V_2 is $\alpha A = B$. But this is not possible. Hence our supposition that T has sides of the same color is wrong.

Case 2: Let V_1V_2 and V_1V_3 have the color α. In a similar way to that of Case 1, we may show that this is not possible.

Lemmas 3 and 4 apply to closed networks, but they also apply to an open network. For, given an open network, we first convert it into a closed network by introducing more triangles so that the "gaps" are filled in. Not until the open network has been converted into a closed network do we start the coloring. Then we may apply Lemmas 3 and 4 so that the closed network gets properly colored. Now, removing the additional triangles will not disturb the color scheme of the rest, which is certainly properly colored. Hence we get

THEOREM 2. *A necessary and sufficient condition that four colors suffice to color a trivalent map M (whether on a closed or an open surface) is that the triangular network of M' be properly colorable.*

In view of Theorem 2, it appears that a further study of the network of the dual of a trivalent map may be of interest.

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A GENERALIZED FIBONACCI SEQUENCE

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1. Introduction. Recently in this MONTHLY [1], [2], [3] there have appeared several problems and results involving the Fibonacci sequence, and consequently one is prompted to offer some comments on a generalized theory.

In the following development $a=\frac{1}{2}(1+\sqrt{5})$, $b=\frac{1}{2}(1-\sqrt{5})$ are the roots of $x^2-x-1=0$ so that $a+b=1$, $a-b=\sqrt{5}$, $ab=-1$. (The values of a and $-b$ are, of course, associated with the classical geometrical problem of the golden section.) For the Fibonacci sequence

$$(\alpha) \qquad 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \cdots,$$

defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$ $(n \geqq 3)$ with $F_1 = F_2 = 1$, it is well known (Daniel Bernoulli, 1732) that the nth term (Fibonacci number) is $F_n = (a^n - b^n)/\sqrt{5}$.

2. A generalized Fibonacci sequence. Suppose we preserve the recurrence relation but alter the first two terms to produce the generalized Fibonacci sequence defined by:

$$(\beta) \qquad H_n = H_{n-1} + H_{n-2} (n \geqq 3), \qquad H_1 = p,\ H_2 = p + q,$$

where p, q are arbitrary integers. That is, the generalized sequence is

$$(\gamma) \qquad p,\ p + q,\ 2p + q,\ 3p + 2q,\ 5p + 3q,\ 8p + 5q,\ 13p + 8q, \cdots .$$

Employing the usual method for difference equations, we deduce, after a little calculation, that

$$(\delta) \qquad H_n = \frac{1}{2\sqrt{5}} (la^n - mb^n),$$

where $l=2(p-qb)$, $m=2(p-qa)$, so that $l+m=2(2p-q)$, $l-m=2q\sqrt{5}$, $\frac{1}{4}lm = p^2-pq-q^2=e$ (say).

Certain results follow almost immediately from (β) and/or (δ), *viz.*,

$$(1) \qquad H_n/H_{n-1} \to a, \qquad H_n/H_{n-i} \to a^i, \qquad H_n/F_n \to p - qb \qquad (\text{as } n \to \infty);$$

$$(2) \qquad 2^{n+1}H_n = \frac{l-m}{\sqrt{5}} \sum_{i=0}^{[\frac{1}{2}n]} 5^i C_{2i}^n + (l + m) \sum_{i=0}^{[\frac{1}{2}n-\frac{1}{2}]} 5^i C_{2i+1}^n;$$

$$(3) \qquad H_{n+2} - 2H_n - H_{n-1} = 0, \qquad H_{n+1} - 2H_n + H_{n-2} = 0;$$

$$(4) \qquad \sum_{i=0}^{n-1} H_{2i+1} = H_{2n} - q, \qquad \sum_{i=1}^{n} H_{2i} = H_{2n+1} - p;$$

$$(5) \qquad \sum_{i=1}^{n} (H_{2i-1} - H_{2i}) = - H_{2n-1} + p - q.$$

Writing $\sigma_n = \sum_{i=1}^n H_i$, $\tau_n = \sum_{i=1}^n \sigma_i$, we have

$$(6) \qquad \sigma_n = H_{n+2} - H_2, \qquad \tau_n = H_{n+4} - (n + 2)H_2 - H_1.$$

From (γ) we observe that

$$(7) \qquad H_{n+1} = qF_n + pF_{n+1}, \qquad H_{n+2} = pF_n + (p + q)F_{n+1}.$$

Putting $n=r$ in (7) and using (β), we find in turn

$$H_{r+3} = (p + q)F_r + (2p + q)F_{r+1} = H_2F_r + H_3F_{r+1},$$
$$H_{r+4} = (2p + q)F_r + (3p + 2q)F_{r+1} = H_3F_r + H_4F_{r+1}, \cdots ,$$

and, in general,

$$(8) \qquad H_{n+r} = H_{n-1}F_r + H_nF_{r+1} \qquad (n \geqq 3).$$

In the ensuing results it is perhaps preferable to use (δ) throughout, in which case the following identities may be advantageously used:

$$(\epsilon)\qquad \begin{gathered} a - \frac{1}{a^3} = -b\sqrt{5}, \qquad b - \frac{1}{b^3} = a\sqrt{5}; \\ a + \frac{1}{a} = \sqrt{5}, \qquad b + \frac{1}{b} = -\sqrt{5}; \\ a^2 - a - 1 = 0, \qquad b^2 - b - 1 = 0. \end{gathered}$$

(No attempt at any meaningful order in these results is implied.)

$$H_{n-1}^2 + H_n^2 = (2p - q)H_{2n-1} - eF_{2n-1}, \tag{9}$$

$$H_{n+1}^2 - H_{n-1}^2 = (2p - q)H_{2n} - eF_{2n}, \tag{10}$$

$$H_{n-1}H_{n+1} - H_n^2 = (-1)^n e, \tag{11}$$

$$H_nH_{n+r+1} - H_{n-s}H_{n+r+s+1} = (-1)^{n+s}eF_sF_{r+s+1}, \tag{12}$$

$$H_n^3 + H_{n+1}^3 = 2H_nH_{n+1}^2 + (-1)^n e, \tag{13}$$

$$H_{n+1-r}H_{n+1+r} - H_{n+1}^2 = (-1)^{n-r}eF_r^2. \tag{14}$$

(Observe that (11) is a special case of (14) when $r=1$ and n is replaced by $n-1$.) Putting $r=n$ in (14), we derive

$$H_{n+1}^2 + eF_n^2 = pH_{2n+1}. \tag{15}$$

Numerous other results may be deduced but, as we are not concerned here with an exhaustive list, we mention only the useful "Pythagorean" theorem:

$$\left\{2H_{n+1}H_{n+2}\right\}^2 + \left\{H_nH_{n+3}\right\}^2 = \left\{2H_{n+1}H_{n+2} + H_n^2\right\}^2, \tag{16}$$

and the remarkable fact that

$$\frac{H_{n+r} + (-1)^r H_{n-r}}{H_n} = F_{r+1} + (-1)^r F_{r-1} = \left(\frac{a}{2}\right)^r + (-1)^r\left(\frac{2}{a}\right)^r, \tag{17}$$

i.e., the expression on the left-hand side of (17) is independent of p, q, and n.

Taking $p=3$, $q=1$, for instance, so that (γ) becomes

$$(\zeta)\qquad 3, 4, 7, 11, 18, 29, 47, 76, 123, \cdots$$

and putting $n=3$, we find that (16) yields $396^2+203^2=445^2$. Again using these values for p and q, and setting $n=7$, $r=2$ in (17), we have $(123+18)/47=3=F_3+F_1$. Note that the simple results $3^2+4^2=5^2$, $5^2+12^2=13^2$, occur when $p=1$, $q=0$, $n=1$, and $p=1$, $q=0$, $n=2$, respectively. When $p=1$, $q=0$, $n=3$, we obtain $8^2+15^2=17^2$ after simplifying and dividing throughout by 4. An interesting question is: Does (16) exhaust all the Pythagorean number triples?

Searching through the available literature on generalizations of (α), one sees that, broadly speaking, the work may be generalized in two main directions. Either the recurrence relation can be generalized and extended (and this has been done in a variety of ways) or the recurrence relation is preserved but the first two Fibonacci numbers are altered—this has been our approach. (Naturally, these two techniques could be combined.) The research to which this article most closely approximates seems to be that of Tagiuri ([5], p. 404), who wrote in Periodico di Matematiche, vol. 16, 1901, on the subject. The results (7), (12), (14), (17) above are due to Tagiuri, who used a slightly different notation from that used here. Doubtless, a good deal of work along these lines has been done by mathematicians not recorded in [5].

3. Special cases. The Fibonacci sequence (α) is obtained from (γ) by putting $p=1$, $q=0$. Making this simplification, so that $l=m=2$, $e=1$, and (δ) reduces to $F_n=(a^n-b^n)/\sqrt{5}$, we obtain, for instance, from (9), (11), and (15), respectively, the well-known results

$$F_{n-1}^2 + F_n^2 = F_{2n-1}, \tag{9'}$$

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n, \tag{11'}$$

$$F_{n+1}^2 + F_n^2 = F_{2n+1}, \tag{15'}$$

attributed to Catalan, Simson, and Lucas, respectively. Obviously, (15′) is obtainable from (9′) by replacing n by $n+1$, but we cannot say the same about (15) and (9).

All the commonly known properties of the Fibonacci sequence are thus deducible from the generalized sequence as special cases when $p=1$, $q=0$. For example, if p is arbitrary and $q=1$ in (γ), we obtain the variant of the Fibonacci sequence discussed by Guest [4]. With his further modification $p=10$, we find that his $F_{11}=89=e$ in our notation and all the results given by him then occur as special cases of the general theory when $p=10$, $q=1$.

A question that may well be asked is: Under what conditions (relating to p and q) is the Fibonacci sequence (α) repeated? Obviously, from (γ) for $H_1=H_2=p$ (*i.e.*, $q=0$), each term of (α) is merely multiplied by p as Guest observed. Furthermore, we must avoid the case when q and p are consecutive Fibonacci numbers for if $p=F_n$, $q=F_{n-1}$, we find that the new sequence is the Fibonacci sequence with the first $n-1$ terms missing, *i.e.*,

$$F_n,\ F_n+F_{n-1},\ 2F_n+F_{n-1},\ 3F_n+2F_{n-1},\ 5F_n+3F_{n-1},\ \cdots,$$

which is the same as $F_n, F_{n+1}, F_{n+2}, F_{n+3}, F_{n+4}, \cdots$. In particular, if $p=8$, $q=5$, the new sequence is 8, 13, 21, 34, 55, *i.e.*, (α) with the first five numbers missing.

On the other hand, if $p=F_{n-1}$, $q=F_n$, we do not obtain the Fibonacci sequence. With $p=5$, $q=8$, for example, we get the sequence 5, 13, 18, 31, 49, 80, 129, $\cdots$.

When $q=np$ (n an integer), we have the sequence

$$p\{1,\ 1+n,\ 2+n,\ 3+2n,\ 5+3n,\ 8+5n,\ 13+8n,\ \cdots\},$$

i.e., p times the generalized sequence with $p=1$, $q=n$.

Turning now to some recent problems and remarks on the ordinary Fibonacci sequence, we can generalize Ivanoff's result [2] thus:

$$\begin{aligned}\sum_{i=0}^{n} C_i^n H_{n-i} &= \frac{1}{2\sqrt{5}}\left\{ l\sum_{i=0}^{n} C_i^n a^{n-i} - m\sum_{i=0}^{n} C_i^n b^{n-i}\right\}\\ &= \frac{1}{2\sqrt{5}}\{l(1+a)^n - m(1+b)^n\}\\ &= \frac{1}{2\sqrt{5}}\{la^{2n} - mb^{2n}\} = H_{2n},\end{aligned}$$

where in the next to the last step we have used (ϵ).

Danese's result [3] is now seen to be a special case of (12) when n is replaced by $n+h$ and $s=h$, $r=k-h-1$ (with $p=1$, $q=0$, of course). The last two results in Ganis [1] are particular cases of Danese's result, as noted in [3]. Venkannayah's problem [3] carries over directly to the generalized case, but Everman's problem [3] is a special case (remembering that $F_5=5$) of a theorem due to Tagiuri, *viz.*, that F_r is a multiple of F_s provided that r is a multiple of s, which is not valid in the generalized sequence. For example, $H_6/H_3=29/7$.

Of the remaining results quoted in [1] (and due originally to Simson, Lucas, and Piccioli), the first and third are generalized in (11) and (6), respectively, and the second, $F_{n+1}=C_0^n+C_1^{n-1}+C_2^{n-2}+\cdots$, could be considered to be generalized in (7).

This paper developed out of an interest in the Fibonacci sequence and a desire to extend the results of Guest's stimulating article. Ever since Fibonacci (Leonardo of Pisa) wrote his Liber Abbaci in 1202, his intriguing sequence has fascinated men through the centuries, not only for its inherent mathematical riches, but also for its applications in art and nature. Indeed, it is almost true to say that the research generated by its nearly amounts to the quantity of offspring generated by the mythical pair of rabbits who started Fibonacci off on the problem!

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