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Kernel structure of Toeplitz-plus-Hankel matrices

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Abstract

The structure of the kernel of block Toeplitz-plus-Hankel matrices $R = [a_{j-k} + b_{j+k}]$, where a_j and b_j are the given $p \times q$ blocks with entries from a given field, is investigated. It is shown that R corresponds to two systems of at most $p + q$ vector polynomials from which a basis of the kernel of R and all other Toeplitz-plus-Hankel matrices with the same parameters a_j and b_j can be built. The main result is an analogue of a known kernel structure theorem for block Toeplitz and block Hankel matrices.

Keywords: Toeplitz matrix; Toeplitz-plus-Hankel matrix

1. Introduction

The kernel (nullspace) $\ker R = \{u\colon Ru = 0\}$ of a Hankel $R = [a_{j+k}]$ or a Toeplitz matrix $R = [a_{j-k}]$ possesses a remarkable structure. It is, provided that it is nontrivial, the span of one or two vectors and their shifts (see [9]). This fact can be formulated in a nice form in polynomial language. If $x = (x_j)_{j=0}^{n}$, then $x(t)$ will denote the polynomial $\sum_{j=0}^{n} x_j t^j$. Now the set of all $u(t)$ with $u \in \ker R$ is given by

$$u(t) = u_1(t)\xi_1(t) + u_2(t)\xi_2(t),$$

where $u_1(t)$ and $u_2(t)$ are two polynomials depending only on the parameters a_j (and not on the size of H) and $\xi_1(t)$, $\xi_2(t)$ run over all polynomials satisfying certain

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degree restrictions. In the case of a square Toeplitz or Hankel matrix only one polynomial $u_1(t)$ generates the kernel. The system $\{u_1(t), u_2(t)\}$ is called *fundamental system* of (a_j). Note that the concept of a fundamental system is also important in the case of a nonsingular Toeplitz or Hankel matrix. In fact, the inverse of a Hankel matrix is, up to constant factor, equal to the Bezoutian of the two polynomials in any fundamental system of the sequence (a_j) (see [9]). The result about the kernel structure was generalized in [7] to block Hankel and Toeplitz matrices.

For infinite Toeplitz matrices $[a_{j-k}]_{j,k=0}^{\infty}$ ($\sum |a_j| < \infty$) generating a Fredholm operator in the sequence space ℓ^p a similar kernel structure result was already obtained in the classical papers on Wiener–Hopf operators [13] for the scalar and for the block case using the Wiener–Hopf factorization of the symbol [2].

The kernel structure result for finite Toeplitz and Hankel matrices found several applications. Let us list some of them. In [9] (see also references therein) it was used to design fast inversion algorithms for Toeplitz and Hankel matrices that are not strongly nonsingular and for which the classical algorithms (Levinson–Trench and Schur–Bareiss) fail. In [5] it was used to study the structure of the Moore–Penrose inverse of a Toeplitz or Hankel matrix and in [6] to design fast algorithms for its computation. Finally, it was applied in [4] to construct matrices which are not similar to a Toeplitz matrix. The kernel structure result for infinite block Toeplitz matrices obtained in a purely algebraic way was used in [12] to construct the Wiener–Hopf factorization in algebras of bounded matrix functions.

These applications are the motivations for the investigation of the kernel structure of other classes of structured matrices. In [3,9] a general approach to describe the kernel of matrices with a displacement structure was presented, but this approach is rather coarse and does not provide the full picture in special cases.

The aim of the present paper is to study the kernel of block Toeplitz-plus-Hankel matrices and to show that a similar result as for block Toeplitz matrices holds. In the scalar case the result can roughly be described as follows: For a given Toeplitz-plus-Hankel matrix $R = T + H$, there exist four polynomials $u_1^+(t)$, $u_2^+(t)$, $u_1^-(t)$ and $u_2^-(t)$ such that the set of all $u(t)$ with $u \in \ker R$ is given by

$$u(t) = u_1^+(t)\xi_1^+(t) + u_2^+(t)\xi_2^+(t) + u_1^-(t)\xi_1^-(t) + u_2^-(t)\xi_2^-(t), \tag{1.1}$$

where $\xi_1^+(t)$, $\xi_2^+(t)$ run over all *symmetric* polynomials and $\xi_1^-(t)$, $\xi_2^-(t)$ over all *skewsymmetric* polynomials satisfying certain degree restrictions. Here we call a polynomial symmetric or skewsymmetric if its coefficient vector is symmetric or skewsymmetric, respectively. A vector $(x_k)_{k=0}^n$ is called symmetric or skewsymmetric if $x_{n-k} = x_k$ or $x_{n-k} = -x_k$ for $k = 0, \dots, n$, respectively. It is remarkable that if we interchange in (1.1) the roles of $\{u_1^+, u_2^+\}$ and $\{u_1^-, u_2^-\}$, then we obtain just the kernel of $T - H$.

The basic idea to prove this result is to use the extension approach for Toeplitz-plus-Hankel matrices, which was, as far as we know, first proposed in [14] and which is based on the observation that the direct sum of the Toeplitz-plus-Hankel matrices $T + H$ and $T - H$ is unitarily equivalent to a block Hankel matrix with

2×2 blocks. Note that this extension idea is also used in other fields of mathematics like the theory of singular integral equations with shift. This allows us to apply the results from [7] about the kernel structure of block Hankel matrices (see also [1]). To obtain our main result we use some concepts that were introduced in [11].

2. Kernel structure of block Hankel matrices

In this section we recall some definitions and results from [7] in a form which is appropriate for our considerations. Before this we introduce some notations that will be used throughout the paper.

Let $\mathbb{F}$ be a field with a characteristic not equal to 2. We denote by $\mathbb{F}^{q,n+1}$ the set of all vectors $(x_j)_{j=0}^n$ with $x_j \in \mathbb{F}^q$. If $x = (x_j)_{j=0}^n \in \mathbb{F}^{q,n+1}$, then $x(t)$ will denote the vector polynomial $\sum_{j=0}^n x_j t^j$.

A sequence of blocks $\mathbf{a} = (a_0, \ldots, a_{N-1})$, $a_j \in \mathbb{F}^{p \times q}$, will be associated with the family of block Hankel matrices

$$H_k(\mathbf{a}) = \left[a_{i+j}\right]_{i=0,\ldots,l-1;\ j=0,\ldots,k-1} \quad (k + l = N + 1),$$

where $k = 1, \ldots, N$. The following theorem is a slightly different formulation of Theorem 2.1 in [7].

Theorem 2.1. *Let a sequence of blocks* $\mathbf{a}$ *be given and* $\delta = q - \operatorname{rank} \mathbf{a}^{\mathrm{T}}$. [1] *Then there exists a uniquely defined* $(p+q)$*-tuple of nonnegative integers* $(d_1, \ldots, d_{p+q})$, $d_1 \leqslant \cdots \leqslant d_{p+q} \leqslant N+1$ *satisfying* $\sum_{j=1}^{p+q} d_j = (N+1)p$ *and* $p+q-\delta$ *vectors* $u_j \in \mathbb{F}^{q,d_j+1}$ *such that, for* $k = 1, \ldots, N$, *the kernel of* $H_k(\mathbf{a})$ *consists of all vectors* u *for which* $u(t)$ *is of the form*

$$u(t) = \sum_{k > d_j} \xi_j(t) u_j(t), \tag{2.1}$$

where ξ_j *is any vector from* $\mathbb{F}^{k-d_j}$. *Furthermore*, $\mathbb{F}^{q,N+1}(t)$ *consists of vector polynomials of the form* (2.1), *where* $\xi_j \in \mathbb{F}^{N+1-d_j}$. *Moreover*, *the vector polynomials* $t^k u_j(t)$, $k = 0, \ldots, N - d_j$, $j = 1, \ldots, p+q$ *are linearly independent.*

Note that δ is the number of d_j that are equal to $N+1$.

The numbers d_j $(j = 1, \ldots, p+q)$ are called *characteristic degrees* and the system $\{u_j\}$ or $\{u_j(t)\}$ $(j = 1, \ldots, p+q-\delta)$ is called a *fundamental system* for $\mathbf{a}$ or for any of the matrices $H_k(\mathbf{a})$.

From Theorem 2.1 we conclude the following.

[1] We identify $\mathbf{a}$ with the corresponding column. Note that $\mathbf{a}^{\mathrm{T}} = H_N(\mathbf{a})$.

Corollary 2.2. *For $k = 1, \dots, N$, the dimension of the kernel of $H_k(\mathbf{a})$ is given by*

$$\dim\ker H_k(\mathbf{a}) = \sum_{k>d_j} (k - d_j).$$

Another consequence is a nonsingularity condition.

Corollary 2.3. *The matrix $H_k(\mathbf{a})$ is nonsingular if and only if $kq = lp$ $(k + l = N + 1)$ and all d_j are equal to k.*

3. Main result

In this section we consider $l \times k$ block Toeplitz-plus-Hankel matrices R_k with $p \times q$ blocks. Let $J_{q,k}$ denote the matrix of the block counteridentity

$$J_{q,k} = \left.\begin{bmatrix} 0 & & I_q \\ & \iddots & \\ I_q & & 0 \end{bmatrix}\right\} k$$

and $I_{q,k}$ the block diagonal matrix with k diagonal blocks I_q, where I_q is the $q \times q$ identity matrix. Then any $l \times k$ block Toeplitz-plus-Hankel matrix R_k can be represented in the form

$$R_k = R_k(\mathbf{a}, \mathbf{b}) = H_k(\mathbf{a}) + H_k(\mathbf{b}) J_{q,k}$$

for some sequences of blocks $\mathbf{a} = (a_0, \dots, a_{N-1})$ and $\mathbf{b} = (b_0, \dots, b_{N-1})$ and $N + 1 = k + l$. Note that this representation is not unique, since the spaces of Toeplitz and Hankel matrices have a nontrivial intersection. Besides $R_k(\mathbf{a}, \mathbf{b})$ we consider the matrix $H_k(\mathbf{a}) - H_k(\mathbf{b}) J_{q,k}$ and the matrix

$$\widetilde{H}_k(\mathbf{a}, \mathbf{b}) = \begin{bmatrix} H_k(\widehat{\mathbf{a}}) & H_k(\widehat{\mathbf{b}}) \\ H_k(\mathbf{b}) & H_k(\mathbf{a}) \end{bmatrix}, \tag{3.1}$$

where $\widehat{\mathbf{a}}$ denotes the sequence $\mathbf{a}$ in reversed order, i.e., $\widehat{\mathbf{a}} = (a_{N-1}, \dots, a_0)$.

There is a relation between the matrices $H_k(\mathbf{a}) \pm H_k(\mathbf{b}) J_{q,k}$ and $\widetilde{H}_k(\mathbf{a}, \mathbf{b})$, which was used, for example, in [8,14] for inversion purposes and which is described in the following.

Let $Q_{q,k}$ denote the unitary matrix

$$Q_{q,k} = \frac{1}{\sqrt{2}} \begin{bmatrix} -J_{q,k} & J_{q,k} \\ I_{q,k} & I_{q,k} \end{bmatrix}.$$

Proposition 3.1. *The matrix $\widetilde{H}_k(\mathbf{a}, \mathbf{b})$ is unitarily equivalent to the direct sum of $H_k(\mathbf{a}) + H_k(\mathbf{b}) J_{q,k}$ and $H_k(\mathbf{a}) - H_k(\mathbf{b}) J_{q,k}$ via*

$$Q_{p,l}^{\mathrm{T}}\, \widetilde{H}_k(\mathbf{a}, \mathbf{b})\, Q_{q,k} = \begin{bmatrix} H_k(\mathbf{a}) - H_k(\mathbf{b}) J_{q,k} & 0 \\ 0 & H_k(\mathbf{a}) + H_k(\mathbf{b}) J_{q,k} \end{bmatrix},$$

where $k + l = N + 1$.

The proof is a direct verification.

From Proposition 3.1 we conclude the following.

Corollary 3.2. *The vector $u \in \mathbb{F}^{q,k}$ belongs to the kernel of $H_k(\mathbf{a}) \pm H_k(\mathbf{b})J_{q,k}$ if and only if $[\begin{smallmatrix} u \\ \pm\widehat{u} \end{smallmatrix}]$ belongs to the kernel of $\widetilde{H}_k(\mathbf{a},\mathbf{b})$, respectively.*

Note that $\widetilde{H}_k(\mathbf{a},\mathbf{b})$ is a 2×2 block matrix where the four blocks are $l \times k$ block Hankel matrices of equal size. Matrices of this kind will be called *Hankel-cross matrices*. Among all Hankel-cross matrices the matrices $\widetilde{H}_k(\mathbf{a},\mathbf{b})$ are characterized by the property to be centro-symmetric.

Recall that a $2l \times 2k$ block matrix R is called *centro-symmetric* if $J_{p,2l} R J_{q,2k} = R$.

Proposition 3.3. *The $2l \times 2k$ Hankel-cross matrix*

$$\widetilde{H}_k = \widetilde{H}_k(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}) = \begin{bmatrix} H_k(\mathbf{c}) & H_k(\mathbf{d}) \\ H_k(\mathbf{b}) & H_k(\mathbf{a}) \end{bmatrix} \tag{3.2}$$

is centro-symmetric if and only if $\mathbf{c} = \widehat{\mathbf{a}}$ and $\mathbf{d} = \widehat{\mathbf{b}}$.

Proof. We have

$$\begin{bmatrix} 0 & J_{p,l} \\ J_{p,l} & 0 \end{bmatrix} \begin{bmatrix} H_k(\mathbf{c}) & H_k(\mathbf{d}) \\ H_k(\mathbf{b}) & H_k(\mathbf{a}) \end{bmatrix} \begin{bmatrix} 0 & J_{q,k} \\ J_{q,k} & 0 \end{bmatrix} = \begin{bmatrix} H_k(\widehat{\mathbf{a}}) & H_k(\widehat{\mathbf{b}}) \\ H_k(\widehat{\mathbf{d}}) & H_k(\widehat{\mathbf{c}}) \end{bmatrix},$$

from which the assertion follows immediately. □

After appropriate permutations of rows and columns a Hankel-cross matrix goes over into a block Hankel matrix with $2p \times 2q$ blocks. Therefore, Theorem 2.1 can be applied. We obtain the following:

Theorem 3.4. *Let sequences of blocks $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ be given and let $\delta = 2q - \operatorname{rank} \widetilde{H}_N$. Then there exists a uniquely defined $2(p+q)$-tuple of nonnegative integers $(d_1, \ldots, d_{2(p+q)})$, $d_1 \leqslant \cdots \leqslant d_{2(p+q)} \leqslant N+1$ satisfying $\sum_{j=1}^{p+q} d_j = 2p(N+1)$ and vectors $\widetilde{u}_j = [\begin{smallmatrix} u_j \\ v_j \end{smallmatrix}]$, $u_j, v_j \in \mathbb{F}^{q,d_j+1}$ $(j = 1, \ldots, 2(p+q) - \delta)$ such that, for $k = 1, \ldots, N$, the kernel of the matrix $\widetilde{H}_k$ given by (3.2) consists of all vectors $\widetilde{u} = [\begin{smallmatrix} u \\ v \end{smallmatrix}]$ for which $\widetilde{u}(t) = [\begin{smallmatrix} u(t) \\ v(t) \end{smallmatrix}]$ is of the form*

$$\widetilde{u}(t) = \sum_{k > d_j} \xi_j(t) \widetilde{u}_j(t), \tag{3.3}$$

where ξ_j is any vector from $\mathbb{F}^{k-d_j}$. Moreover, the coefficient vectors of $t^s \widetilde{u}_j(t)$, $s = 0, \ldots, N - d_j$, $j = 1, \ldots, 2(p+q)$ form a basis of $\mathbb{F}^{q,2N+2}$.

We now specify this result for the centro-symmetric case, i.e., for matrices $\widetilde{H}_k(\mathbf{a},\mathbf{b})$. In this case

$$\delta = 2p - \operatorname{rank} \begin{bmatrix} \widehat{\mathbf{a}}^{\mathrm{T}} & \widehat{\mathbf{b}}^{\mathrm{T}} \\ \mathbf{b}^{\mathrm{T}} & \mathbf{a}^{\mathrm{T}} \end{bmatrix} = 2p - \operatorname{rank} \widetilde{H}_N(\mathbf{a}, \mathbf{b}).$$

In the sequel δ will always denote this integer.

A subspace of $\mathbb{F}^{q,k}$ will be called *flip invariant* if it is an invariant subspace for $J_{q,k}$. For example, the kernel of a centro-symmetric matrix is flip invariant. A vector will be called *flip invariant* if it is symmetric or skewsymmetric.

Let W be a flip invariant subspace of $\mathbb{F}^{q,k}$. Then W can be represented as a direct sum

$$W = W_+ \oplus W_-,$$

where W_+ consists of all symmetric and W_- of all skewsymmetric vectors in W.

Definition [11]. The integer

$$\operatorname{sgn} W = \dim W_+ - \dim W_-$$

will be called the *signature* of W.

The whole space $\mathbb{F}^{q,k}$ has the signature zero if k is even and q if k is odd. Thus the space $\mathbb{F}^{q,2k}$ has signature zero. The row space of the $2p \times Nq$ matrix $\widetilde{H}_N(\mathbf{a}, \mathbf{b})$ is flip invariant. Let $\sigma = \sigma(\mathbf{a}, \mathbf{b})$ denote its signature. Then the signature of the subspace $\ker \widetilde{H}_N(\mathbf{a}, \mathbf{b})$ equals $-\sigma$. Obviously, $|\sigma| \leqslant \delta$.

It follows from the construction of a fundamental system (see [7]) that in the case of a centro-symmetric block Hankel matrix the vectors of this system can be chosen as flip invariant. Such a system splits into a symmetric part $\{u_1^+, \ldots, u_{r_+}^+\}$ and a skewsymmetric part $\{u_1^-, \ldots, u_{r_-}^-\}$, where $r_+ + r_- = 2(p+q) - \delta$. We are going to show that $r_+ = r_- = p + q + \sigma$.

For a vector $\widetilde{u} = \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$, $u, v \in \mathbb{F}^{q,m+1}$ and $k \geqslant m+1$, $\mathscr{M}_k(\widetilde{u})$ will denote the space of all vectors $w \in \mathbb{F}^{q,k}$ with the property that $w(t) = \xi(t)\widetilde{u}(t)$ for some $\xi \in \mathbb{F}^{k-m+1}$. The dimension of $\mathscr{M}_k(\widetilde{u})$ is equal to $k - m + 1$.

The following obvious fact is important for our considerations (cf. [7,11]).

Lemma 3.5. *Let $\widetilde{u} = \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$ be flip invariant, i.e., $v = \widehat{u}$ or $v = -\widehat{u}$. Then $\mathscr{M}_k(\widetilde{u})$ is a flip invariant subspace and has the signature*

$$\operatorname{sgn} \mathscr{M}_k(\widetilde{u}) = \begin{cases} 0, & k - m + 1 \text{ even}, \\ 1, & k - m - 1 \text{ odd}, \ \widetilde{u} \text{ symmetric}, \\ -1, & k - m - 1 \text{ odd}, \ \widetilde{u} \text{ skewsymmetric}. \end{cases}$$

Let $r_{\mathrm{e},+}$ and $r_{\mathrm{o},+}$ denote the number elements in the symmetric part of a fundamental system of $\widetilde{H}_k(\mathbf{a}, \mathbf{b})$ with even or odd characteristic degree, respectively, and let $r_{\mathrm{e},-}$ and $r_{\mathrm{o},-}$ be defined analogously.

Proposition 3.6. *If N is odd, then*

$$r_{\mathrm{e},+} = r_{\mathrm{e},-} \quad \textit{and} \quad r_{\mathrm{o},+} = r_{\mathrm{o},-} - \sigma.$$

If N is even, then

$$r_{\mathrm{o},+} = r_{\mathrm{o},-} \quad \textit{and} \quad r_{\mathrm{e},+} = r_{\mathrm{e},-} - \sigma.$$

In particular, $r_+ = r_- - \sigma$.

Proof. Let N be odd and let $\{\widetilde{u}_j\}$ be a fundamental system for $\widetilde{H}_k(\mathbf{a}, \mathbf{b})$ that consists of symmetric and skewsymmetric vectors. In view of Theorem 3.4, the space $\mathbb{F}^{q,2N+2}$ is the direct sum of the subspaces $\mathscr{M}_{N+1}(\widetilde{u}_j)$. Since the signature of $\mathbb{F}^{q,2N+2}$ equals zero, the sum of the signatures of the subspaces $\mathscr{M}_{N+1}(\widetilde{u}_j)$ must be zero. According to Lemma 3.5, the signature of $\mathscr{M}_{N+1}(\widetilde{u}_j)$ is nonzero only if d_j is even and in this case it is ± 1, depending on whether u_j is even or odd. Hence $r_{\mathrm{e},+} = r_{\mathrm{e},-}$.

To obtain the equality for the other pair we consider the subspace $\ker \widetilde{H}_N(\mathbf{a}, \mathbf{b})$ of $\mathbb{F}^{q,2N}$. This subspace has the signature $-\sigma$. Furthermore, it is the direct sum of the subspaces $\mathscr{M}_N(\widetilde{u}_j)$. This means the sum of the signatures of the subspaces $\mathscr{M}_N(\widetilde{u}_j)$ must be $-\sigma$. Hence, $r_{\mathrm{o},+} = r_{\mathrm{o},-} - \sigma$. For even N the proof is analogous. □

Corollary 3.7. *If* $\delta = 0$, *then* $r_+ = r_- = p + q$.

We now can describe the structure of the kernel of the matrices $\widetilde{H}_k = \widetilde{H}_k(\mathbf{a}, \mathbf{b})$. From Theorem 3.4 and Proposition 3.6 we conclude the following.

Theorem 3.8. *Let two sequences of blocks* $\mathbf{a}$ *and* $\mathbf{b}$ *be given. Then there exist two uniquely defined* $(p+q)$*-tuples of nonnegative integers* $(d_1^+, \dots, d_{p+q}^+)$ *and* $(d_1^-, \dots, d_{p+q}^-)$ *satisfying* $d_1^\pm \leqslant \cdots \leqslant d_{p+q}^\pm \leqslant N+1$ *and* $\sum_{j=1}^{p+q}(d_j^+ + d_j^-) = 2p(N+1)$ *and vectors*

$$\widetilde{u}_j^\pm = \begin{bmatrix} u_j^\pm \\ \pm\widehat{u}_j^\pm \end{bmatrix}, \quad u_j^\pm \in \mathbb{F}^{q,d_j+1}, \ j = 1, \dots, r_\pm,$$

where $r_+ + r_- = p + q - \delta$ *and* $r_+ - r_- = -\sigma$, *such that, for* $k = 1, \dots, N$, *the kernel of the matrix* $\widetilde{H}_k(\mathbf{a}, \mathbf{b})$ *consists of all vectors* $\widetilde{u} = \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$ *for which* $\widetilde{u}(t) = \left[\begin{smallmatrix} u(t) \\ v(t) \end{smallmatrix}\right]$ *is of the form*

$$\widetilde{u}(t) = \sum_{k > d_j^+} \xi_j^+(t)\widetilde{u}_j^+(t) + \sum_{k > d_j^-} \xi_j^-(t)\widetilde{u}_j^-(t), \tag{3.4}$$

where $\xi_j^\pm$ *is any vector from* $\mathbb{F}^{k-d_j^\pm}$. *Moreover, the coefficient vectors of* $t^s\widetilde{u}_j(t)$, $s = 0, \dots, N - d_j$, $j = 1, \dots, 2(p+q)$ *form a basis of* $\mathbb{F}^{q,2N+2}$.

We now can state our main result.

Theorem 3.9. *Let two sequences of blocks* $\mathbf{a}$ *and* $\mathbf{b}$ *be given. Then there exist two uniquely defined* $(p+q)$*-tuples of nonnegative integers* $(d_1^+, \dots, d_{p+q}^+)$ *and* $(d_1^-, \dots, d_{p+q}^-)$ *satisfying* $d_1^\pm \leqslant \cdots \leqslant d_{p+q}^\pm \leqslant N+1$ *and* $\sum_{j=1}^{p+q}(d_j^+ + d_j^-)$

$= 2p(N+1)$ *and vectors* $u_j^{\pm} \in \mathbb{F}^{q,d_j^{\pm}+1}$ *such that, for* $k = 1, \ldots, N$, *the kernel of* $H_k(\mathbf{a}) + H_k(\mathbf{b})J_{q,k}$ *consists of all vectors u for which* $u(t)$ *is of the form*

$$u(t) = \sum_{k>d_j^+} \xi_j^+(t)u_j^+(t) + \sum_{k>d_j^-} \xi_j^-(t)u_j^-(t), \tag{3.5}$$

where $\xi_j^{\pm}$ *is any vector from* $\mathbb{F}_{\pm}^{k-d_j^{\pm}}$. *The kernel of* $H_k(\mathbf{a}) - H_k(\mathbf{b})J_{q,k}$ *consists of all vectors u for which* $u(t)$ *is of the form*

$$u(t) = \sum_{k>d_j^+} \xi_j^-(t)u_j^+(t) + \sum_{k>d_j^-} \xi_j^+(t)u_j^-(t) \tag{3.6}$$

with $\xi_j^{\pm} \in \mathbb{F}_{\pm}^{k-d_j^{\mp}}$.

Proof. According to Corollary 3.2 u belongs to the kernel of $H_k(\mathbf{a}) + H_k(\mathbf{b})J_{q,k}$ if $[u^{\mathrm{T}}\ \widehat{u}^{\mathrm{T}}]^{\mathrm{T}}$ belongs to the kernel of $\widetilde{H}_k(\mathbf{a},\mathbf{b})$. In view of Theorem 3.7 this is equivalent to the existence of polynomials $\xi_j(t)^{\pm}$ satisfying the degree restrictions mentioned in Theorem 3.7 such that

$$u(t) = \sum_{j=1}^{r_+} \xi_j^+(t)u_j^+(t) + \sum_{j=1}^{r_-} \xi_j^-(t)u_j^-(t)$$

and

$$\widehat{u}(t) = \sum_{j=1}^{r_+} \xi_j^+(t)\widehat{u}_j^+(t) - \sum_{j=1}^{r_-} \xi_j^-(t)\widehat{u}_j^-(t).$$

Since the vectors $\xi_j^{\pm}$ are unique, we conclude that $\widehat{\xi}_j^+ = \xi_j^+$ and $\widehat{\xi}_j^- = -\xi_j^-$, which means that the ξ_j^+ are symmetric and the ξ_j^- skewsymmetric. □

The dimension of $\mathbb{F}_+^k$ equals $[(k+1)/2]$ and the dimension of $\mathbb{F}_-^k$ equals $[k/2]$, where $[\cdot]$ denotes the integer part. Therefore, the following is true.

Corollary 3.10. *The dimensions of the kernels of* $H_k(\mathbf{a}) \pm H_k(\mathbf{b})J_{q,k}$ *are given by*

$$\dim\ker(H_k(\mathbf{a}) + H_k(\mathbf{b})J_{q,k}) = \sum_{k>d_j^+} \left[\frac{k-d_j^+ + 1}{2}\right] + \sum_{k>d_j^- + 1} \left[\frac{k-d_j^-}{2}\right],$$

$$\dim\ker(H_k(\mathbf{a}) - H_k(\mathbf{b})J_{q,k}) = \sum_{k>d_j^+ + 1} \left[\frac{k-d_j^+}{2}\right] + \sum_{k>d_j^-} \left[\frac{k-d_j^- + 1}{2}\right].$$

The system of vectors $\{u_j^+\}$ $(j = 1, \dots, p+q)$ will be called the $(+)$-*part* and the system $\{u_j^-\}$ the $(-)$-*part* of a *fundamental system* for the pair of block sequences $(\mathbf{a}, \mathbf{b})$. The integers $d_j^\pm$ will be called *characteristic degrees* of $(\mathbf{a}, \mathbf{b})$.

Let us point out that the characteristic degrees are associated with the pair of sequences $(\mathbf{a}, \mathbf{b})$ rather than with the matrix $R_k(\mathbf{a}, \mathbf{b})$. This is because $R_k(\mathbf{a}, \mathbf{b})$ might have different characteristic degrees depending on the representation of $R_k(\mathbf{a}, \mathbf{b})$ in the form $R_k(\mathbf{a}, \mathbf{b}) = H_k(\mathbf{a}) + H_k(\mathbf{b})J_{q,k}$. This concerns, however, only the numbers d_j^-. The numbers d_j^+ are uniquely defined by $R_k(\mathbf{a}, \mathbf{b})$.

Let us give a simple example. The matrix $2I_2$ can be represented with $\mathbf{a} = (0, 0, 0)$ and $\mathbf{b} = (0, 2, 0)$. In this case all four characteristic degrees are equal to 2. But this matrix can also be represented with $\mathbf{a} = (1, 0, 1)$ and $\mathbf{b} = (0, 1, 0)$. In this case we have $d_1^+ = d_2^+ = 2$, $d_1^- = 0$ and $d_2^- = 4$.

Corollary 3.11. *Let d_j^+ and d_j^- $(j = 1, \dots, r_\pm)$ denote the characteristic degrees of* $(\mathbf{a}, \mathbf{b})$. *Then*:

1. *The kernel of $H_k(\mathbf{a}) + H_k(\mathbf{b})J_{q,k}$ is trivial if and only if $d_j^+ \geqslant k$ and $d_j^- \geqslant k-1$ for all j.*
2. *The kernel of $H_k(\mathbf{a}) - H_k(\mathbf{b})J_{q,k}$ is trivial if and only if $d_j^- \geqslant k$ and $d_j^+ \geqslant k-1$ for all j.*

Corollary 3.12. *For the kernel dimensions of $H_k(\mathbf{a}) + H_k(\mathbf{b})J_{q,k}$ and $H_k(\mathbf{a}) - H_k(\mathbf{b})J_{q,k}$ the relation*

$$|\dim\ker(H_k(\mathbf{a}) + H_k(\mathbf{b})J_{q,k}) - \dim\ker(H_k(\mathbf{a}) - H_k(\mathbf{b})J_{q,k})| \leqslant p + q$$

holds.

For the scalar case $p = q = 1$ this result was obtained in [3].

4. Special cases

4.1. Nonsingular Toeplitz-plus-Hankel matrices

In this section we consider the case of a nonsingular scalar Toeplitz-plus-Hankel matrix $R_n = R_n(\mathbf{a}, \mathbf{b}) = H_n(\mathbf{a}) + H_n(\mathbf{b})J_n$. It was shown in [10] that R_n is nonsingular if and only if the dimension of the kernel of R_{n+2} equals 4, and the inverse of R_n can be described with the help of a basis of the kernel of R_{n+2} and a basis of the kernel of $R_{n-2}^{\mathrm{T}} = R_{n+2}(\mathbf{a}, \widehat{\mathbf{b}})$. We now show how such a basis can be constructed from a fundamental system of $(\mathbf{a}, \mathbf{b})$.

According to Corollary 3.11, for a nonsingular R_n four cases are possible:

1. $d_1^+ = d_2^+ = d_1^- = d_2^- = n$,
2. $d_1^+ = d_2^+ = n,\ d_1^- = n-1, d_2^- = n+1$,

3. $d_1^+ = n,\ d_2^+ = n+1,\ d_1^- = n-1,\ d_2^- = n,$
4. $d_1^+ = d_2^+ = n+1,\ d_1^- = d_2^- = n-1.$

If now $\{u_1^+, u_2^+, u_1^-, u_2^-\}$ is a corresponding fundamental system, then the coefficient vectors of the following polynomials form a basis of R_{n+2}, respectively:

1. $(t+1)u_1^+(t),\ (t+1)u_2^+(t),\ (t-1)u_1^-(t),\ (t-1)u_2^-(t),$
2. $(t+1)u_1^+(t),\ (t+1)u_2^+(t),\ (t^2-1)u_1^-(t),\ u_2^-(t),$
3. $(t+1)u_1^+(t),\ u_2^+(t),\ (t^2-1)u_1^-(t),\ (t-1)u_2^-(t),$
4. $u_1^+(t),\ u_2^+(t),\ (t^2-1)u_1^-(t),\ (t^2-1)u_2^-(t).$

4.2. Block Hankel and Toeplitz matrices

In the case of a pure block Hankel (or block Toeplitz) matrix $H_k(\mathbf{a}) = R_k(\mathbf{a}, 0)$ the matrix $\widetilde{H}_k(\mathbf{a}, 0)$ splits into the direct sum of the two block Hankel matrices $H_k(\mathbf{a})$ and $H_k(\widehat{\mathbf{a}})$. Hence, if d_j are the characteristic degrees, then $d_j^+ = d_j$, $d_j^- = d_j$ are the characteristic degrees of $(\mathbf{a}, 0)$. Furthermore, if $\{u_j\}$ is a fundamental system of $\mathbf{a}$, then a fundamental system of $(\mathbf{a}, 0)$ is given by

$$u_j^+ = u_j^- = u_j \quad (j = 1, \ldots, p+q-\delta).$$

Thus Theorem 2.1 follows from Theorem 3.9. In this sense our result is a generalization of the corresponding result for block Hankel matrices.

4.3. Centro-symmetric block Toeplitz-plus-Hankel matrices

We now consider the case that the block Toeplitz-plus-Hankel matrix $R_k = H_k(\mathbf{a}) + H_k(\mathbf{b})J_{q,k}$ is centro-symmetric. i.e., $J_{p,l}R_kJ_{q,k} = R_k$. It is easily checked that then $\mathbf{a}$ and $\mathbf{b}$ can be chosen such that $\widehat{\mathbf{a}} = \mathbf{a}$ and $\widehat{\mathbf{b}} = \mathbf{b}$. The matrix $\widetilde{H}_k(\mathbf{a}, \mathbf{b})$ now takes the form

$$\widetilde{H}(\mathbf{a}, \mathbf{b}) = \begin{bmatrix} H_k(\mathbf{a}) & H_k(\mathbf{b}) \\ H_k(\mathbf{b}) & H_k(\mathbf{a}) \end{bmatrix}.$$

Let $P_{q,k}$ denote the unitary matrix

$$P_{q,k} = \frac{1}{\sqrt{2}} \begin{bmatrix} -I_{q,k} & I_{q,k} \\ I_{q,k} & I_{q,k} \end{bmatrix}.$$

Then we have

$$P_{p,l}^{\mathrm{T}}\, \widetilde{H}_k(\mathbf{a}, \mathbf{b})\, P_{q,k} = \begin{bmatrix} H_k(\mathbf{a}-\mathbf{b}) & 0 \\ 0 & H_k(\mathbf{a}+\mathbf{b}) \end{bmatrix}$$

where $k + l = N + 1$.

From this equality we conclude that $u \in \mathbb{F}^{q,k}$ belongs to the kernel of $H_k(\mathbf{a}+\mathbf{b})$ if and only if $[\begin{smallmatrix} u \\ u \end{smallmatrix}]$ belongs to the kernel of $\widetilde{H}(\mathbf{a}, \mathbf{b})$, and $u \in \ker H_k(\mathbf{a}-\mathbf{b})$ if and only if $[\begin{smallmatrix} u \\ -u \end{smallmatrix}] \in \ker \widetilde{H}(\mathbf{a}, \mathbf{b})$. Note that if u is symmetric, then $[\begin{smallmatrix} u \\ u \end{smallmatrix}]$ is symmetric and $[\begin{smallmatrix} u \\ -u \end{smallmatrix}]$ is skewsymmetric, if u is skewsymmetric, then $[\begin{smallmatrix} u \\ u \end{smallmatrix}]$ is skewsymmetric and $[\begin{smallmatrix} u \\ -u \end{smallmatrix}]$ is symmetric.

Let u_j be flip invariant and belong to a fundamental system of $\mathbf{a}+\mathbf{b}$. If u_j is symmetric, then this vector contributes to the $(+)$-part of a fundamental system of $(\mathbf{a}, \mathbf{b})$ and if u_j is skewsymmetric, then it contributes to the $(-)$-part of a fundamental system. Analogously, if v_j belongs to a fundamental system of $\mathbf{a}-\mathbf{b}$, then it contributes to the $(-)$-part if v_j is symmetric and to the $(+)$-part if it is skewsymmetric.

From now on we assume, for simplicity of formulation, that the block rows $(\mathbf{a} \pm \mathbf{b})^{\mathrm{T}}$ have full rank. This is equivalent to the condition $\delta = 0$.

In [7] the following is proved (Theorem 5.1).

Proposition 4.1. *A centro-symmetric block Hankel or Toeplitz matrix has a fundamental system consisting of flip invariant vectors. If r_+ denotes the number of symmetric and r_- the number of skewsymmetric vectors in a fundamental system, then $r_+ - r_- = q - p$.*

From this we can conclude the following theorem.

Theorem 4.2. *Suppose that* $\mathbf{a}$ *and* $\mathbf{b}$ *are symmetric. Let u_j^+ $(j = 1, \dots, r_+)$ be the symmetric vectors and u_j^- $(j = 1, \dots, r_-)$ the skewsymmetric vectors of a fundamental system of* $\mathbf{a}+\mathbf{b}$ $(r_+ + r_- = p + q)$, *and let $\{v_j^+\}$, $\{v_j^-\}$ be the corresponding systems for* $\mathbf{a}-\mathbf{b}$. *Then the union of $\{u_j^+\}$ and $\{v_j^-\}$ forms the $(+)$-part of a fundamental system and the union of $\{u_j^-\}$ and $\{v_j^+\}$ forms the $(-)$-part of a fundamental system for* $(\mathbf{a}, \mathbf{b})$.

Recall that the $(+)$-part of a fundamental system of $(\mathbf{a}, \mathbf{b})$ is the $(-)$-part of a fundamental system of $(\mathbf{a}, -\mathbf{b})$ and the $(-)$-part of a fundamental system of $(\mathbf{a}, \mathbf{b})$ is the $(+)$-part of a fundamental system of $(\mathbf{a}, -\mathbf{b})$.

Corollary 4.3. *Let $\delta_j^{\pm}$ denote the characteristic degrees for* $\mathbf{a}+\mathbf{b}$, *where the δ_j^+ correspond to symmetric and the δ_j^- correspond to the skewsymmetric vectors in this system and let $\varepsilon_j^{\pm}$ be analogously defined for* $\mathbf{a}-\mathbf{b}$. *Then δ_j^+ and ε_j^- are the characteristic degrees d_j^+ of* $(\mathbf{a}, \mathbf{b})$ *and δ_j^- and ε_j^- are the characteristic degrees d_j^-.*

One consequence of this corollary is that for a scalar nonsingular centro-symmetric Toeplitz-plus-Hankel matrix Case 2 in Section 4.1 is not possible.

4.4. Centro-skewsymmetric block Toeplitz-plus-Hankel matrices

We now consider centro-skewsymmetric block Toeplitz-plus-Hankel matrices $R_k = H_k(\mathbf{a}) + H_k(\mathbf{b}) J_{q,k}$. Again we assume that the block rows $(\mathbf{a} \pm \mathbf{b})^{\mathrm{T}}$ have full rank.

The block sequences $\mathbf{a}$ and $\mathbf{b}$ can be chosen such that $\widehat{\mathbf{a}} = -\mathbf{a}$ and $\widehat{\mathbf{b}} = -\mathbf{b}$ and the matrix $\widetilde{H}_k(\mathbf{a}, \mathbf{b})$ takes the form

$$\widetilde{H}_k(\mathbf{a}, \mathbf{b}) = \begin{bmatrix} -H_k(\mathbf{a}) & -H_k(\mathbf{b}) \\ H_k(\mathbf{b}) & H_k(\mathbf{a}) \end{bmatrix}.$$

Hence

$$P_{p,l}^{\mathrm{T}}\, \widetilde{H}_k(\mathbf{a}, \mathbf{b})\, P_{q,k} = \begin{bmatrix} 0 & H_k(\mathbf{a}+\mathbf{b}) \\ H_k(\mathbf{a}-\mathbf{b}) & 0 \end{bmatrix}.$$

As in the centro-symmetric case, we conclude from this relation that $u \in \ker H_k(\mathbf{a} + \mathbf{b})$ if and only if $[\begin{smallmatrix} u \\ u \end{smallmatrix}] \in \ker \widetilde{H}_k(\mathbf{a}, \mathbf{b})$, and $u \in \ker H_k(\mathbf{a} - \mathbf{b})$ if and only if $[\begin{smallmatrix} u \\ -u \end{smallmatrix}] \in \ker \widetilde{H}_k(\mathbf{a}, \mathbf{b})$.

The following proposition shows that there is an essential difference between the centro-symmetric and the centro-skewsymmetric case.

Proposition 4.4. *A centro-skewsymmetric block Hankel matrix $H_k(\mathbf{a})$ has a fundamental system consisting of symmetric vectors.*

Proof. The proof follows the same lines as the proof of Proposition 3.6. As in the centro-symmetric case (see [7]), it can be shown that a skewsymmetric $\mathbf{a}$ has a fundamental system consisting of flip invariant vectors. Let $r_{\mathrm{e},+}$ and $r_{\mathrm{o},+}$ denote the number of symmetric vectors in this system with even or odd characteristic degree, respectively.

First we consider the case of an odd N. The dimension of the kernel of $\mathbf{a}^{\mathrm{T}}$ is equal to $Nq - p$. Since $\mathbf{a}$ is skewsymmetric and the signature of $\mathbb{F}^{q,N}$ is equal to q, the subspace $\ker \mathbf{a}^{\mathrm{T}}$ has the signature $p + q$. On the other side, this subspace is the direct sum of the $\mathscr{M}_N(\widetilde{u}_j)$ $(j = 1, \ldots, p + q)$. With the help of Lemma 3.5 we conclude that $r_{\mathrm{e},+} = p + q$ and $r_{\mathrm{o},+} = r_{\mathrm{o},-} = r_{\mathrm{e},-} = 0$. In other words, all u_j are symmetric and have an even characteristic degree.

Now let N be even. Then the signature of $\mathbb{F}^{q,N+1}$ is equal to q. By Theorem 2.1, $\mathbb{F}^{q,N+1}$ is the direct sum of the subspaces $\mathscr{M}_{N+1}(u_j)$. With the help of Lemma 3.5 we conclude that $r_{\mathrm{e},+} - r_{\mathrm{e},-} = q$. The signature of $\mathbb{F}^{q,N}$ is in this case equal to zero, thus the signature of $\ker \mathbf{a}^{\mathrm{T}}$ is equal to p. Hence $r_{\mathrm{o},+} - r_{\mathrm{o},-} = p$. From the two relations we conclude that $r_{\mathrm{e},-} = r_{\mathrm{o},-} = 0$, $r_{\mathrm{e},+} = q$ and $r_{\mathrm{o},+} = p$. This means that we have again only symmetric vectors in the fundamental system. $\square$

Thus, the following centro-skewsymmetric analogue of Theorem 4.2 holds. It surprises that its formulation is much simpler than the centro-symmetric version.

Theorem 4.5. *Suppose that $\mathbf{a}$ and $\mathbf{b}$ are skewsymmetric. Let $\{u_j\}$ $(j = 1, \ldots, p + q)$ be a fundamental system of $\mathbf{a} + \mathbf{b}$ and $\{v_j\}$ be a fundamental system of $\mathbf{a} - \mathbf{b}$. Then $\{u_j\}$ is the $(+)$-part and $\{v_j\}$ is the $(-)$-part of a fundamental system of $(\mathbf{a}, \mathbf{b})$.*

Let us discuss finally the nonsingularity of an $n \times n$ centro-skewsymmetric Toeplitz-plus-Hankel matrix in the scalar case $p = q = 1$. Clearly, for this it is necessary that n is even. Furthermore, it is shown in [11] that the characteristic degrees of a square centro-skewsymmetric Toeplitz (or Hankel) matrix are even. This means the Cases 2–4 in Section 4.1 are not possible and we come to a somehow surprising conclusion: If $H_n(\mathbf{a}) + H_n(\mathbf{b})J_n$ is a nonsingular, centro-skewsymmetric Toeplitz-plus-Hankel matrix, then all four characteristic degrees of $(\mathbf{a}, \mathbf{b})$ are equal to n. This means, in particular, that $H_n(\mathbf{a}) - H_n(\mathbf{b})J_n$ is also nonsingular.

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