Ulam Quaterly – Volume 3, Number 1, 1994

Applications of Modified Pell Numbers to Representations

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1 Background

Define the sequence $\{q_n\}$ for all integers n by the recurrence

$$q_{n+2} = 2q_{n+1} + q_n \qquad (q_0 = 1, q_1 = 1), \tag{1.1}$$

and the associated *Pell sequence* $\{P_n\}$ for all integers by the recurrence

$$P_{n+2} = 2P_{n+1} + P_n \qquad (P_0 = 0, P_1 = 1). \tag{1.2}$$

For a few basic relationships connecting P_n and q_n, see, for instance, [1] and [14].

Closely related to $\{q_n\}$ is the *Pell-Lucas sequence* $\{Q_n\}$ which has been extensively analyzed in a series of publications (e.g. [8], [11]), principally in relation to $\{P_n\}$, but also in its own right. In fact, $Q_n = 2q_n$. Consequently, the known properties of $\{Q_n\}$ are easily transferable to $\{q_n\}$.

Here, we are concerned only with $\{q_n\}$ and more especially, with the problem of representing any integer by sums of the numbers q_n.

For this purpose, we require the extension of the sequence $\{q_n\}$ to negative values of n. Inevitably, a study of $\{q_n\}$ involves familiarity with P_n. Using (1.1) and (1.2), we readily derive the following tabulation:

n	$\cdots$	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	$\cdots$
q_n	$\cdots$	99	-41	17	-7	3	-1	1	1	3	7	17	41	99	$\cdots$
P_n	$\cdots$	-70	29	-12	5	-2	1	0	1	2	5	12	29	70	$\cdots$

Fig. 1: Values of $q_n, P_n \quad (-6 \le n \le 6)$.

Clearly

$$q_{-n} = (-1)^n q_n \tag{1.3}$$

and

$$P_{-n} = (-1)^{n+1} P_n. \tag{1.4}$$

Also,

$$q_n = P_n + P_{n-1} = P_{n+1} - P_n = \frac{Pn + 1 + P_{n-1}}{2} \tag{1.5}$$

while

$$2P_n = q_n + q_{n-1} = q_{n+1} - q_n = \frac{q_{n+1} + q_{n-1}}{2}. \tag{1.6}$$

Explicit *Binet forms* for q_n and P_n are

$$q_n = \frac{\alpha^n + \beta^n}{2}, \tag{1.7}$$

and

$$P_n = \frac{\alpha^n + \beta^n}{\alpha - \beta} \tag{1.8}$$

where α,β are the roots of the characteristic equation $x^2 - 2x + 1 = 0$ of the recurrence relations (1.1) and 1.2), i.e.,

$$\begin{cases} \alpha = 1 + \sqrt{2} \\ \beta = 1 - \sqrt{2} \end{cases} \quad \text{so } \alpha + \beta = 2, \ \ \alpha - \beta = 2\sqrt{2} \ , \alpha\beta = -1. \tag{1.9}$$

With negative subscripts, (1.6) becomes

$$2P_{-n} = q_{-n} + q_{-n-1} = q_{-n+1} - q_{-n} = \frac{q_{-n+1} + q_{-n-1}}{2}. \tag{1.10}$$

The Name of the Sequence $\{q_n\}$ References to the numbers q_n in Sloane [13] are associated with Thébault [14] in 1949, and earlier in 1916 with an unspecified writer in [12]. Both P_n and q_n were designated *Eudoxus numbers* by Budden [2] in 1969, though I have no independent information of this claim.

Lucas [10] makes no specific reference to q_n so far as I am aware, but he does use the numbers $2q_n$ which he designates $V_n - a$ generic symbol of his, and couples them with P_n (his U_n) as *Suites de Pell.* Because of these historical origins, I have called $2q_n$, which I label Q_n, the *Pell-Lucas numbers.* (Perhaps, then, p_n might be named the "quasi Pell-Lucas numbers"?)

All things considered, I accept the nomenclature of Bruckman [1], who refers to q_n as *modified Pell numbers*, as suitably apt, and to this I have adhered.

2 Properties of $\{q_n\}$

Some properties of $\{q_n\}$ *per se* which are relevant to our study include the *Simson formula*

$$q_{n+1}q_{n-1} - q_n^2 = 2(-1)^{n+1} \tag{2.1}$$

and the summations

$$\sum_{i=1}^{n} q_{2i} = \frac{q_{2n+1}-1}{2} \tag{2.2}$$

$$\sum_{i=1}^{n} q_{2i-1} = \frac{q_{2n}-1}{2} \tag{2.3}$$

$$\sum_{i=1}^{n} q_i = \frac{q_{n+1}+q_n}{2} - 1 = P_{n+1} - 1 \quad \text{by (1.6)} \tag{2.4}$$

$$\sum_{i=0}^{n-1} (-1)^i q_{-i} = \frac{(-1)^n(q_{-n}-q_{-n+1})}{2} = (-1)^{n+1}P_{-n} \quad \text{by (1.10)} \tag{2.5}$$

$$\begin{aligned} q_n &= 2(q_{n-1}+q_{n-3}+\cdots+q_4+q_2)+1 \quad n \text{ odd } (q_0 = 1) \\ &= 2(q_{n-1}+q_{n-3}+\cdots+q_3+q_1)+1 \quad n \text{ even (c.f. (2.2), (2.3))} \end{aligned} \tag{2.6}$$

$$\sum_{i=1}^{n} P_i = \frac{-q_{n+1}-1}{2} \tag{2.7}$$

Checking on the validity of these results is left to the vigilance of the reader. Moreover,

$$\sum_{i=0}^{n} q_{-2i} = \frac{-q_{-2n-1}+1}{2} \tag{2.8}$$

$$\sum_{i=1}^{n} q_{-2i+1} = \frac{-q_{-2n}+1}{2}. \tag{2.9}$$

Presence of 0 as the starting point in (2.8) is to be especially noted.

3 Representation of Positve Integers By $\{q_n\}$, $n \geq 0$

A. Minimal Representation

Theorem 3.1 *The representation of a positive integer $N > 0$ in the form*

$$N = \sum_{i=1}^{\infty} \alpha_i q_i \quad (\alpha_i = 0, 1 \text{ or } 2) \tag{3.1}$$

where

$$\alpha_i = 2 \Rightarrow \alpha_{i-1} = 0 \tag{3.2}$$

is unique *and* minimal.

Proof This is similar to that for $\{P_n\}$ in [7], with appropriate adjustments. E.g., use (2.6). Alternatively, consult the proof given in outline after Theorem 2.

By the phrase minimal representation we mean the representation with the least number of numbers q_i occurring in the sum (3.1), subject to the proviso (3.2). Such a representation may be called a *Zeckendorf representation* [7]. Figure 4 gives the minimal representations of N by $\{q_n\} : 1 \leq N \leq 50$. Absence of q_0 in (3.1) ought to be compared with the situation in (3.3) for Theorem 3.2.

A "Greedy" Algorithm. Remarks similar to those in [7] regarding a "greedy algorithm" for $\{P_n\}$ are also applicable in the case of $\{q_n\}$. As an illustration, from Figure 1 we have

$$350 - q_7 = 111,\ 111 - q_5 = 70,\ 70 - q_5 = 29,\ 29 - q_4 = 12,\ 12 - q_3 = 5,$$
$$5 - q_2 = 2,\ 2 - q_1 = 1,\ 1 - q_1 = 0, \text{ so that}$$
$$350 = q_7 + 2q_5 + q_4 + q_3 + q_2 + 2q_1.$$

B. Maximal Representation. By a *maximal representation*, we mean the greatest number of q_i occurring in the sum (3.3) below, in the context of the criteria (3.4).

Introduction of q_0. For maximality, we must introduce $q_0 = 1$. Otherwise 2, for example, could not be expressed maximally ($2 = 1.q_0 + 1.q_1$). Lucas numbers L_n similarly require the use of $L_0 = 2$ in the theory of maximal representations [15]. But, for uniqueness, we define $1 = q_i$. Furthermore, in the ensuing MinMax theory (section 4) we require $q_i = 1$ (not $q_0 = 1$) since q_0 is absent in considerations of minimality.

Pertinent to our usage is the fact that in section 4 $2q_0 (= 2)$ is never used, only $1q_0$.

Thus, the special purpose of q_0 for maximality is to fill in the gap ($2 = 3 - 1$) between $q_1 = 1$ and $q_2 = 3$.

Theorem 3.2 *Every positive integer $N > 0$ has a unique representation in the form*

$$N = \sum_{i=0}^{m} \beta_i q_i \quad (\beta_i = 0, 1, \text{ or } 2) \tag{3.3}$$

where

$$\begin{cases} \beta_i = 0 & \Rightarrow \quad \beta_{i-1} = 2 \\ \beta_m = 1 & \text{or} \quad 2. \end{cases} \tag{3.4}$$

Proof of this Theorem has a number of lemma as a prologue:

First consider the sequence of coefficients of q_i in (3.3) of *length* $k \geq 1$, namely,

$$(\beta_0, \beta_1, \beta_2, \cdots, \beta_{k-1}) \tag{3.5}$$

subject to the criteria (3.4).

$$\begin{aligned} \text{Write } S_k &\equiv \text{the number of sequences (3.5) with (3.4) attached} \\ r_k &\equiv \text{the range of values of } N \text{ for } S_k \\ N_k^{min} &\equiv \text{the minimum number of } r_k \\ N_k^{max} &\equiv \text{the maximum number of } r_k \\ I_k &\equiv \text{the number of integers in } r_k. \end{aligned}$$

Data relevant to these symbols for the number N in (3.3) are:

k	S_k	r_k	N_k^{min}	N_k^{max}	I_k
1	S_1	1			
2	S_2	$2, 3$	$2P_1$	$2P_2 - 1$	$2q_1$
3	S_3	$4, \cdots, 9$	$2P_2$	$2P_3 - 1$	$2q_2$
4	S_4	$10, \cdots, 23$	$2P_3$	$2P_4 - 1$	$2q_3$
5	S_5	$24, \cdots, 57$	$2P_4$	$2P_5 - 1$	$2q_4$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
k	S_k	$2P_{k-1} \cdots, 2P_k - 1$	$2P_{k-1}$	$2P_k - 1$	$2q_{k-1}$

Fig. 2: Basic Data for Theorem 2

Let us elaborate a little on this information.

Lemma 3.1 $2P_{k-1} \leq N \leq 2P_k - 1 \quad (k \geq 2)$.

Proof: For a sequence (3.5) of length $k \geq 2$, the maximum number N_k^{max} which it can represent is given by

$$\underbrace{(1, 2, 2, 2, \cdots, 2)}_{k \text{ digits}}$$

corresponding to

$$\begin{aligned} N_k^{max} &= 1.q_0 + 2\sum_{i=1}^{k-1} q_i \\ &= 1 + 2\left[\frac{q_k + q_{k-1}}{2} - 1\right] \quad \text{by } (1.1), (2.4) \\ &= q_k + q_{k-1} - 1 \\ &= 2P_k - 1 \quad \text{by } (1.6). \end{aligned} \tag{3.6}$$

The minimum number N_k^{min} for a sequence of length k is obtained by the following reasoning:

After the number in (3.6), the next number in order is $(2P_k - 1) + 1 = 2P_k$ which occurs as the first number in the (next) sequence of length $k+1$. Accordingly, the minimum number in the sequence of length k is derived from $2P_k$ by replacing k by $k-1$, i.e., $N_k^{min} = 2P_{k-1}$. Hence

$$\underbrace{2P_{k-1} \leq Nleq 2P_k - 1}_{r_k} \quad (k \geq 2).$$

Corollary 3.1 *When $k - 1$, we are left merely with the number* 1.

Lemma 3.2 $S_k = 2q_{k-1} \quad (k \geq 2)$.

Proof From Lemma 3.1, the number of numbers I_k included in the range r_k, which is the same as the number of sequences S_k, is

$$\begin{aligned} S_k &= (2P_k - 1) - \{(2Pp_{k-1} - 1\} = I_k \\ &= 2(P_k - P_{k-1}) \\ &= 2q_{k-1} \quad \text{by } (1.5). \end{aligned}$$

Lemma 3.3 *k is uniquely determined by $N(\beta_k \neq 0)$.*

This is obvious from Figure 2. As an example, consider

$$\begin{aligned} N &= 1000 \\ &\Rightarrow 816 \leq 1000 \leq 1969 \quad (= 1970 - 1) \\ &\Rightarrow 2P_8 \leq 1000 \leq 2P_9 - 1 \\ &\Rightarrow k = 9 \end{aligned}$$

(with $r_9 = 1969 - 815 = (1970 - 1) - (816 - 1) = 1154 = 2q_s$).

Lemma 3.4 *β_k ($\neq 0$) is uniquely determined by N.*

This is clearly so, since, from (3.3), $N - \beta_k q_k$ is a specific number.

For instance, $N = 50 \Rightarrow \begin{cases} N - 2q_4 & = & 1+2+6+7 & = & 16 & (\beta_4 = 2) \\ N - q_4 & = & 1+2+6+7+17 & = & 33 & (\beta_4 = 1). \end{cases}$

Proof of Theorem 3.1 Assembling all the evidence which has followed the enunciation of Theorem 3.1 (namely, Figure 2 and Lemmas 3.1-3.4), we are led to accept the validity of the Theorem.

Observation We remark that the numbers N_k^{max} are identical with the *subsidiary MinMax numbers* N_{k-1} ($k \geq 1$) for Pell numbers [5]. The reason for this is that, by (3.6), $N_k^{max} = 2P_k - 1 = N_{k-1}$ by [5].

Alternative Proof (Outline) of Theorem 3.1 This may be set out to parallel the treatment in Theorem 3.1 by using similar techniques. Firstly, consider the sequence of length k in (3.7)

$$(\alpha_1, \alpha_2, \cdots, \alpha_k).$$

The minimum number represented by this sequence is given by

$$\underbrace{(0, 0, 0, \cdots, 1)}_{k \text{ digits}}$$

i.e., q_k.

The maximum number is given by

$$(2, 0, 2, 0, \cdots, 2, 0) \text{ or } (0, 2, 0, 2, \cdots 0, 2)$$

depending on the parity of k, i.e., $q_{2k+1} - 1$ or $q_{2k} - 1$ from (2.2) and (2.3) respectively. Eventually, we may assert (replacing S_k in Lemma 3.2 by s_k):

Lemma 3.5 $q_k \leq N \leq q_{k+1} - 1$

Lemma 3.6 $s_k = 2P_k$.

Lemma 3.7 *k is uniquely determined by N ($\alpha_k \neq 0$).*

Lemma 3.8 α_k $(\neq 0)$ *is uniquely determined by* N.

Gathering together these results, we establish the validity of Theorem 3.1.

Remarks

(i) The ranges in the above treatment for Theorem 3.1 are:

$1, 2;\ \ 3, \cdots, 6;\ \ 7, \cdots, 16;\ \ 17, \cdots, 40;\ \ \cdots.$

(ii) Figure 5 gives representations of N for $1 \leq N \leq 50$.

(iii) $\begin{cases} \text{In Figure 4, 2 is always preceded by 0 (except in the first column).} \\ \text{In Figure 5, 0 is always preceded by 2 (except in the last column).} \end{cases}$

That is, there is a type of **duality** in the enunciations of Theorems 3.1 and 3.2 -cf. the criteria (3.2) and (3.4). In a similar context for Fibonacci numbers, Brown [9] refers to a "Dual Zeckendorf Theorem".

4 The MinMax Sequence $\{Q_n\}$.

Comparing the data in Figures 4 and 5 we discern that, for certain values of N, the minimal and maximal representations are identical. These may be designated as the *MinMax numbers* for $\{q_n\}$. But what are these numbers? Inspection of Figures 4 and 5, buttressed by an argument paralleling that used in [5] relating to $\{P_n\}$, establishes that the *MinMax sequence* $\{Q_n\}$ consists of those numbers whose representations (3.1) and (3.3) have coefficients α_i and β_i of q_i which are all unity, i.e. for which $\alpha_i = \beta_i = 1$, where $i = 1, 2, 3, \cdots$.

Write

$$Q_n = \sum_{i=1}^{n} q_i. \tag{4.1}$$

Take

$$Q_0 = 0. \tag{4.2}$$

Then by (2.4) and (1.8),

$$Q_n = P_{n+1} - 1 \tag{4.3}$$

that is,

$$Q_n = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} - 1 \quad (\textit{Binet}\ \text{form}) \tag{4.4}$$

Assembling this information in order, we find that the first few members of the MinMax sequence $\{Q_n\}$ for $\{q_n\}$ are:

$$\begin{array}{c|ccccccccccc} n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \cdots \\ \hline Q_n & 1 & 4 & 11 & 28 & 69 & 168 & 407 & 984 & 2377 & 5740 & \cdots \end{array} \tag{4.5}$$

Using (4.3) with (1.2), or (4.4), one discovers the recursion

$$Q_{n+2} = 2Q_n + Q_n + 2, \tag{4.6}$$

the *Simson formula* analogue for $\{Q_n\}$

$$Q_{n+1}Q_{n-1} - Q_n^2 = (-1)^{n+1} - 2P_n, \tag{4.7}$$

and

$$\sum_{i=1}^{n} Q_i = \frac{q_{n+2}-1}{2} - (n+1). \tag{4.8}$$

Generating function for $\{Q_n\}$ is

$$(1+x)(1-3x+x^2+x^3)^{-1} = \sum_{n=1}^{\infty} Q_n x^{n-1}. \tag{4.9}$$

Other relationships of interest include

$$Q_n - Q_{n-1} = q_n \tag{4.10}$$

$$Q_n + Q_{n+1} = q_{n+2} - 2 \tag{4.11}$$

$$Q_n - Q_{n-2} = 2P_n \tag{4.12}$$

$$Q_n + Q_{n+2} = 2q_{n+2} - 2 \tag{4.13}$$

$$Q_n^2 - Q_{n-1}^2 = q_n(q_{n+1} - 2) \tag{4.14}$$

$$Q_n^2 - Q_{n-2}^2 = 4P_n(q_n - 1) \tag{4.15}$$

$$Q_n^2 + Q_{n+1}^2 = P_{2n+3} - 2q_{n+2} + 2. \tag{4.16}$$

Discoveries of further properties of $\{Q_n\}$, e.g. divisibility properties, may be unearthed *ad infinitum, ad nauseam* according to one's fortitude and motivation.

Worthy of recording is the following relationship, where $\{N_n\}$ is the subsidiary MinMax sequence of $\{M_n\}$ - see [5] -:

$$\begin{aligned} 2Q_n &= 2P_{n+1} - 2 && \text{from (4.3)} \\ &= (2P_{n+1} - 1) - 1 \\ &= N_n - 1 && \text{from [5]} \end{aligned}$$

5 The Subsidiary MinMax Sequence $\{R_n\}$

Suppose we introduce the *subsidiary MinMax sequence* $\{R_n\}$ of $\{Q_n\}$ defined recursively by

$$R_n = Q_{n+1} + Q_{n-1} \quad (R_0 = 0, Q_{-1} = -1). \tag{5.1}$$

Values of R_n are, from (4.5) thus:

$$\begin{array}{c|ccccccccccc} n & = & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots & \\ \hline R_n & = & 4 & 12 & 32 & 80 & 196 & 476 & 1152 & 2784 & \cdots & \\ & = & 4(1 & 3 & 8 & 20 & 49 & 119 & 288 & 696 & \cdots) & \\ & = & 4M_n. \end{array} \tag{5.2}$$

where $\{M_n\}$ is the *MinMax sequence* for $\{P_n\}$ examined in [5].

No undue surprise should emanate from this fact, for, with the notation and results of [5],

$$\begin{aligned} R_n &= (M_{n+1} + M_n) + (M_{n-1} + M_{n-2}) = Q_{n+1} + Q_{n-1} \\ &= (2M_n + M_{n-1} + 1) + M_n + M_{n-1} + (M_n - 2M_{n-1} - 1) \\ &= 4M_n \end{aligned}$$

Properties of $\{M\}$ enumerated in [5] may accordingly be transferred to $\{R_n\}$, provided the appropriate modifications are made. Thus, for instance, we have the recurrence relation

$$R_{n+2} = 2R_{n+1} + R_n + 4 \quad (R_0 = 0), \tag{5.3}$$

$$4(1 - 3x + x^2 + x^3)^{-1} = \sum_{n=1}^{\infty} R_n x^{n-1} \quad \text{(generating function)}, \tag{5.4}$$

$$R_{n+1}R_{n-1} - R_n^2 = 8((-1)^n - q_n) \quad \text{(Simson's formula)}, \tag{5.5}$$

and

$$R_n = \alpha^{n+1} + \beta^{n+1} - 2 \quad \text{(Binetform)}. \tag{5.6}$$

Divisibility attributes of $\{M_n\}$ mentioned in [5] automatically carry over to $\{R_n\}$. Of course, the quality of primeness is absent.

Neither the sequence $\{R_n\}$ nor the sequence $\{q_n\}$ is listed in [13].

References to many seminal contributions to representations involving Fibonacci and Lucas numbers (e.g. these by Zeckendorf and Lekkerkerker) are to be found in [5].

6 Negatively Subscripted Q_n.

Though it is not meaningful in the context of representations to extend $\{Q_n\}$ through negative values of n, let us nonetheless complete the mathematical theory by considering the numbers Q_{-n}, $n > 0$. Imagine that the recursive statement (4.6) is applied to negative subscripts. Then the following table results:

n	$\cdots$	9	8	7	6	5	4	3	2	1	0	1	$\cdots$
Q_{-n}	$\cdots$	-409	170	-71	28	-13	4	-3	0	-1	0	1	$\cdots$

(6.7)

Inherent in (6.1) is the recursion

$$Q_{-n+2} = 2Q_{-n+1} + Q_{-n} + 2. \tag{6.8}$$

Many other characteristic features of $\{Q_{-n}\}$ are deducible, e.g. (cf. (4.3)),

$$Q_{-n} = P_{-n+1} - 1. \tag{6.9}$$

Replacing n by $-n$ in (4.4), (4.7), and (4.9) readily produces expressions for the Binet form, Simson's formula, and the generating function, respectively.

Similar avenues for development exist in the case of R_{-n}, $n > 0$.

Each of Q_{-n} and R_{-n}, where $n > 0$, opens up fertile new territory for exploration.

However, we must restrict over freedom of choice to our stated goal: the representations of the integers by q_n, where n may be positive or negative.

7 Representation of Any Integer by $\{q_{-n}\}$, $n > 0$.

To demonstrate the truth of Theorem 3.1 below, two options are available to us, namely,

(i) to follow the techniques for Fibonacci numbers used in [3], and

(ii) to modify the proof in [3] to suit our purposes.

Initially, (i) was attempted but its procedures seemed too intricate to pursue. Possibly it is still amenable to mathematical discipline. However, the treatment defined in (ii) appeared quicker and generally more desirable (cf. [6], [4]).

Heavy reliance will be placed on (2.5) in what follows.

But first we need to note the following comments.

Representation of Zero (0) Evidently the integer 0 is not representable by $\{q_n\}$ since no q_i, where $i = 0, 1, 2, \cdots$ is zero. To represent 0, we need to introduce a negatively subscripted q_n, via q_{-1}:

$$0 = 1.q_{-1} + 1.q_0. \tag{7.1}$$

Theorem 7.1 *The representation of any integer N as*

$$N = \sum_{i=0}^{\infty} a_i q_{-i} \quad (a_i = 0, 1, \mathbf{or}\ 2) \tag{7.2}$$

where

$$a_i = 2 \Rightarrow a_{i+1} = 0, \tag{7.3}$$

is unique and minimal.

Proof Suppose there are two different representations

$$N = \sum_{i=0}^{h} a_i q_{-i} \quad a_k \neq 0, a_i = 2 \Rightarrow a_{i+1} = 0 \tag{7.4}$$

$$N = \sum_{i=0}^{m} b_i q_{-i} \quad b_m \neq 0, b_i = 2 \Rightarrow b_{i+1} = 0 \tag{7.5}$$

Case I Assume $h = m$. Conceivably, the numbers in (7.3) and (7.4) are the same but their coeffiecients a_i, b_i are generally different.

Write

$$c_i = a_i - b_i \quad (c_i = 1, \pm 1, \pm 2;\ i = 0, 1, 2, \cdots, m). \tag{7.6}$$

Subtract (7.4) from (7.3). After simplification using (7.5), we derive

$$c_m q_{-m} + \sum_{i=0}^{m-1} c_i q_{-i} = 0 \quad (m \geq 1). \tag{7.7}$$

Employing (2.5), we see that for a maximum or a minimum sum (7.6) i.e., $c_i = \pm 2$ where $i = 0, 1, 2, \cdots, m-1$, we must have

$$c_m q_{-m} + (-1)^m (q_{-m} - q{-}m + 1) = 0 \quad (m \geq 1) \tag{7.8}$$

in which the notation of (1.10) may alternatively be used. Concentrate now on $c_m q_{-m}$ because this term reigns supreme over the sums (7.6) and (7.7).

m **even** ($q_{-m} > 0$) Equation (7.7) now yields

$$(c_m + 1)q_{-m} - q_{-m+1} = 0 \quad (m \geq 2). \tag{7.9}$$

So, with $m \geq 2$,

$$\begin{array}{lllll} c_m & = & 0 \Rightarrow & q_{-m} & = & q_{-m+1} \\ c_m & = & 1 \Rightarrow & q_{-m-1} & = & 0 \\ c_m & = & 2 \Rightarrow & q_{-m} & = & q_{-m-1}. \end{array}$$

For m odd ($q_{-m} < 0$: Under these circumstances (7.7) becomes

$$(c_m - 1)q_{-m} + q_{-m+1} = 0 \quad (m \geq 1). \tag{7.10}$$

Then, for $m \geq 1$,

$$\begin{array}{llllll} c_m & = & 0 \Rightarrow & q_{-m} & = & q_{-m+1} \text{ as before} \\ c_m & = & 1 \Rightarrow & q_{-m+1} & = & 0 \\ c_m & = & 2 \Rightarrow & q_{-m} & = & -q_{-m+1}. \end{array}$$

None of these can possibly be valid, as a little checking discloses. Similar reasoning can be applied for $c_m = -1, -2$. Consequently, the assumption in Case I is untrue.

Summary of Case I conclusions If $h = m$, then $a_i = b_i$ where $i = 0, 1, 2, \cdots, m$. That is, (7.3) and (7.4) are identical, so the representation (7.1) with (7.2) is unique.

Case II Assume $h > m$. Consider the set of coefficients of q_{-i} of length $k+1$ in (7.1), namely,

$$(a_0, a_1, a_2, \cdots, a_{k-1}, a_k). \tag{7.11}$$

For a minimum sum, we must have the arrangement

$$(0, 2, 0, 2, 0, 2, \cdots, 0, 2) \tag{7.12}$$

while for a maximum sum we have

$$(2, 0, 2, 0, 2, 0, \cdots, 2, 0) \tag{7.13}$$

Now replace the symbolism used in Figure 2 by the corresponding asterisked symbolism, e.g., r_k is replaced by r_k^*. Then the appropriate data may be tabulated in this manner (cf. Figure 2 and the discussion germane to it) with the aid of (2.8) and (2.9):

k	r_k^*	N_k^{*min}	N_k^{*max}
$\begin{cases} 0 \\ 1 \end{cases}$	$-2, \cdots, 2$	$-q_{-2}+1=-2$	$-q_{-1}+1=2$
$\begin{cases} 1 \\ 2 \end{cases}$	$-16, \cdots, 8$	$-q_{-4}+1=-16$	$-q_{-3}+1=8$
$\begin{cases} 2 \\ 3 \end{cases}$	$-98, \cdots, 42$	$-q_{-6}+1=-98$	$-q_{-7}+1=240$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\begin{cases} m \\ m+1 \end{cases}$	$-x, \cdots, y$	$-q_{-2(m+1)}+1=-x$	$-q_{-2m-1}+1=y$

k	I_k^*
$\begin{cases} 0 \\ 1 \end{cases}$	$2P_2+1=5$
$\begin{cases} 1 \\ 2 \end{cases}$	$2P_4+1=25$
$\begin{cases} 2 \\ 3 \end{cases}$	$2P_8+1=817$
$\vdots$	$\vdots$
$\begin{cases} m \\ m+1 \end{cases}$	$2P_{2(m+1)}+1=x+y+1$

Basic Data for Theorem 7.1

Appealing to (2.8) and (2.9), we may check this information thus:

$$\begin{aligned} I_k^* &= N_k^{*max} - N_k^{*min} + 1 \quad \text{for the zero representation} \\ &= (-q_{-2m-1}+1) - (-q_{-2(m+1)}+1) + 1 \\ &= q_{-2m-2} - q_{-2m-1} + 1 \\ &= 2P_{2(m+1)} + 1 \quad \text{by } (1.4), (1.10). \end{aligned}$$

Evidently, each number N, as it occurs for the first time in Figure 3, is represented uniquely and minimally, E.g.,

$$-10 = (1q_0 + 0q_{-1} + 1q_{-2} + 2q_{-3}) + 0q_{-4} + 0q_{-5} + 0q_{-6} + \cdots$$

has a unique minimal representation $1q_0 + 1q_{-2} + 2q_{-3}$, i.e., the sequence $(1, 1, 0, 2)$. We conclude that h not greater than m and similarly that h not less than m . Consequently, $h = m$. Hence, Case I and the Summary, are true.

Collecting together all the arguments above, we agree that the validity of the Theorem has been established. (Consult Figure 6 and 7 for details of the numerical mechanism of Theorem 7.1)

There can be no maximal representation of a number by means of negatively subscripted q_n. Arguments for this salient feature are analogous to those used in [6] for negatively subscripted P_n. Having asserted this, we may now mentally review our attainments. These confirm that our stated objectives have indeed been achieved.

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N^+	q_1	q_2	q_3	q_4	q_5	N^+	q_1	q_2	q_3	q_4	q_5
1	1					26	2		1	1	
2	2					27		1	1	1	
3		1				28	1	1	1	1	
4	1	1				29	2	1	1	1	
5	2	1				30		2	1	1	
6		2				31			2	1	
7			1			32	1		2	1	
8	1		1			33	2		2	1	
9	2		1			34				2	
10		1	1			35	1			2	
11	1	1	1			36	2			2	
12	2	1	1			37		1		2	
13		2	1			38	1	1		2	
14			2			39	2	1		2	
15	1		2			40		2		2	
16	2		2			41					1
17				1		42	1				1
18	1			1		43	2				1
19	2			1		44		1			1
20		1		1		45	1	1			1
21	1	1		1		46	2	1			1
22	2	1		1		47		2			1
23		2		1		48			1		1
24			1	1		49	1		1		1
25	1		1	1		50	2		1		1

Fig. 3: Minimal Representations of Positive Integers by Sums of the Numbers q_i where $i = 1, 2, 3 \ldots$

N^+	q_0	q_1	q_2	q_3	q_4	N^+	q_0	q_1	q_2	q_3	q_4
1		1				26	1	2	2		1
2	1	1				27	1	2		1	1
3	1	2				28		1	1	1	1
4		1	1			29	1	1	1	1	1
5	1	1	1			30	1	2	1	1	1
6	1	2	1			31		1	2	1	1
7		1	2			32	1	1	2	1	1
8	1	1	2			33	1	2	2	1	1
9	1	2	2			34	1	2		2	1
10	1	2		1		35		1	1	2	1
11		1	1	1		36	1	1	1	2	1
12	1	1	1	1		37	1	2	1	2	1
13	1	2	1	1		38		1	2	2	1
14		1	2	1		39	1	1	2	2	1
15	1	1	2	1		40	1	2	2	2	1
16	1	2	2	1		41		1	2		2
17	1	2		2		42	1	1	2		2
18		1	1	2		43	1	2	2		2
19	1	1	1	2		44	1	2		1	2
20	1	2	1	2		45		1	1	1	2
21		1	2	2		46	1	1	1	1	2
22	1	1	2	2		47	1	2	1	1	2
23	1	2	2	2		48		1	2	1	2
24		1	2		1	49	1	1	2	1	2
25	1	1	2		1	50	1	2	2	1	2

Fig. 4: Maximal Representations of Positive Intergers by Sums of the Numbers q_i, where $i = 0, -1, -2, \ldots$

0 1 1

N^+	q_0	q_{-1}	q_{-2}	q_{-3}	q_{-4}	q_{-5}	q_{-6}	N^+	q_0	q_{-1}	q_{-2}	q_{-3}	q_{-4}	q_{-5}	q_{-6}
1	1							26		1		1	2		
2	2							27				1	2		
3		1						28	1			1	2		
4	1		1					29	2			1	2		
5	2		1					30			1	1	2		
6			2					31	1		1	1	2		
7	1		2					32		2			2		
8	2		2					33		1			2		
9		1		1	1			34					2		
10				1	1			35	1				2		
11	1		1	1				36	2				2		
12	2		1	1				37			1		2		
13			1	1	1			38	1		1		2		
14	1		1	1	1			39	2		1		2		
15		2			1			40			2		2		
16		1			1			41	1		2		2		
17					1			42	2		2		2		
18	1				1			43		1		2		1	1
19	2				1			44				2		1	1
20			1		1			45	1			2		1	1
21	1		1		1			46	2			2		1	1
22	2		1		1			47			1	2		1	1
23			2		1			48	1		1	2		1	1
24	1		2		1			49	2		1	2		1	1
25	2		2		1			50			2	2		1	1

Fig. 5: Minimal Representations of Positive Intergers by Sums of the Numbers q_i, where $i = 0, -1, -2, \ldots$.

0	1	1											
N^+	q_0	q_{-1}	q_{-2}	q_{-3}	q_{-4}	q_{-5}	N^+	q_0	q_{-1}	q_{-2}	q_{-3}	q_{-4}	q_{-5}
-1		1					-26		2			1	1
-2		2					-27	1		1	1	1	1
-3	1		1	1			-28			1	1	1	1
-4			1	1			-29	2			1	1	1
-5	2			1			-30	1			1	1	1
-6	1			1			-31				1	1	1
-7				1			-32		1		1	1	1
-8		1		1			-33		2		1	1	1
-9		2		1			-34	1		2			1
-10	1		1	2			-35			2			1
-11			1	2			-36	2		1			1
-12	2			2			-37	1		1			1
-13	1			2			-38			1			1
-14				2			-39	2					1
-15		1		2			-40	1					1
-16		2		2			-41						1
-17	1		2		1	1	-42		1				1
-18			2		1	1	-43		2				1
-19	2		1		1	1	-44	1		1	1		1
-20	1		1		1	1	-45			1	1		1
-21			1		1	1	-46	2			1		1
-22	2				1	1	-47	1			1		1
-23	1				1	1	-48				1		1
-24					1	1	-49		1		1		1
-25		1			1	1	-50		2		1		1

Fig. 6: Minimal Representations of Negative Integers by Sums of the Numbers q_i, where $i = 0, -1, -2, \ldots$.