# Matrix Representations of Toeplitz-plus-Hankel Matrix Inverses

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ABSTRACT

Inverses of Toeplitz-plus-Hankel matrices and, more generally, T + H-Bezoutians are represented as sums of products of triangular Toeplitz and Hankel matrices. The parameters occurring in these representations can be determined with the help of (1) solutions of "fundamental equations," (2) solutions of a certain homogeneous equation, and (3) columns and rows of the inverse matrix.

## INTRODUCTION

According to the well-known Gohberg-Semencul formulas [1], the inverse of a Toeplitz matrix $[t_{i-j}]$ can be represented as product sum of triangular Toeplitz matrices (for other formulas and further discussion we refer to [2]). A similar representation involving triangular Toeplitz and Hankel matrices exists, of course, also for the inverses of regular Hankel matrices $[s_{i+j}]$.

On the other hand, it has been observed that Bezoutian matrices occurring in stability theory have the same representations. From this emerges the fact that the inverse of a Toeplitz matrix is a Toeplitz Bezoutian (or unit-circle Bezoutian) and the inverse of a Hankel matrix is a Hankel (or real-line) Bezoutian. The converse is also true. Such representations of Bezoutians can be found in [3] and, in more detail, in [4].

In our paper [5] the concept of a T + H-Bezoutian was introduced, and it was shown that a regular matrix is a T + H-Bezoutian iff it is the inverse of a Toeplitz-plus-Hankel matrix (T + H-matrix). In the present note we give some




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matrix representations for T + H-Bezoutians, which are in particular representations of T + H-matrix inverses. These representations can be transformed into a form involving mainly triangular Toeplitz and Hankel matrices. We discuss furthermore the problem how to determine the parameters occurring in the representations of T + H-matrix inverses. In particular we clarify under which conditions these inverses are determined by some (four) columns or rows of the inverse matrix.

## 1. T + H-BEZOUTIANS

It is convenient to introduce the concept of a generating function of a matrix. Suppose $A = [a_{ij}]_{i=0}^{m-1}{}_{j=0}^{n-1}$; then the polynomial $A(\lambda, \mu)$ in two variables defined by

$$A(\lambda, \mu) = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} a_{ij} \lambda^i \mu^j$$

is called the generating function of $A$.

DEFINITION. An $m \times n$ matrix $B$ is said to be a T + H-Bezoutian iff

$$B(\lambda, \mu) = \frac{1}{(\lambda - \mu)(1 - \lambda\mu)} \sum_{i=1}^{4} g_i(\lambda) f_i(\mu) \tag{1.1}$$

for some vectors $g_i \in \mathbf{C}^{m+2}$ and $f_i \in \mathbf{C}^{n+2}$ $(i = 1, 2, 3, 4)$.

The concept of a T + H-Bezoutian generalizes the concepts of T- and H-Bezoutians (cf. [2]).

An equivalent definition can be given using the transformation $\nabla$ defined by

$$\nabla A := \left[a_{i-1,j} + a_{i-1,j-2} - a_{i,j-1} - a_{i-2,j-1}\right]_{i=0}^{m+1}{}_{j=0}^{n+1} \tag{1.2}$$

for $A = [a_{ij}]$, acting from $\mathbf{C}^{m \times n}$ into $\mathbf{C}^{(m+2)\times(n+2)}$. Here we put $a_{ij} = 0$ for

$i < 0$, $i \geqslant m$, $j < 0$, and $j \geqslant n$. It is easily checked that

$$(\nabla A)(\lambda, \mu) = (\lambda - \mu)(1 - \lambda\mu)A(\lambda, \mu). \tag{1.3}$$

Hence the matrix $B$ is a T + H-Bezoutian iff

$$\operatorname{rank} \nabla B \leqslant 4,$$

i.e., iff $\nabla B$ admits a representation

$$\nabla B = \sum_{i=1}^{4} g_i f_i^T \tag{1.4}$$

with $g_i \in \mathbf{C}^{m+2}$ and $f_i \in \mathbf{C}^{n+2}$. We show now how the matrix $A$ can be reconstructed from $\nabla A$.

Proposition 1.1. *The entries $a_{ij}$ of the matrix $A$ ($i = 0,\dots, m-1$; $j = 0,\dots, n-1$) can be evaluated with the help of the entries of the matrix $\nabla A = [d_{ij}]_{i=0}^{m+1}{}_{j=0}^{n+1}$ by the following recursion:*

$$\begin{aligned} a_{i0} &= +d_{i+1,0},\ a_{i,n-1} = d_{i+1,n+1}, \\ a_{0j} &= -d_{0,j+1},\ a_{1j} = -d_{1,j+1} + a_{0,j+1} + a_{0,j-1}, \\ a_{ij} &= -d_{i,j+1} + a_{i-1,j+1} + a_{i-1,j-1} - a_{i-2,j} \\ &\qquad (i = 2,\dots, m-1;\quad j = 1,\dots, n-2). \end{aligned} \tag{1.5}$$

Proposition 1.1 provides the reconstruction of $A$ with the help of the first two rows, the first column, and the last column. Clearly, modifications of these formulas may be obtained.

## 2. MATRIX REPRESENTATIONS OF T + H-BEZOUTIANS

The aim of this section is to transform the recursion (1.5) into matrix representations. Since we are mainly interested in T + H-Bezoutians, we may restrict ourselves to the case of $n \times n$ matrices $A$ for which $\nabla A$ is represented in the form

$$\nabla A = \sum_{p=1}^{r} b_p c_p^T.$$

Using generating functions this can be rewritten as

$$A(\lambda, \mu) = \frac{1}{(\lambda - \mu)(1 - \lambda\mu)} \sum_{p=1}^{r} b_p(\lambda) c_p(\mu) \tag{2.1}$$

with $b_p(\lambda) = \sum_{j=0}^{n+1} b_{pj}\lambda^j$, $c_p(\mu) = \sum_{j=0}^{n+1} c_{pj}\mu^j$.

LEMMA 2.1. *Suppose the generating function of a matrix $A$ is given by (2.1). Then the entries $a_{jk}$ ($j, k = 0, \ldots, n-1$) of $A$ can be computed with the help of the following formulas*:

$$a_{jk} = -\sum_{p=1}^{r} \sum_{s=0}^{j} b_{p,j-s} \sum_{t=0}^{s} c_{p,s+k+1-2t}, \tag{2.2}$$

$$a_{jk} = \sum_{p=1}^{r} \sum_{s=0}^{k} c_{p,k-s} \sum_{t=0}^{s} b_{b,j+s+1-2t}, \tag{2.3}$$

$$a_{jk} = \sum_{p=1}^{r} \sum_{s=0}^{n-k-1} c_{p,k+s+2} \sum_{t=0}^{s} b_{p,j+s+1-2t}, \tag{2.4}$$

$$a_{jk} = -\sum_{p=1}^{r} \sum_{s=0}^{n-j-1} b_{p,j+s+2} \sum_{t=0}^{s} c_{p,s+k+1-2t}, \tag{2.5}$$

*where we put $b_{pi} = 0$, $c_{pi} = 0$ if $i \notin \{0, 1, \ldots, n+1\}$.*

*Proof.* Using the recursion (1.5) with

$$d_{ij} := \sum_{p=1}^{r} b_{pj} c_{pk},$$

we can prove the formula (2.2) by induction. Obviously, (2.2) holds for $j = 0$.

Suppose (2.2) is true for some $j$ and all $k$. We show that it is also true for $j+1$ and all $k$. By assumption we have

$$a_{j,k-1} = -\sum_{p=1}^{r} \sum_{s=0}^{j} b_{p,j-s} \sum_{t=0}^{s} c_{p,k+s-2t},$$

$$a_{j,k+1} = -\sum_{p=1}^{r} \sum_{s=0}^{j} b_{p,j-s} \sum_{t=0}^{s} c_{p,k+s+2-2t},$$

and

$$a_{j-1,k} = -\sum_{p=1}^{r} \sum_{s=0}^{j-1} b_{p,j-s-1} \sum_{t=0}^{s} c_{p,k+s+1-2t},$$

$$= -\sum_{p=1}^{r} \sum_{s=0}^{j} b_{p,j-s} \sum_{t=0}^{s-1} c_{p,k+s-2t}.$$

Inserting these into the recursion

$$a_{j+1,k} = -\sum_{p=1}^{r} b_{p,j+1} c_{p,k+1} + a_{j,k-1} + a_{j,k+1} - a_{j-1,k},$$

we obtain

$$a_{j+1,k} = -\sum_{p=1}^{r} \left[ b_{p,j+1} c_{p,k+1} + \sum_{s=0}^{j} b_{p,j-s} \left( \sum_{t=0}^{s} c_{p,k+s+2-2t} + c_{p,k-s} \right) \right]$$

$$= -\sum_{p=1}^{r} \left( \sum_{s=0}^{j+1} b_{p,j-s+1} \sum_{t=0}^{s} c_{p,k+s+1-2t} \right),$$

which is just (2.2) for $j+1$. With a similar recursion formula obtained from (1.5) the relation (2.4) is shown. Taking into account that the generating function of $A^T$ has the form

$$-\sum_{p=1}^{r}\frac{c_p(\lambda)b_p(\mu)}{(\lambda-\mu)(1-\lambda\mu)},$$

the formulas (2.3), (2.5) are obtained directly from (2.2), (2.4), respectively. ∎

Our next step is the translation of the relations (2.2)–(2.5) into matrix equalities. For this we introduce some notation. For $a=[a_j]_0^{n+1}$ we denote

$$T_+(a)=\begin{bmatrix} a_0 & & & \\ a_1 & a_0 & & 0 \\ \vdots & & \ddots & \\ a_{n-1} & a_{n-2} & \cdots & a_0 \end{bmatrix},\qquad T_-(a)=T_+(a)^T, \tag{2.6}$$

$$W_+(a)=\left[\sum_{t=0}^{j}a_{j+k+1-2t}\right]_{j,k=0}^{n-1},\qquad W_-(a)=W_+(a)^T.$$

Then (2.2)–(2.5) can be rewritten in the form

$$A=-\sum_{p=1}^{r}T_+(b_p)W_+(c_p), \tag{2.7}$$

$$A=\sum_{p=1}^{r}W_-(b_p)T_-(c_p), \tag{2.8}$$

$$A=\sum_{p=1}^{r}W_-(b_p)T_-(\hat{c}_p)J, \tag{2.9}$$

$$A=-\sum_{p=1}^{r}JT_+(\hat{b}_p)W_+(c_p), \tag{2.10}$$

where $J_n(x_0,\dots,x_{n-1}):=(x_{n-1},\dots,x_0)$, $J:=J_n$, and $\hat{b}:=J_{n+2}b$. For a more refined representation we introduce the matrices

$$z=\left.\begin{bmatrix} 0 & & & & & 1 & & & & & 0\\ & & & & 1 & 0 & 1 & & & & \\ & & & 1 & 0 & 1 & 0 & 1 & & & \\ & & \iddots & & & & & & \ddots & & \\ 0 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 0\end{bmatrix}\right\}n,$$

$$D(a)=\left.\begin{bmatrix} a_0 & a_1 & \cdots & a_{n+1} & & & & 0\\ & a_0 & a_1 & \cdots & a_{n+1} & & & \\ & & \ddots & \ddots & & & \ddots & \\ 0 & & & a_0 & a_1 & & \cdots & a_{n+1}\end{bmatrix}\right\}n.$$

Then we have

$$W_+(a)=ZD(a)^T. \tag{2.11}$$

Thus we have proved the following theorem.

THEOREM 2.1. *Let $B$ be a* T+H*-Bezoutian* (1.1). *Then $B$ can be represented in the following forms*:

$$B=-\sum_{p=1}^{4}T_+(g_p)ZD(\hat{f}_p)^T, \tag{2.12}$$

$$B=\sum_{p=1}^{4}D(\hat{g}_p)Z^TT_-(f_p), \tag{2.13}$$

$$B=\sum_{p=1}^{4}D(\hat{g}_p)Z^TT_-(\hat{f}_p)J, \tag{2.14}$$

$$B=-\sum_{p=1}^{4}JT_+(\hat{g}_p)ZD(\hat{f}_p)^T. \tag{2.15}$$

The formulas (2.12)–(2.15) can be easily transformed into other expressions in which occur mainly triangular T- and H-matrices. For this we define the vectors

$$e = [1,0,1,0,\dots]^T, \qquad a' = [a_1,\dots,a_n]^T$$

and the matrix

$$R = T_+(0,1,0,1,\dots).$$

Then

$$Z = [RJ, e, R],$$

and therefore

$$W_+(a) = R[JT_+(\hat{a}) + T_-(a)] + ea'^T,$$

$$W_-(a) = [T_-(\hat{a}) + T_+(a)J]RJ + a'e^T.$$

Let us finally remark that after comparing the formulas (2.2)–(2.5) one can obtain a large variety of relations between the entries of T + H-Bezoutians.

## 3. INVERSES OF T + H-MATRICES

Inverses of (regular) T + H-matrices are T + H-Bezoutians. This was proved in our paper [5]. The question is now how to determine the vectors $b_p$ and $c_p$ $(p = 1,2,3,4)$. We discuss three possibilities:

(1) using solutions of "fundamental equations,"
(2) using solutions of homogeneous equations,
(3) using columns and/or rows of the inverse.

Some discussions concerning (1) and (2) have been already presented in [5]; the results concerning (3) seem to be new.

*3.1.*

Suppose the matrix $A$ is given by

$$A = \left[t_{i-j} + s_{i+j}\right]_{i,j=0}^{n-1}, \tag{3.1}$$

where $t_i$ and $s_i$ are complex numbers. The equations

$$Ax_1 = g_1, \qquad Ax_2 = g_2, \qquad Ax_3 = e_0, \qquad Ax_4 = e_{n-1}, \tag{3.2}$$

$$A^T y_1 = e_0, \qquad A^T y_2 = e_{n-1}, \qquad A^T y_3 = f_1, \qquad A^T y_4 = f_2 \tag{3.3}$$

with $g_1 := (t_{1+i} + s_{i-1})_0^{n-1}$, $g_2 := (t_{i-n} + s_{i+n})_0^{n-1}$, $f_1 := (t_{-1-i} + s_{i-1})_0^{n-1}$, $f_2 := (t_{n-i} + s_{i+n})_0^{n-1}$ are called fundamental, since their solutions completely determine the inverse matrix $A^{-1}$. Here $s_{-1}$, $s_{2n-1}$, $t_n$, $t_{-n}$ denote arbitrary numbers, and $e_k$ is the $(k+1)$th unit vector.

THEOREM 3.1 [5]. *Let one of the systems* (3.2) *or* (3.3) *be solvable. Then* $A$ *is regular and*

$$A^{-1}(\lambda, \mu) = \frac{1}{(\lambda - \mu)(1 - \lambda\mu)} \sum_{i=1}^{4} b_i(\lambda) c_i(\mu),$$

*where*

$$\begin{aligned} b_1(\lambda) &= -1 + \lambda x_1(\lambda), & c_1(\lambda) &= \lambda y_1(\lambda), \\ b_2(\lambda) &= \lambda x_2(\lambda) - \lambda^{n+1}, & c_2(\lambda) &= \lambda y_2(\lambda), \\ b_3(\lambda) &= \lambda x_3(\lambda), & c_3(\lambda) &= 1 - \lambda y_3(\lambda), \\ b_4(\lambda) &= \lambda x_4(\lambda), & c_4(\lambda) &= -\lambda y_4(\lambda) + \lambda^{n+1}. \end{aligned} \tag{3.4}$$

This theorem together with Theorem 2.1 leads to matrix representations of T + H-matrix inverses.

REMARK. Let us note that the inverse of $A$ can be constructed using only the solutions $x_i$ $(i=1,2,3,4)$. In fact the following recursion holds:

$$A^{-1}e_{k+1}=\left(S_n+S_n^T\right)A^{-1}e_k-A^{-1}e_{k-1}$$
$$+\left(x_1e_0^T+x_2e_{n-1}^T-x_3f_1^T-x_4f_2^T\right)A^{-1}e_k \qquad (3.5)$$

(see [5]). Here $S_n$ denotes the forward shift in $\mathbf{C}^n$.

3.2.

Now let us show how the solutions of the fundamental equations (3.2) can be obtained from the kernel of the matrix

$$\tilde{A}=\left[t_{i-j}+s_{i+j}\right]_{i=1\ j=-1}^{n-2\ n}. \qquad (3.6)$$

When $A$ is regular, then $\tilde{A}$ has full rank. Consequently, its kernel dimension is equal to 4. Let $\{w_1, w_2, w_3, w_4\}$ be an arbitrary basis of the kernel of $\tilde{A}$. We introduce the matrices

$$\Phi:=\begin{bmatrix} -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & -1 \\ 0 & & a_1 & & 0 \\ 0 & & a_n & & 0 \end{bmatrix}, \qquad W:=\begin{bmatrix} w_1 & w_2 & w_3 & w_4 \end{bmatrix},$$

where $a_1$ denotes the first and $a_n$ the last row of $A$. Then the $4\times 4$ matrix

$$C:=\Phi W$$

is regular. Suppose $C^{-1}=[\alpha_{ij}]_1^4$ and $\tilde{w}_i:=\sum_{j=1}^4\alpha_{ji}w_j$; the set $\{\tilde{w}_1, \tilde{w}_2, \tilde{w}_3, \tilde{w}_4\}$ is a basis of $\ker\tilde{A}$ as well. Furthermore, $\Phi\tilde{w}_i=e_i$. That means, in particular, that the vectors $\tilde{w}_i$ have the form

$$\tilde{w}_1=\begin{bmatrix} -1 \\ x_1 \\ 0 \end{bmatrix}, \qquad \tilde{w}_2=\begin{bmatrix} 0 \\ x_2 \\ -1 \end{bmatrix}, \qquad \tilde{w}_3=\begin{bmatrix} 0 \\ x_3 \\ 0 \end{bmatrix}, \qquad \tilde{w}_4=\begin{bmatrix} 0 \\ x_4 \\ 0 \end{bmatrix}, \qquad (3.7)$$

where $x_i$ $(i=1,2,3,4)$ are just the solutions of the fundamental equations (3.2). Let us summarize.

PROPOSITION 3.1. *When* $\{w_1, w_2, w_3, w_4\}$ *is a basis of* $\ker \tilde{A}$ *and*

$$[\tilde{w}_1 \quad \tilde{w}_2 \quad \tilde{w}_3 \quad \tilde{w}_4] := [w_1 \quad w_2 \quad w_3 \quad w_4](\Phi W)^{-1},$$

*then the vectors* $\tilde{w}_i$ *have the form* (3.7) *and* $x_i$ *are the fundamental solutions of* (3.2).

We discuss the problem whether the inverse $A^{-1}$ of a T + H-matrix can be constructed with the help of some columns (and rows) of $A^{-1}$. The Gohberg-Semencul theorem [1] states that the inverse of a Toeplitz matrix can be constructed with the help of the first and last columns if the left upper element of $A^{-1}$ is different from zero. Another theorem, due to Gohberg and Krupnik [6], states that $A^{-1}$ can be constructed from the first two columns if the left lowest element is nonzero. We are going to construct the inverse of a T + H-matrix with the help of the first two and last two columns. The idea is to represent the solutions of the fundamental equations by means of columns or rows.

Let $A$ be a regular T + H-matrix given by (3.1), let $\tilde{A}$ be defined by (3.6), and let $u_1$, $u_2$ denote the first two columns of $A^{-1}$ and $v_1$, $v_2$ the last two columns. The vectors $u_1$ and $v_1$ are already fundamental solutions: $u_1 = x_3$, $v_1 = x_4$. It remains to construct the solutions $x_1$ and $x_2$.

Obviously,

$$\tilde{A}\begin{bmatrix} 0 & 0 & 0 & 0 \\ u_1 & v_1 & u_2 & v_2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.8}$$

We evaluate now $\tilde{A}\tilde{u}_1$ for

$$\tilde{u}_1 = \begin{bmatrix} u_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ u_1 \end{bmatrix}.$$

Introducing the matrices

$$H' = [s_{i+j}]_0^{n-3} {}_0^{n-1}, \qquad T' = [t_{i-j}]_0^{n-3} {}_0^{n-1},$$

$$H'' = [s_{i+j}]_2^{n-1} {}_0^{n-1}, \qquad T'' = [t_{i-j}]_2^{n-1} {}_0^{n-1},$$

we have the two representations

$$A = \begin{bmatrix} H' + T' \\ * \\ * \end{bmatrix} = \begin{bmatrix} * \\ * \\ H'' + T'' \end{bmatrix}.$$

Hence

$$\tilde{A} = [\, H' + T'' \quad * \quad * \,] = [\, * \quad * \quad H'' + T' \,].$$

We obtain

$$\tilde{A}\tilde{u}_1 = (H' + T'')u_1 + (H'' + T')u_1 = e_0. \tag{3.9}$$

Comparing (3.9) with (3.8), we conclude that the vector $z_1$ defined by

$$z_1 = \begin{bmatrix} u_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ u_1 \end{bmatrix} - \begin{bmatrix} 0 \\ u_2 \\ 0 \end{bmatrix} \tag{3.10}$$

belongs to $\ker \tilde{A}$. Analogously, the vector $z_2$ defined by

$$z_2 = \begin{bmatrix} v_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ v_1 \end{bmatrix} - \begin{bmatrix} 0 \\ v_2 \\ 0 \end{bmatrix} \tag{3.11}$$

belongs to $\ker \tilde{A}$.

We introduce now the number

$$\Delta = \det \begin{bmatrix} (u_1)_0 & (v_1)_0 \\ (u_1)_{n-1} & (v_1)_{n-1} \end{bmatrix}. \tag{3.12}$$

Here $(u)_k$ denotes the $(k+1)$th component of the vector $u$ $(k = 0, \dots, n-1)$. Under the assumption $\Delta \neq 0$ we may introduce

$$\begin{aligned} \tilde{x}_1 &= \frac{1}{\Delta}\left[ (v_1)_0 z_1 - (u_1)_0 z_2 \right], \\ \tilde{x}_2 &= \frac{1}{\Delta}\left[ -(v_1)_{n-1} z_1 + (u_1)_{n-1} z_2 \right]. \end{aligned} \tag{3.13}$$

Then $\tilde{x}_1, \tilde{x}_2 \in \ker \tilde{A}$,

$$(\tilde{x}_1)_0 = \frac{1}{\Delta}\left[(v_1)_0(u_1)_0 - (u_1)_0(v_1)_0\right] = 0,$$

$$(\tilde{x}_1)_{n+1} = \frac{1}{\Delta}\left[(v_1)_0(u_1)_{n-1} - (u_1)_0(v_1)_{n-1}\right] = -1,$$

and, analogously,

$$(\tilde{x}_2)_0 = -1, \qquad (\tilde{x}_2)_{n+1} = 0.$$

Hence $\tilde{x}_1$, $\tilde{x}_2$ have the form

$$\tilde{x}_1 = \begin{bmatrix} 0 \\ x_1 \\ -1 \end{bmatrix}, \qquad \tilde{x}_2 = \begin{bmatrix} -1 \\ x_2 \\ 0 \end{bmatrix},$$

where $x_1$ and $x_2$ are just the solutions of the corresponding fundamental equations (3.2). We have arrived at the following statement.

THEOREM 3.2. *Let $u_1$, $u_2$ denote the first two and $v_2$, $v_1$ the last two columns of $A^{-1}$, where $A$ is a regular* T + H*-matrix. If $\Delta$, defined by* (3.12), *is nonzero, then the fundamental solutions $x_1$, $x_2$, $x_3$, $x_4$ can be obtained from $u_1$, $u_2$, $v_1$, $v_2$ with the help of the relations*

$$x_1 = \frac{1}{\Delta}\left[(v_1)_0 z_1' - (u_1)_0 z_2'\right],$$

$$x_2 = \frac{1}{\Delta}\left[-(v_1)_{n-1} z_1' + (u_1)_{n-1} z_2'\right],$$

$$x_3 = u_1, \qquad x_4 = v_1,$$

*where*

$$\begin{bmatrix} * \\ z_1' \\ * \end{bmatrix} = \begin{bmatrix} u_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ u_1 \end{bmatrix} - \begin{bmatrix} 0 \\ u_2 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} * \\ z_2' \\ * \end{bmatrix} = \begin{bmatrix} v_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ v_1 \end{bmatrix} - \begin{bmatrix} 0 \\ v_2 \\ 0 \end{bmatrix}.$$

COROLLARY. *If $\Delta \neq 0$, then the inverse of a $T+H$-matrix is uniquely determined by the first two and last two columns.*

For constructing matrix representations as in Section 2, one will need also the first and last two rows of the inverse matrix. The corresponding relations are obtained from those of Theorem 3.2, replacing the matrix $A$ by its transpose.

REMARK. The condition $\Delta \neq 0$ is equivalent to the regularity of the $(n-2)\times(n-2)$ submatrix

$$A' = \left[s_{i+j} + t_{i-j}\right]_1^{n-2}$$

of $A$.

In the paper [1] a formula is presented for constructing the inverse of the Toeplitz matrix $T' = [t_{i-j}]_0^{n-2}$ with the help of the first and last columns of the matrix $T^{-1}$, $T = [t_{i-j}]_0^{n-1}$. An analogous formula can be constructed for the matrix $A'$ defined above. In fact, as noted in Section 3.2, the inverse of $A'$ can be constructed with the help of the kernel of the matrix

$$\tilde{A}' = \left[s_{i+j} + t_{i-j}\right]_2^{n-3}\,{}_0^{n-1}.$$

It is easily checked now that the vectors $u_1$, $u_2$, $v_2$, $v_1$—the first two and the last two columns of $A^{-1}$—form a basis of $\ker \tilde{A}'$. As described in Proposition 3.1, the fundamental solutions for the matrix $A'$ can now easily be evaluated.

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