{1, ..., n-r-(i-$

$1)r\}$, then set $x_1 = y_1$, $x_2 = y_2 + r$, $x_3 = y_3 + 2r$, ..., $x_i = y_i + r(i-1)$. The number of ways to choose the y_j is $\binom{n-r-(i-1)r}{i} = \binom{n-ri}{i}$. Since the equations above provide a bijection between the x_i and the y_i, this is also the number of ways to choose the x_i and thus the number of ways to designate leftmost cells of illegal tiles.

Now to give tilings with an illegal tile on the left edge of the board weight 2, we simply count those tilings again and add them to the total, so that they get counted twice. If one illegal tile has its left edge flush with the left edge of the board, then what remains is to choose $i-1$ cells to serve as left endpoints for the remaining illegal tiles, where we choose from $\{r+2, ..., n-r\}$. (The first $r+1$ cells belong to the leftmost illegal tile, of course.) By the same reasoning as above, the number of ways to do this is $\binom{(n-r-(i-2)r)-(r+1)}{i-1} = \binom{n-ri-1}{i-1}$.

Thus in total $a_{n,i} = \binom{n-ri}{i} + \binom{n-ri-1}{i-1}$. □

The next theorem also involves subtracting away "illegal" tilings. It allows us to consider much more general n, but the identity is recursive.

Theorem 2.6. *For $n \geq 2r-1$,*

$$g_n = 2^{r-1} g_{n-r} + \sum_{k=1}^{r-1} \left(\sum_{i=1}^{r-k} 2^{r-1-i} \right) g_{n-r-k}.$$

Proof. The left-hand side counts tilings of a length n board with tiles of length up to r.

To show that the right-hand side counts the same quantity, we start with a board of length n. Assuming no tile crosses the interface between cell r and cell $r+1$, we can tile its first r cells and its last $n-r$ cells separately. These can be (safely) tiled in 2^{r-1} and g_{n-r} ways, respectively, giving a total of $2^{r-1} g_{n-r}$ such tilings.

We must add in the tilings which *do* have a tile crossing the line after cell r. Consider tilings with a tile of length $i+k$ crossing the r, $r+1$ interface, where i is the number of cells the tile extends past the interface to the left, and k is the number of cells the tile extends past the interface to the right. To the right of the crossing tile there are then g_{n-r-k} possible tilings, and to the left of the crossing tile there are 2^{r-1-i} possible tilings.

Note that k can range from 1 to $r-1$ (if k were any longer the interface-crossing tile would be illegally long, as i must have length at least 1), while for fixed k, i can range from 1 to $r-k$. Thus the total number of tilings with an interface-crossing tiling is $\sum_{k=1}^{r-1} (\sum_{i=1}^{r-k} 2^{r-1-i}) g_{n-r-k}$.

Thus the total number of tilings of a length n board with tiles of length at most r is

$$g_n = 2^{r-1} g_{n-r} + \sum_{k=1}^{r-1} \sum_{i=1}^{r-k} 2^{r-1-i} g_{n-r-k}.$$

□

The next theorem follows a similar approach.

Theorem 2.7. *For $n \geq r$,*

$$g_n = 2^{r-1} g_{n+1-r} - \sum_{k=2}^{r} 2^{k-2} g_{n-r-k+1}.$$

Proof. The left-hand side counts the number of ways to tile a length n board with tiles of length at most r.

To obtain the right-hand side, we divide the board into two parts. The first $r-1$ cells can be tiled in 2^{r-2} ways, and the remaining $n-(r-1)$ cells can be tiled in g_{n-r+1} ways. We have not yet addressed what happens at the interface between cell $r-1$ and cell r. We can either keep or remove the dividing line here, giving us an extra factor of 2 and a total of $2^{r-1} g_{n-r+1}$ tilings. However, removing the dividing line between $r-1$ and r will occasionally result in the creation of a tile of illegal length.

We now count the number of such illegal tilings. First consider the rightmost $n-r+1$ cells. (That is, the cells r through n.) If the tile beginning at cell r has length k, the remaining cells to its right can be tiled in $g_{n-r-k+1}$ ways. Now remove the line between cells $r-1$ and r. For this to create a tile of illegal length, we must have had a tile of length $r+1-k$ ending on cell $r-1$. The internal $r-k$ lines of this tile are fixed, so that we have $(r-1)-(r-k) = k-2$ choices to make about the remaining lines. Thus the left side can be tiled in 2^{k-2} ways. Note that this only makes sense for $2 \leq k$, since if k is 1, removing the line between $r-1$ and r cannot create an illegal tile, no matter how the first $r-1$ cells have been tiled.

Summing over all possible k, we find that the number of illegal tilings is $\sum_{k=2}^{r} 2^{k-2} g_{n-r-k+1}$. Subtracting this from the total number of tilings created in this way gives the number of legal tilings,

$$g_n = 2^{r-1} g_{n+1-r} - \sum_{k=2}^{r} 2^{k-2} g_{n-r-k+1}.$$

□

The next theorem generalizes a well-known "sum of squares" identity for Fibonacci numbers.

Theorem 2.8. *For $r \geq 2$,*

$$\sum_{k=0}^{n} g_k^2 + \sum_{i=2}^{r-1} \sum_{k=i}^{n} g_k g_{k-i} = g_n g_{n+1}.$$

Proof. Consider an $(n+1)$-board laid parallel to an n-board, such that the left edges of the two boards align and the $(n+1)$-board extends one cell to the right past the right edge of the n-board.

There are g_{n+1} ways to tile the longer board, and g_n ways to tile the shorter, so that there are $g_{n+1}g_n$ ways to tile the pair simultaneously.

We now show that the left-hand side of the equation counts the same quantity. Let s be the rightmost cell of either board which is *not* covered by a domino, and let $k+1$ be its position within its board. Note that since n and $n+1$ have different parity, s is always uniquely determined by k.

Suppose that s is covered by a square. Then the cells to its right on its board must be covered by dominoes and the cells to its left can be tiled in g_k ways. Likewise, cells $k+1$ and beyond of the board not containing s must be covered by dominoes, with g_k ways to tile the remaining k cells. Thus the two boards can be tiled in g_k^2 ways. The cell s can be positioned anywhere from 1 to $n+1$, so k can range from 0 to n. The total number of tilings where s is covered by a square is thus $\sum_{k=0}^{n} g_k^2$.

Otherwise, s is covered by a tile of length $i+1$, where $2 \leq i \leq r-1$. Then the cells to the right of s must still be covered by dominoes, the tile covering s also covers the first i cells to its left, and the remaining $k-i$ cells can be tiled in g_{k-i} ways. As before, cells $k+1$ and beyond of the board not containing s must be covered by dominoes, and the remaining cells can be tiled in g_k ways, giving $g_{k-i}g_k$ tilings. Summing over all possible i and k gives $\sum_{i=2}^{r-1} \sum_{k=i}^{n} g_k g_{k-i}$ tilings where s is not covered by a square. (Note that $k \geq i$, as s must necessarily be the rightmost cell of the tile that covers it.)

Thus in total the number of tilings of the pair of boards is

$$\sum_{k=0}^{n} g_k^2 + \sum_{i=2}^{r-1} \sum_{k=i}^{n} g_k g_{k-i} = g_n g_{n+1}.$$

□

Theorem 2.9. *For $n > 0$, $m > 0$, $r \geq 3$,*

$$g_{n+m-r+1} = g_{n-r+1}g_{m-r+1} + g_{n-r+1}g_{m-r} + g_{n-r}g_{m-r+1} + \sum_{i=1}^{r-2} g_{n-i}g_{m-r+i+1} - \sum_{i=2}^{r-2}\sum_{j=1}^{i-1} g_{n-i}g_{m-r+i-j+1}.$$

Proof. We count the number of ways to tile a board of length $n + m - r + 1$. Obviously this is $g_{n+m-r+1}$, the left side of the equation.

To see that the right-hand side also counts this, we begin by considering the first $r - 2$ potential breaks after cell $n - r + 2$. That is, the $r - 2$ gridlines starting with the right edge of cell $n - r + 2$ and ending with the right edge of cell $n - 1$.

For each such gridline, we count the number of tilings which have a break at that line. In general, given a break at cell $n - i$, there are g_{n-i} ways to tile the leftmost $n - i$ cells. This leaves $m - r + i + 1$ cells to be tiled to the right of the break, which can be done in $g_{m-r+i+1}$ ways, so that there are $g_{n-i}g_{m-r+i+1}$ tilings with a break at cell $n - i$. Summing over all $r - 2$ potential breakpoints under consideration gives $\sum_{i=1}^{r-2} g_{n-i}g_{m-r+i+1}$ tilings.

Since it is possible for a tiling to have breaks at more than one of the $r - 2$ special lines, the sum just constructed counts many tilings multiple times. We must subtract away each tiling the appropriate number of times; if a tiling has breaks at exactly k of the $r - 2$ special lines, it will have been counted k times, so we must subtract it $k - 1$ times.

To do this, we consider, for each tiling, what happens in the region of the board bounded by the first and last of the $r - 2$ special lines. If a tiling has breaks at exactly k of the $r - 2$ lines, then those k breaks bound $k - 1$ tiles within this region. (We do not count tiles which overlap this region but are not wholly contained within it.) Thus we can achieve the appropriate subtraction by counting the number of tilings with a particular length tile in a particular position, for each possible tile and position within the special region. For example, a tiling with exactly three breaks within the region—say at the right edges of cells $n - 4$, $n - 2$, and $n - 1$—will be subtracted twice: once for having a domino on cells $n - 3$ and $n - 2$, and once for having a square on cell $n - 1$. Note that for tilings with exactly one break within the region, no subtraction is needed, and since no tiles are fully contained in the special region, no subtraction will be performed.

We now perform this subtraction. Consider tilings with a tile of length j covering cells $n - i + 1$ through $n - i + j$. There are g_{n-i} ways to tile the board to the left of this tile, and $g_{m-r+i-j+1}$ ways to tile the board to the right of this tile. Thus there are $g_{n-i}g_{m-i-j+1}$ such tilings. We sum over all such tiles within the region bounded by the $r-2$ special lines. The leftmost of these lines is the right edge of cell $n-r+2$; thus the first cell the special tile can include is $n-r+3$, and so i can be at most $r-2$. The last cell the tile can include is $n-1$, so i must be at least 2. Since tilings start at cell $n-i+1$ and must end at or before cell $n-1$, j can be at most $i-1$. Thus j ranges from 1 to $i-1$, and we have the double sum

$$\sum_{i=2}^{r-2}\sum_{j=1}^{i-1} g_{n-i}g_{m-r+i-j+1}.$$

Thus far we have shown that the number of tilings with one or more breaks at $r-2$ specially designated lines is

$$\sum_{i=1}^{r-2} g_{n-i}g_{m-r+i+1} - \sum_{i=2}^{r-2}\sum_{j=1}^{i-1} g_{n-i}g_{m-r+i-j+1}.$$

All that remains is to add in the tilings which don't have breaks at *any* of the $r-2$ special lines. There are three possible cases. First, the $r-2$ lines can be covered by a single tile of length $r-1$ covering cells $n-r+2$ through n. Note that no shorter tile could cover all $r-2$ lines, and that this is the only way to position a tile of length $r-1$ to cover $r-2$ lines. Tiling the board to the left and the right of the tile, respectively, we see that there are $g_{n-r+1}g_{m-r+1}$ such tilings. The other possibility is that the $r-2$ lines are covered by a tile of length r. Since this tile is longer than necessary by 1, it can begin either at cell $n-r+1$ or at cell $n-r+2$ and still cover all $r-2$ of the special lines. In the former case the rest of the board can be tiled in $g_{n-r}g_{m-r+1}$ ways; in the latter case there are $g_{n-r+1}g_{m-r}$ possible tilings. Thus in total there are $g_{n-r+1}g_{m-r+1} + g_{n-r}g_{m-r+1} + g_{n-r+1}g_{m-r}$ tilings which have no breaks at any of the special points.

The total number of tilings of a board of length $n+m-r+1$ is thus

$$\begin{aligned} &g_{n-r+1}g_{m-r+1} + g_{n-r}g_{m-r+1} + g_{n-r+1}g_{m-r} \\ &\quad + \sum_{i=1}^{r-2} g_{n-i}g_{m-r+i+1} - \sum_{i=2}^{r-2}\sum_{j=1}^{i-1} g_{n-i}g_{m-r+i-j+1}, \end{aligned}$$

as desired. □

Chapter 3

Further Work

Future work will focus primarily on using the combinatorial interpretation to produce interesting r-Fibonacci identities that go beyond those presented by Howard and Cooper. Additionally, Howard and Cooper present a number of congruence relationships involving r-Fibonacci numbers that may lend themselves to combinatorial interpretation.

3.1 Binomial Coefficients

One area of research that seems particularly ripe for generalization is the connection between Fibonacci numbers and binomial coefficients. The following theorem is well known.

Theorem 3.1. *For $n \geq 0$,*

$$f_n = \sum_{k=0}^{\frac{n}{2}} \binom{n-k}{k}.$$

To generalize this result to r-Fibonacci numbers we require a new definition.

Definition 3.1. *For $s \geq 1$, $n \geq 0$ and $k \geq 0$ all integers, we define the s-restricted multichoose function $\binom{n}{k}_s$ to be the number of ways of choosing a k-element subset from a parent set of n elements, where elements can be chosen for the subset a maximum of s times. Note that $\binom{n}{k}_1 = \binom{n}{k}$, $\binom{n}{k}_0 = 1$.*

With this definition in hand, we can state and prove the following theorem:

Theorem 3.2. *For $n \geq 0$, $r \geq 2$,*

$$g_n = \sum_{k\geq 0} \binom{n-k}{k}_{r-1},$$

where g_n is the generalized r-Fibonacci sequence.

Proof. The left side counts the number of ways to tile a length n board with tiles of length at most r.

For the right side we condition on the number of tiles in the tiling. Consider a tiling with $n-k$ tiles. We can create such a tiling by starting with $n-k$ squares and lengthening them. Each lengthening step increases the length of a single tile by exactly 1. There must therefore be k lengthening steps in total. Each tile can be lengthened a maximum of $r-1$ times—any more and it would become too long. Thus to count the number of possible tilings we count the number of ways to choose i tiles from a set of $n-i$, where each tile can be chosen at most $r-1$ times. This is, by definition, $\binom{n-k}{k}_{r-1}$. Ranging over all possible k, the total number of tilings is

$$g_n = \sum_{k=0} \binom{n-k}{k}_{r-1}$$

□

This easy generalization suggests that other, more complicated identities involving Fibonacci numbers and binomial coefficients might also be generalized to identities on r-Fibonacci numbers and s-binomial coefficients.

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