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GENERATING FUNCTIONS FOR POWERS OF A CERTAIN GENERALISED SEQUENCE OF NUMBERS

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1. Introduction. Two fundamental sequences in the theory of second-order recurrences are the Fibonacci sequence $\{f_n\}$ and the Lucas sequence $\{a_n\}$ defined by:

	n	: 0	1	2	3	4	5	6	$\cdots$
(1)	$\{f_n\}$	: 1	1	2	3	5	8	13	$\cdots$
(2)	$\{a_n\}$	: 2	1	3	4	7	11	18	$\cdots$

where

(3) $$f_n = f_{n-1} + f_{n-2} = \frac{\alpha_1^{n+1} - \beta_1^{n+1}}{\alpha_1 - \beta_1} \qquad (n \geq 2)$$

(4) $$a_n = a_{n-1} + a_{n-2} = \alpha_1^n + \beta_1^n$$

in which α_1, β_1 are the roots of $x^2 - x - 1 = 0$, so that

(5) $$\alpha_1 = \frac{1 + \sqrt{5}}{2}, \beta_1 = \frac{1 - \sqrt{5}}{2}, \alpha_1 + \beta_1 = 1, \alpha_1\beta_1 = -1, \alpha_1 - \beta_1 = \sqrt{5}.$$

Classical extensions of these are the sequences $\{u_n\}$, $\{v_n\}$ defined by:

(6) $$\{u_n\} \equiv \{u_n(p, q)\} : u_0 = 1, u_1 = p, u_n = pu_{n-1} - qu_{n-2} \qquad (n \geq 2)$$

(7) $$\{v_n\} \equiv \{v_n(p, q)\} : v_0 = 2, v_1 = p, v_n = pv_{n-1} - qv_{n-2}$$

with $p^2 \neq 4q$ and p, q arbitrary integers. (Comments on the degenerate case $p^2 = 4q$ are made towards the end of §4.)

Another extension is defined in [4], with some obvious notational alterations, and for arbitrary integers r, s, by:

(8) $$\{h_n\} \equiv \{h_n(r, s)\} : h_0 = r, h_1 = r + s, h_n = h_{n-1} + h_{n-2} = rf_n + sf_{n-1} (n \geq 2).$$

Sequences (6) and (7) have long exercised interest; see for instance, Bessel-Hagen [1], Lucas [8] and Tagiuri [10], and, for historical details, Dickson [3]. Important particular cases of them are [8],

the Fermat sequences

$$\begin{cases} \{u_n(3, 2)\} \equiv \{2^{n+1} - 1\} : 1 \ 3 \ 7 \ 15 \ 31 \ \cdots \\ \{v_n(3, 2)\} \equiv \{2^n + 1\} \quad : 2 \ 3 \ 5 \ 9 \ 17 \ \cdots \end{cases}$$

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and
the Pell sequences

$$\begin{cases} \{u_n(2, -1)\} & : 1 \ 2 \ 5 \ 12 \ 29 \ \cdots \\ \{v_n(2, -1)\} & : 2 \ 2 \ 6 \ 14 \ 34 \ \cdots \end{cases}$$

Of course,

(9) $$\begin{cases} u_n(1, -1) = f_n = h_n(1, 0) \\ v_n(1, -1) = a_n = h_n(2, -1). \end{cases}$$

Details of the properties of (8), including extensions to the field of complex numbers, may be found in a series of papers [4]–[7].

2. Generating functions. Recently, much attention has been focused on the generating functions for powers of numbers in these various sequences. Generating functions for the first-powers of the sequences (that is, for the actual sequences) are known, namely:

(10) $$\begin{cases} f_1(x) = \sum_{n=0}^{\infty} f_n x^n = (1 - x - x^2)^{-1} \\ a_1(x) = \sum_{n=0}^{\infty} a_n x^n = (2 - x)(1 - x - x^2)^{-1} \\ u_1(x) = \sum_{n=0}^{\infty} u_n x^n = (1 - px + qx^2)^{-1} \\ v_1(x) = \sum_{n=0}^{\infty} v_n x^n = (2 - px)(1 - px + qx^2)^{-1} \\ h_1(x) = \sum_{n=0}^{\infty} h_n x^n = (r + sx)(1 - x - x^2)^{-1}. \end{cases}$$

Designating any of the sequences (1), (2), (6), (7), (8) and, later, (12) by $\{s_n\}$ with corresponding first-power generating function $s_1(x) = \sum_{n=0}^{\infty} s_n x^n$, we define the generating function for the k-th powers (k integer) of s_n as

(11) $$s_k(x) = \sum_{n=0}^{\infty} (s_n)^k x^n \equiv \sum_{n=0}^{\infty} s_n^k x^n.$$

Using combinatorial method, Riordan [9] found a recurrence for $f_k(x)$; see (38) below. This was soon generalised by Carlitz [2], who added direct expressions for the generating functions of $u_k(x)$ and $v_k(x)$; see (34) and (35) below. Following Riordan's reasoning, the author obtained a recurrence for $h_k(x)$; see (36) below. It was easily verified that this agreed with Carlitz's results in the only two sequences which $\{u_n\}$ and $\{v_n\}$, and $\{h_n\}$, have in common, namely, $\{f_n\}$ and $\{a_n\}$; see (9) above. It was suggested by the referee that it would be worthwhile combining Carlitz's results and my own into one theory—I am very grateful for this suggestion and for the helpful advice offered with it.

3. Generalisations. Consider the sequence

(12) $$\{w_n\} \equiv \{w_n(a, b; p, q)\} : w_0 = a,\ w_1 = b,\ w_n = pw_{n-1} - qw_{n-2} \quad (n \geq 2)$$

for arbitrary integers a, b.

Clearly, this is an extension of (6) and (7), and contains (8) if we put $p = -q = 1$, $a = r$, $b = r + s$. That is,

(13) $$\begin{cases} w_n(1, p; p, q) = u_n(p, q) \\ w_n(2, p; p, q) = v_n(p, q) \\ w_n(r, r + s; 1, -1) = h_n(r, s) \\ w_n(1, 1; 1, -1) = f_n \\ w_n(2, 1; 1, -1) = a_n. \end{cases}$$

Only those properties of w_n essential to our objective stated in §2 will be introduced here. Other properties will be developed in a further paper.

Let α, β be the roots of $x^2 - px + q = 0$ so that

(14) $$\begin{cases} \alpha = \dfrac{p + \sqrt{p^2 - 4q}}{2},\ \beta = \dfrac{p - \sqrt{p^2 - 4q}}{2},\ \alpha + \beta = p,\ \alpha\beta = q, \\ \alpha - \beta = \sqrt{p^2 - 4q}. \end{cases}$$

Then $p = -q = 1$ implies $\alpha = \alpha_1$, $\beta = \beta_1$ in (5).

Using standard difference methods, we determine

(15) $$w_n = A\alpha^n + B\beta^n$$

with

(16) $$A = \frac{b - a\beta}{\alpha - \beta},\ B = \frac{a\alpha - b}{\alpha - \beta} \qquad (A + B = a).$$

Special cases are (13) the well-known results (see [8])

(17) $$u_n = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}$$

and

(18) $$v_n = \alpha^n + \beta^n.$$

Another special case is, of course, h_n (see [4]).

Allowing unrestricted integral values of n, notice that

(19) $$u_{-n} = -q^{-n+1} u_{n-2}$$

and

(20) $$v_{-n} = q^{-n} v_n,$$

whence $u_{-1} = 0$, $qv_{-1} = p$ (see (29)). Also note that $h_{-1} = s$, $f_{-1} = 0$, $a_{-1} = -1$.

From (15), (16), (17) and $\alpha + \beta = p$ follows the alternative form

$$w_n = au_n + (b - ap)u_{n-1}\,. \tag{21}$$

It is easy to confirm that

$$w_1(x) = \sum_{n=0}^{\infty} w_n x^n = \frac{a + (b - pa)x}{1 - px + qx^2}. \tag{22}$$

Our problem is to find an expression for

$$w_k(x) = \sum_{n=0}^{\infty} w_n^k x^n. \tag{23}$$

Now, a little calculation involving (22) shows that

$$(w_m - qw_{m-1}x)w_1(x) = [a + (b - pa)x] \sum_{n=0}^{\infty} w_{m+n} x^n. \tag{24}$$

Comparison of coefficients of x^n yields

$$aw_{m+n} + (b - pa)w_{m+n-1} = w_m w_n - qw_{m-1}w_{n-1}. \tag{25}$$

Putting firstly $m = n$ in (25) and then $m = n + 2$ and replacing n by $n - 1$, we obtain

$$aw_{2n} + (b - pa)w_{2n-1} = w_n^2 - qw_{n-1}^2 = w_{n+1}w_{n-1} - qw_n w_{n-2} \tag{26}$$

whence

$$\begin{aligned} w_{n+1}w_{n-1} - w_n^2 &= q(w_n w_{n-2} - w_{n-1}^2) \\ &= q^k(w_{n-k+1}w_{n-k-1} - w_{n-k}^2) \end{aligned}$$

by repeated application of (26), that is,

$$w_{n+1}w_{n-1} - w_n^2 = q^{n-1}e \tag{27}$$

where, by (12), (14) and (16),

$$\begin{cases} e = q(w_1 w_{-1} - w_0^2) \\ \quad = pab - qa^2 - b^2 \qquad (=(p^2 - 4q)AB) \end{cases} \tag{28}$$

in which

$$w_{-1} = A\alpha^{-1} + B\beta^{-1} = \frac{pa - b}{q}. \tag{29}$$

The number e characterises the various special sequences, for example, $e = -q$, $p^2 - 4q$, $r^2 - rs - s^2$ for $\{u_n\}$, $\{v_n\}$, $\{h_n\}$ respectively; hence, $e = 1$, 5 for $\{f_n\}$, $\{a_n\}$ respectively.

Following Carlitz [2], write

$$\begin{aligned} W(x, z) &= \sum_{k=1}^{\infty} (1 - \alpha^k x)(1 - \beta^k x) w_k(x) \frac{z^k}{k} \\ &= \sum_{k=1}^{\infty} (1 - \alpha^k x)(1 - \beta^k x) \frac{z^k}{k} \sum_{j=0}^{\infty} w_j^k x^j. \end{aligned} \tag{30}$$

For brevity, we do not reproduce the reasoning Carlitz used to expand his expression which corresponds to (30). Making the necessary adjustments as we proceed, we parallel his argument.

Essentially, all that is needed is the replacement in equation (4.2) of 2 in the first term by v_0, p in the second term by qv_{-1} and then v_0, qv_{-1} and $d = p^2 - 4q$ by w_0, qw_{-1} and e to get

$$\begin{aligned} W(x, z) = {} & -\log(1 - w_0 z) + x \log(1 - qw_{-1}z) \\ & + x \sum_{k=1}^{\infty} z^k \sum_{i=1}^{[k/2]} \frac{(-1)^i}{i} e^i a_{ki} w_{k-2i}(q^i x) \end{aligned} \tag{31}$$

where a_{ki} is defined by

$$(1 - px + qx^2)^{-i} = \sum_{k=2i}^{\infty} a_{ki} x^{k-2i}. \tag{32}$$

Comparing coefficients of z^k and using (12) and (29), we deduce that

$$(1 - \alpha^k x)(1 - \beta^k x) w_k(x) = a^k - x(pa - b)^k + kx \sum_{i=1}^{[k/2]} \frac{(-1)^i}{i} e^i a_{ki} w_{k-2i}(q^i x). \tag{33}$$

This readily reduces to the special cases given by Carlitz [2], namely,

$$(1 - \alpha^k x)(1 - \beta^k x) u_k(x) = 1 + kx \sum_{i=1}^{[k/2]} \frac{(-1)^i}{i} (-q)^i a_{ki} u_{k-2i}(q^i x), \tag{34}$$

and

$$(1 - \alpha^k x)(1 - \beta^k x) v_k(x) = 2^k - p^k x + kx \sum_{j=1}^{[k/2]} \frac{(-1)^j}{j} (p^2 - 4q)^j a_{kj} v_{k-2j}(q^j x), \tag{35}$$

A further special case is

$$\begin{aligned} (1 - \alpha^k x)(1 - \beta^k x) h_k(x) = {} & r^k - (-s)^k x \\ & + kx \sum_{i=1}^{[k/2]} \frac{(-1)^i}{i} (r^2 - rs - s^2)^i a_{ki} h_{k-2i}((-1)^i x), \end{aligned} \tag{36}$$

where in (36) a_{ki} is given by

$$(1 - x + x^2)^{-i} = \sum_{k=2i}^{\infty} a_{ki} x^{k-2i}. \tag{37}$$

More particularly, from (34) or (36) we derive Riordan's result [9], namely,

(38) $$(1-\alpha_1^k x)(1-\beta_1^k x)f_k(x) = 1 + kx\sum_{j=1}^{[k/2]}\frac{(-1)^j}{j}a_{kj}f_{k-2j}((-1)^j x),$$

while from (35) or (36) we derive

(39) $$(1-\alpha_1^k x)(1-\beta_1^k x)a_k(x) = 2^k - x + kx\sum_{j=1}^{[k/2]}\frac{(-1)^j}{j}5^j a_{kj}a_{k-2j}((-1)^j x).$$

Note that

(40) $$(1-\alpha^k x)(1-\beta^k x) = 1 - v_k x + q^k x^2$$

and that

(41) $$(1-\alpha_1^k x)(1-\beta_1^k x) = 1 - a_k x + (-1)^k x^2.$$

4. **The coefficients of** $w_k(x)$. Without much difficulty, it may be shown that

(42) $$w_k(x) = \sum_{n=0}^{\infty}(A\alpha^n + B\beta^n)^k x^n = \sum_{s=0}^{k}\binom{k}{s}A^{k-s}B^s(1-\alpha^{k-s}\beta^s x)^{-1}.$$

Define, as in [2],

(43) $$D_k(x) = \prod_{s=0}^{k}(1-\alpha^{k-s}\beta^s x)$$

and write

(44) $$w_k(x) \quad \frac{W_k(x)}{D_k(x)}$$

where $W_k(x)$ is a polynomial of degree $\le k$ ($k \ge 1$).
Using (15), we deduce

(45) $$w_{k+1}(x) = Aw_k(\alpha x) + Bw_k(\beta x).$$

Then

(46) $$\frac{W_{k+1}(x)}{D_{k+1}(x)} = A\frac{W_k(\alpha x)}{D_k(\alpha x)} + B\frac{W_k(\beta x)}{D_k(\beta x)}$$

whence, by (43),

(47) $$W_{k+1}(x) = A(1-\beta^{k+1}x)W_k(\alpha x) + B(1-\alpha^{k+1}x)W_k(\beta x).$$

Put

(48) $$W_k(x) = \sum_{s=0}^{k} W_{ks}x^s$$

and it follows from (47) on equating coefficients of x^j that

(49) $$W_{k+1,j} = w_j W_{kj} - q^{k+1}w_{-(k-j+2)}W_{k,j-1}$$

which, using (19) and (20), reduces to the special cases in [2].
Expand (43) as in [2] to obtain

(50) $$D_k(x) = \sum_{j=0}^{k}(-1)^j q^{\frac{1}{2}j(j-1)}\begin{Bmatrix}k\\j\end{Bmatrix}x^j$$

where

(51) $$\begin{Bmatrix}k\\j\end{Bmatrix} = \frac{u_k u_{k-1}\cdots u_{k-j+1}}{u_0 u_1\cdots u_{j-1}},\quad \begin{Bmatrix}k\\0\end{Bmatrix} = 1.$$

Further, from (43),

(52) $$D_k(q^{-k}x^{-1}) = (-1)^{k+1}q^{-\frac{1}{2}k(k+1)}x^{-k-1}D_k(x),$$

and from (42),

(53) $$w_k(q^{-k}x^{-1}) = a^k - w_k^*(x)$$

in which, by (16),

$$a^k = (A+B)^k = \sum_{s=0}^{k}\binom{k}{s}A^{k-s}B^s$$

and

(54) $$w_k^*(x) = \sum_{s=0}^{k}\binom{k}{s}A^{k-s}B^s(1-\alpha^s\beta^{k-s}x)^{-1} = \sum_{n=0}^{\infty}(q^n w_{-n})^k x^n.$$

Along with (53), the first of these expressions gives

(55) $$w_k(x) + w_k(q^{-k}x^{-1}) = a^k + (\alpha-\beta)x\sum_{s=0}^{k}\frac{\binom{k}{s}(qAB)^s A^{k-2s}u_{k-1-2s}}{1-q^s v_{k-2s}x + q^k x^2}$$

which may be cast in the alternative form

(56) $$w_k(x) + w_k(q^{-k}x^{-1}) = a^k + (2b-pa)x\sum_{t=0}^{[k/2]}\binom{k}{t}(qAB)^t\frac{A^{k-2t}-B^{k-2t}}{A-B}\cdot\frac{u_{k-1-2t}}{1-q^t v_{k-2t}x + q^k x^2}$$

using (19) and $(\alpha-\beta)(A-B) = 2b - pa$.
At once, the expression for $v_k(x) + v_k(q^{-k}x^{-1})$ found in [2] follows since $2b - pa = 0$ then. Likewise, an expression for $u_k(x) + u_k(q^{-k}x^{-1})$ may be obtained. (Here the factor $2b - pa$ reduces to p.) Also, from (42), we may write down an expression for the function $w_k(q^{-k}x^{-1})/w_k(x)$ leading to the form for $u_k(q^{-k}x^{-1})/u_k(x)$ described in [2], and to the form for $v_k(q^{-k}x^{-1})/v_k(x)$.
Combining (44), (52) and (53), we obtain

(57) $$(-1)^{k+1}q^{\frac{1}{2}k(k+1)}x^{k+1}W_k(q^{-k}x^{-1}) = a^k D_k(x) - W_k^*(x),$$

where $W_k^*(x) = w_k^*(x)D_k(x) = \sum_{s=0}^{k} W_{ks}^* x^s$ bears the same relationship to $W_k(x)$ as $w_k^*(x)$ does to $w_k(x)$ ((44) and (48)). Equate coefficients of x^j. Then

(58) $$(-1)^{k+1} q^{\frac{1}{2}k(k+1)-k(k-j+1)} W_{k,k-j+1} = (-1)^j a^k q^{\frac{1}{2}j(j-1)} \begin{Bmatrix} k \\ j \end{Bmatrix} - W_{kj}^* .$$

Next it follows from (44), (48) and (50) that

(59) $$W_{kj} = \sum_{r=0}^{j} (-1)^r q^{\frac{1}{2}r(r-1)} \begin{Bmatrix} k \\ r \end{Bmatrix} w_{j-r}^k .$$

Observe that in (59) $j > k$ implies $W_{kj} = 0$ by (48). Hence (49) leads to

(60) $$W_{k+1,k+1} = -q^{k+1} w_{-1} W_{kk}$$

so that, by repeated application of (60),

(61) $$W_{kk} = (-1)^k q^{\frac{1}{2}k(k-1)} w_{-1}^k .$$

Appropriate substitutions in §4 for $u_k(x)$ and $v_k(x)$, namely (by (13), (14) and (15)), $a = 1$, $b = p$, $A = \alpha(\alpha - \beta)^{-1}$, $B = -\beta(\alpha - \beta)^{-1}$ and $a = 2$, $b = p$, $A = B = 1$ respectively, yield the specialised expressions given in [2], as expected; for example, $U_{kk} = 0$, since $u_{-1} = 0$, and $V_{kk} = (-1)^k p^k q^{\frac{1}{2}k(k-1)}$.

Corresponding results likewise follow for $h_k(x)$ if we put $a = r$, $b = r + s$, $p = 1$, $q = -1$, $A = (r + s - r\beta_1)/\sqrt{5}$ and $B = (r\alpha_1 - r - s)/\sqrt{5}$. Hence, for example, $H_{kk} = (-1)^{\frac{1}{2}k(k+1)} (-s)^k$ leading to $F_{kk} = 0$ $(s = 0)$ and $A_{kk} = (-1)^{\frac{1}{2}k(k+1)}$ $(s = -1)$, the further specialisations of U_{kk} and V_{kk}.

Furthermore, put

(62) $$\theta_j(x) = \sum_{k=j}^{\infty} W_{kj} x^{k-j}.$$

Use of (62) and substitution from the recurrence (49) give

$$\begin{aligned}
\alpha B \theta_{j-1}(\alpha x) + \beta A \theta_{j-1}(\beta x) &= \sum_{k=j-1}^{\infty} \{\alpha B W_{k,j-1}(\alpha x)^{k-j+1} + \beta A W_{k,j-1}(\beta x)^{k-j+1}\} \\
&= \sum_{k=j-1}^{\infty} (A\beta^{k-j+2} + B\alpha^{k-j+2}) W_{k,j-1} x^{k-j+1} \\
&= \sum_{k=j-1}^{\infty} q^{k-j+2} w_{-(k-j+2)} W_{k,j-1} x^{k-j+1} \\
&= q^{-j+1} \sum_{k=j-1}^{\infty} (w_j W_{kj} - W_{k+1,j}) x^{k-j+1} \\
&= q^{-j+1}(w_j x \theta_j(x) - \theta_j(x)).
\end{aligned}$$

Therefore

(63) $$q^{j-1}(\alpha B \theta_{j-1}(\alpha x) + \beta A \theta_{j-1}(\beta x)) = (w_j x - 1)\theta_j(x),$$

which allows us to compute $\theta_j(x)$ by recurrence.

Since by (12) and (59), $W_{k0} = w_0^k = a^k$, we calculate that

(64) $$\begin{cases}
\theta_0(x) = \sum_{k=0}^{\infty} W_{k0} x^k = W_{00} + W_{10}x + W_{20}x^2 + \cdots = \dfrac{1}{1 - w_0 x} = \dfrac{1}{1 - ax} \\
\theta_1(x) = \sum_{k=1}^{\infty} W_{k1} x^{k-1} = W_{11} + W_{21}x + W_{31}x^2 + \cdots \\
\qquad = \dfrac{b - pa + a^2 qx}{(1 - w_0\alpha x)(1 - w_0\beta x)(1 - w_1 x)}.
\end{cases}$$

(The presence of the term $b - pa$ in the numerator of $\theta_1(x)$ (64) explains why there is no constant in the numerator of the Carlitz function $\phi_1(x)$ (for which $a = 1$, $b = p$).)

Generally, $\theta_j(x)$ is a rational function of x with denominator

$$D_j(w_0 x) D_{j-1}(w_1 x) \cdots D_0(w_j x).$$

Suitable substitutions produce the specialisations listed in [2], where we note, among other things, the link with work on q-Bernoulli and Eulerian numbers.

Throughout, we have assumed that $p^2 \neq 4q$. Suppose now that $p^2 = 4q$ whence $\alpha = \beta = p/2$. The simplest such degenerate case occurs when $p = 2$, $q = 1$ $(\alpha = \beta = 1)$ for which (7), for instance, reduces to the trival sequence v_n (2, 1): 2, 2, , 2, $\cdots$.

Using (16) and the Carlitz result [2] that $u_n \to (n + 1)(p/2)^n$, we obtain the degenerate forms

(15)′ $$w_n \to \left(\frac{p}{2}\right)^n \left\{\frac{2bn}{p} - a(n - 1)\right\}$$

(23)′ $$w_k(x) = \sum_{n=0}^{\infty} \left[\left(\frac{p}{2}\right)^n \left\{\frac{2bn}{p} - a(n - 1)\right\}\right]^k x^n$$

and

(33)′ $$\left(1 - \left(\frac{p}{2}\right)^k x\right)^2 w_k(x) = a^k - x(pa - b)^k + 2x\left(\frac{p}{2}\right)^k \sum_{r=1}^{[k/2]} \binom{k}{2r}\left(a - \frac{2b}{p}\right)^{2r} w_{k-2r}\left(\frac{p^{2r}}{2^{2r}} x\right)$$

where in (33)′ we have used

(40)′ $$(1 - \alpha^k x)(1 - \beta^k x) = \left(1 - \left(\frac{p}{2}\right)^k x\right)^2.$$

Results (15)′, (23)′ and (33)′ extend corresponding results in [2]. Note the degenerate form of (28), namely, $e = -(pa/2 - b)^2$. But observe that $\{h_n\}$ can have no such degenerate cases since $p^2 - 4q$ then has the value 5 ($\neq 0$).

Finally, we give some of the simplest numerical cases involving $w_k(x)$.

Computation in (42) gives

$$w_2(x) = \frac{a^2 + \{b^2 - a^2(p^2 - q)\}x + q(b - pa)^2x^2}{(1 - \alpha^2x)(1 - \alpha\beta x)(1 - \beta^2x)}. \tag{65}$$

From (22) and (65) it is easy to write down expressions for $w_1(q^{-1}x^{-1})$ and $w_2(q^{-2}x^{-1})$. Explicit forms of the above functions are then

$$\begin{cases} w_1(x) + w_1(q^{-1}x^{-1}) = a + \dfrac{(2b - pa)x}{1 - v_1x + qx^2}, \\[2mm] w_2(x) + w_2(q^{-2}x^{-1}) = a^2 + \dfrac{(2b - pa)pax}{1 - v_2x + q^2x^2}, \\[2mm] \dfrac{w_1(q^{-1}x^{-1})}{w_1(x)} = \dfrac{(b - pa + qax)x}{a + (b - pa)x}, \\[2mm] \dfrac{w_2(q^{-2}x^{-1})}{w_2(x)} = -\dfrac{[(b - pa)^2 + q\{b^2 - a^2(p^2 - q)\}x + a^2q^3x^2]x}{a^2 + \{b^2 - a^2(p^2 - q)\}x + q(b - pa)^2x^2}, \end{cases} \tag{66}$$

whence, for instance, results (6.11) and (7.7) of [2] for $k = 1, 2$ follow immediately. It is worth noting that the right-hand side of the second formula in (66) involves a simplification (namely, division of numerator and denominator by the factor $1 - qx$) which is generally not typical of the expression for $w_k(x) + w_k(q^{-k}x^{-1})$.

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AN ITERATED LOGARITHM LAW FOR LOCAL TIME

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1. **Introduction.** P. Erdös posed to the author the problem of finding strong limit laws for the maximal multiplicity in n steps of a simple random walk in one dimension (as $n \to \infty$). The analogous problem for dimension two and higher had been solved by Erdös and Taylor [3]. We consider here the continuous time analogue, namely we derive limit laws for the maximum (over the space variable) of the local time of Brownian motion. This problem can be very well treated with the help of two recent papers by Knight [4] and Ray [7]. The corresponding result for random walks (Theorem 3) is stated without proof even though its proof is not a mere "translation to the discrete case" of the proofs of Theorem 1 and 2.

Throughout this note we use the following notation. $X(t, \omega)$ is a standard Brownian motion ($X(0, \omega) = 0$ with probability 1, $E\,X(t, \omega) = 0$, $E\,X^2(t, \omega) = t$) and $f(x, t) = f(x, t, \omega)$ is its local time, i.e. the continuous (jointly in x and t) function which satisfies for each measurable set E

$$\int_E f(x, t, \omega)\, dx = \int_0^t \chi_E(X(\tau, \omega))\, d\tau \tag{1.1}$$

where $\chi_E(\cdot)$ is the characteristic function of the set E. The existence of a continuous local time was shown by Trotter [8]. Finally

$$f(t) = f(t, \omega) = \sup_x f(x, t, \omega).$$

When convenient we shall not write the argument ω.

Theorem 1.

$$\limsup_{t\to\infty} \frac{f(t, \omega)}{\sqrt{t \log\log t}} = \limsup_{t\to\infty} \frac{f(0, t, \omega)}{\sqrt{t \log\log t}} = \sqrt{2}$$

with probability 1.

Theorem 2. *There exists a constant γ such that*

$$\frac{q_0}{2} \leq \gamma \leq \frac{q_0^2}{\sqrt{2}}$$

and

$$\liminf_{t\to\infty} \frac{f(t, \omega)\sqrt{\log\log t}}{\sqrt{t}} = \gamma$$

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