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# On Some Properties of Horadam Polynomials

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**Abstract**

In this study, we introduce a new generalization of the second order polynomial sequences. Namely, we define the Horadam polynomials sequence. Afterwards, we investigate the some properties of the Horadam polynomials.

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## 1 Introduction

For $n \geq 0$, the second order linear recurrence sequence $h_n(a, b; p, q)$, or briefly $h_n$, is defined by

$$h_{n+2} = ph_{n+1} + qh_n, \tag{1}$$

recurrence relations and

$$h_0 = a, \quad h_1 = b \tag{2}$$

initial conditions.

This sequence was introduced, in 1965, by Horadam [2, 3], and it generalizes many sequences (see [7]). Examples of such sequences are Fibonacci numbers sequence $(F_n)_{n\geq 0}$, Lucas numbers sequence $(L_n)_{n\geq 0}$, Pell numbers sequence $(P_n)_{n\geq 0}$, Pell-Lucas numbers sequence $(Q_n)_{n\geq 0}$, Jacobsthal numbers sequence $(J_n)_{n\geq 0}$ and Jacobsthal-Lucas numbers sequence $(j_n)_{n\geq 0}$.

The characteristic equation of recurrence relation (1) is

$$t^2 - pt - q = 0. \tag{3}$$

This equation has two real roots;

$$\alpha = \frac{p + \sqrt{p^2 + 4q}}{2}, \quad \beta = \frac{p - \sqrt{p^2 + 4q}}{2}. \tag{4}$$

The generating function of Horadam sequence is

$$g(t) = \frac{a + t(b - ap)}{1 - pt - qt^2}. \tag{5}$$

Using the generating function (5), the Binet's formula for $h_n$ is obtained as follows

$$h_n = A\alpha^n + B\beta^n = A\left(\frac{p + \sqrt{p^2 + 4q}}{2}\right)^n + B\left(\frac{p - \sqrt{p^2 + 4q}}{2}\right)^n \tag{6}$$

where

$$A = \frac{b - a\beta}{\sqrt{p^2 + 4q}}, \quad B = \frac{a\alpha - b}{\sqrt{p^2 + 4q}}. \tag{7}$$

In [6], the authors give the summations of Horadam numbers as follows

$$\sum_{k=0}^{n-1} h_k = \frac{1}{p + q - 1}(h_n + qh_{n-1} + pa - a - b). \tag{8}$$

where $p + q \neq 1$ and $p, q \geq 0$.

The explicit formula for Horadam sequence is given as

$$h_n = a\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} p^{n-2k} q^k + \left(\frac{b}{p} - a\right) \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k} p^{n-2k} q^k \tag{9}$$

where $n \geq 1$ and $a, b, p, q \in \mathbb{Z}$ (see [1]).

In [9, 11], the authors investigate the some properties of Horadam numbers and give the relations between Chebyshev polynomials and Horadam numbers.

Now, We define the Horadam Polynomials which is generalized Horadam numbers and second order polynomials sequences.

# 2 Horadam Polynomials

## 2.1 Definitions

For $n \geq 3$, Horadam polynomials sequence $h_n(x, a, b; p, q)$, or briefly $h_n(x)$, is defined by

$$h_n(x) = pxh_{n-1}(x) + qh_{n-2}(x) \tag{10}$$

recurrence relations

$$h_1(x) = a, \quad h_2(x) = bx \tag{11}$$

initial conditions.

The characteristic equation of recurrence relation (10) is

$$t^2 - pxt - q = 0. \tag{12}$$

This equation has two real roots;

$$\alpha = \frac{px + \sqrt{p^2x^2 + 4q}}{2}, \qquad \beta = \frac{px - \sqrt{p^2x^2 + 4q}}{2}.$$

From (10, 11), we give the following table.

| $n$ | $h_n(x, a, b; p, q)$ |
|---|---|
| 1 | $a$ |
| 2 | $bx$ |
| 3 | $bpx^2 + aq$ |
| 4 | $bp^2x^3 + (apq + bq)x$ |
| 5 | $bp^3x^4 + (ap^2q + 2bpq)x^2 + aq^2$ |
| 6 | $bp^4x^5 + (ap^3q + 3bp^2q)x^3 + (2apq^2 + bq^2)x$ |
| 7 | $bp^5x^6 + (ap^4q + 4bp^3q)x^4 + (3ap^2q^2 + 3bpq^2)x^2 + aq^3$ |
| 8 | $bp^6x^7 + (ap^5q + 5bp^4q)x^5 + (4ap^3q^2 + 6bp^2q^2)x^3 + (3apq^3 + bq^3)x$ |
| $\vdots$ | $\vdots$ |

Table 2.1. Horadam Polynomials

Particular cases of Horadam polynomials sequence are

- If $a = b = p = q = 1$, the Fibonacci polynomials sequence is obtained

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x); \quad F_1(x) = 1, \ F_2(x) = x.$$

- If $a = 2, \quad b = p = q = 1$, the Lucas polynomials sequence is obtained

$$L_{n-1}(x) = xL_{n-2}(x) + L_{n-3}(x); \quad L_0(x) = 2, \ L_1(x) = x.$$

- If $a=q=1, \quad b=p=2$, the Pell polynomials sequence is obtained

$$P_n(x)=2xP_{n-1}(x)+P_{n-2}(x)\,; \quad P_1(x)=1,\ P_2(x)=2x.$$

- If $a=b=p=2,\ q=1$, the Pell-Lucas polynomials sequence is obtained

$$Q_{n-1}(x)=2xQ_{n-2}(x)+Q_{n-3}(x)\,; \quad Q_0(x)=2,\ Q_1(x)=2x.$$

- If $a=b=p=x=1,\ q=2y$, the Jacobsthal polynomials sequence is obtained

$$J_n(y)=J_{n-1}(y)+2yJ_{n-2}(y)\,; \quad J_1(y)=1,\ J_2(y)=1.$$

- If $a=2, \quad b=p=x=1,\ q=2y$, the Jacobsthal-Lucas polynomials sequence is obtained

$$j_{n-1}(y)=j_{n-2}(y)+2yj_{n-3}(y)\,; \quad j_0(y)=2,\ j_1(y)=1.$$

- If $a=1,\ b=p=2,\ q=-1$, the Chebyshev polynomials of second kind sequence is obtained

$$U_{n-1}(x)=2xU_{n-2}(x)-U_{n-3}(x)\,; \quad U_0(x)=1,\ U_1(x)=2x.$$

- If $a=b=1,\ p=2,\ q=-1$, the Chebyshev polynomials of first kind sequence is obtained

$$T_{n-1}(x)=2xT_{n-2}(x)-T_{n-3}(x)\,; \quad T_0(x)=1,\ T_1(x)=x.$$

- If $x=1$, The Horadam numbers sequence is obtained

$$h_{n-1}(1)=ph_{n-2}(1)+qh_{n-3}(1)\,; \quad h_0(1)=a,\ h_1(1)=b$$

We can find the more information associated with these polynomials sequences in [4, 5, 7, 8].

We obtain the generating function of Horadam polynomials sequence as

$$g(x,t)=\frac{a+xt(b-ap)}{1-pxt-qt^2}. \tag{13}$$

Binet's formula for the Horadam polynomials sequence $h_n(x)$ to be represented by the roots $\alpha$ and $\beta$ of equation (12)

$$h_n(x)=A_1\alpha^{n-1}+A_2\beta^{n-1} \tag{14}$$

where

$$A_1 = \frac{bx - a\beta}{\sqrt{p^2x^2 + 4q}}, \quad A_2 = \frac{a\alpha - bx}{\sqrt{p^2x^2 + 4q}}. \tag{15}$$

Taking $x = 1$ in (13,14), we have the generating function and Binet's formula for the Horadam numbers sequence which is given in (5,6). Similarly, using (13,14), we have the generating functions and Binet's formulas for the Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas, Chebyshev polynomials (see [4, 5, 7, 8, 11]).

We have the expilicit formula for the Horadam polynomials as

$$h_{n+1}(x) = a\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} (px)^{n-2k} q^k + \left(\frac{b}{p} - a\right) \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k} (px)^{n-2k} q^k. \tag{16}$$

Now, we give the following Lemma associated with expilicit formula.

**Lemma 1** *For $n \geq 1$*

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k} \binom{n-k}{k} (x)^{n-2k} = 2\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} (x)^{n-2k} - \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k} (x)^{n-2k}. \tag{17}$$

It's note that, taking $x = 1$ in (16), we obtain the explicit formula for Horadam sequence which is given in (9). Using (16,17), we have the explicit formula for the other second order polynomials sequences as follows;

$$F_{n+1}(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} (x)^{n-2k} \qquad L_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k} \binom{n-k}{k} (x)^{n-2k}$$

$$P_{n+1}(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} (2x)^{n-2k} \qquad Q_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k} \binom{n-k}{k} (2x)^{n-2k}$$

$$J_{n+1}(y) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} (2y)^{k} \qquad j_n(y) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k} \binom{n-k}{k} (2y)^{k}$$

$$U_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} (2x)^{n-2k} \quad T_n(x) = \frac{n}{2} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k}{n-k} \binom{n-k}{k} (2x)^{n-2k}.$$

# 3 Some properties

**Proposition 2** *Let $h_n(x)$ be nth Horadam polynomial. Then*

$$\sum_{k=1}^{n-1} h_k(x) = \frac{h_n(x) + qh_{n-1}(x) - a - x(b - ap)}{px + q - 1}. \tag{18}$$

**Proof.** Using the Binet's formula for the Horadam polynomials, we have

$$\begin{aligned}
\sum_{k=1}^{n-1} h_k(x) &= \sum_{k=1}^{n-1} A_1\alpha^{k-1} + A_2\beta^{k-1} \\
&= A_1 \sum_{k=1}^{n-1} \alpha^{k-1} + A_2 \sum_{k=1}^{n-1} \beta^{k-1} \\
&= A_1 \left(\frac{1-\alpha^{n-1}}{1-\alpha}\right) + A_2 \left(\frac{1-\beta^{n-1}}{1-\beta}\right) \\
&= \frac{A_1 + A_2 - (A_1\beta + A_2\alpha) - (A_1\alpha^{n-1} + A_2\beta^{n-1}) - q\left(A_1\alpha^{n-2} + A_2\beta^{n-2}\right)}{1-px-q} \\
&= \frac{h_1(x) - a\left(\alpha^2-\beta^2\right) + bx(\alpha-\beta) - h_n(x) - qh_{n-1}(x)}{1-px-q}.
\end{aligned}$$

Therefore, we obtain

$$\sum_{k=1}^{n-1} h_k(x) = \frac{h_n(x) + qh_{n-1}(x) - a - x(b-ap)}{px+q-1}.$$

■

Some particular cases are;

- Taking $x = 1$ in (18), we obtain the sum of Horadam numbers as

$$\sum_{k=1}^{n-1} h_{k-1} = \frac{1}{p+q-1}(h_n + qh_{n-1} + pa - a - b).$$

- Taking $a = b = p = q = 1$ in (18), we have the sum of the Fibonacci polynomials as

$$\sum_{k=1}^{n-1} F_k(x) = \frac{F_n(x) + F_{n-1}(x) - 1}{x}.$$

- Taking $a = 1,\ b = p = 2$ and $q = -1$ in (18), we have the sum of the Chebyshev polynomials of second kind as

$$\sum_{k=1}^{n-1} U_{k-1}(x) = \frac{U_{n-1}(x) - U_{n-2}(x) - 1}{2x-2}.$$

**Proposition 3** *Let $h_n(x)$ be nth Horadam polynomial. Then*

$$\sum_{k=1}^{n} h_k^2(x) = \left(\frac{h_{n+1}(x)\left(h_{n+1}(x)-q^2h_{n-1}(x)\right)}{p^2x^2+2q-q^2-1}\right) - \left(\frac{A_1A_2\left(2\frac{1-(-q)^k}{1+q}+(-q)^k\right)-x^2(b-ap)^2+a^2}{p^2x^2+2q-q^2-1}\right).$$

**Proof.** Using the Binet's formula (14), the proof is clear. ■

**Proposition 4** *(Catalan's Identity) Let $h_n(x)$ be nth Horadam polynomial. Then*

$$h_n^2(x) - h_{n+r}(x)\,h_{n-r}(x) = \frac{(-q)^{n-r-1}\left(bxh_{r+1}(x)-ah_{r+2}(x)\right)^2}{b^2x^2-abpx^2-a^2q} \tag{19}$$

*where $n \geq 0$ and $n \geq r$.*

**Proof.** Using the Binet's formula (14) to left hand side (LHS), we have

$$\begin{aligned}
(LHS) &= \left(A_1\alpha^{n-1}+A_2\beta^{n-1}\right)^2 - \left(A_1\alpha^{n+r-1}+A_2\beta^{n+r-1}\right)\left(A_1\alpha^{n-r-1}+A_2\beta^{n-r-1}\right) \\
&= A_1A_2(\alpha\beta)^{n-1}\left(2-\alpha^r\beta^{-r}-\beta^r\alpha^{-r}\right) \\
&= A_1A_2(-q)^{n-1}\left(2-\frac{\alpha^r}{\beta^r}-\frac{\beta^r}{\alpha^r}\right) \\
&= A_1A_2(-q)^{n-1}\frac{-1}{(-q)^r}\left(\alpha^r-\beta^r\right)^2 \\
&= \left(b^2x^2-abpx^2-a^2q\right)(-q)^{n-r-1}\left(\frac{\alpha^r-\beta^r}{\alpha-\beta}\right)^2.
\end{aligned}$$

From

$$\frac{\alpha^r-\beta^r}{\alpha-\beta} = \frac{bxh_{r+1}(x)-ah_{r+2}(x)}{b^2x^2-abpx^2-a^2q},$$

we obtain

$$h_n^2(x) - h_{n+r}(x)\,h_{n-r}(x) = \frac{(-q)^{n-r-1}\left(bxh_{r+1}(x)-ah_{r+2}(x)\right)^2}{b^2x^2-abpx^2-a^2q}.$$

■

As applications of the above proposition we obtain the following results.

**Corollary 5** *(Cassini's Identity) Let $h_n(x)$ be nth Horadam polynomial. Then*

$$h_n^2(x) - h_{n+1}(x)\,h_{n-1}(x) = (-q)^{n-2}\left(b^2x^2 - abpx^2 - a^2q\right) \tag{20}$$

*where $n \geq 0$.*

**Proof.** Taking $r = 1$ in Catalan's identity (19), the proof is completed. ■

Some particular cases are,

- Taking $x = 1$ in (20), we have the Cassini's identity for Horadam numbers as
$$h_n^2 - h_{n+1}h_{n-1} = (-q)^{n-2}\left(b^2 - abp - a^2q\right).$$
- Taking $a = b = p = q = 1$ in (20), we have the Cassini's identity for Fibonacci polynomials as
$$F_n^2(x) - F_{n+1}(x)\,F_{n-1}(x) = (-1)^{n-1}.$$
- Taking $a = q = 1,\ b = p = 2$ in (20), we have the Cassini's identity for Pell polynomials as
$$P_n^2(x) - P_{n+1}(x)\,P_{n-1}(x) = (-1)^{n-1}.$$

**Proposition 6** *(d'Ocagnes's Identity) Let $h_n(x)$ be nth Horadam polynomial. Then*

$$h_m(x)\,h_{n+1}(x) - h_{m+1}(x)\,h_n(x) = \frac{(bxh_{m-n+1}(x) - ah_{m-n+2}(x))}{(-q)^{1-n}} \tag{21}$$

*where $n \leq m$ integers.*

It's note that, taking $n-1$ instead of $m$ in (21), we obtain the Cassini's identity for Horadam polynomials (20).

Also, taking $a = b = p = q = 1$ in (21), we obtain the d'Ocagne's identity for Fibonacci polynomials as

$$F_m(x)\,F_{n+1}(x) - F_{m+1}(x)\,F_n(x) = (-1)^{n-1}F_{m-n}(x)$$

(see [7]).

**Proposition 7** *(Honsberger's formula) Let $T = h_{m+n}(x)$ and $h_n(x)$ be nth Horadam polynomial. Then*

$$\begin{aligned} T &= \frac{((bp)x^2 - ap^2x^2 - aq)\,h_{m+1}(x) + qx\,(b - ap)\,h_m(x))\,h_n(x)}{b^2x^2 - abpx^2 - a^2q} \\ &\quad + \frac{q\,(x\,(b - ap)\,h_{m+1}(x) - aqh_m(x))\,h_{n-1}(x)}{b^2x^2 - abpx^2 - a^2q} \end{aligned} \tag{22}$$

*where $n, m \geq 2$.*

Some particular cases are:

- If we take $a = b = p = q = 1$ in (22), we obtain the Honsberger's formula for Fibonacci polynomials as

$$F_{m+n}(x) = F_{m+1}(x) F_n(x) + F_m(x) F_{n-1}(x)$$

- If we take $a = q = 1, \quad b = p = 2$ in (22), we obtain the Honsberger's formula for Pell polynomials as

$$P_{m+n}(x) = P_{m+1}(x) P_n(x) + P_m(x) P_{n-1}(x)$$

- If we take $a = 1, \quad b = p = 2$ and $q = -1$ in (22), we obtain the Honsberger's formula for the Chebyshev polynomials of second kind as

$$U_{m+n}(x) = U_m(x) U_n(x) - U_{m-1}(x) U_{n-1}(x)$$

**Proposition 8** *Let* $K = h_m(x) h_n(x) - h_{m-r}(x) h_{n+r}(x)$ *and* $h_n(x)$ *be nth Horadam polynomials. Then*

$$K = \frac{(bxh_{r+1}(x) - ah_{r+2}(x))(bxh_{n+r-m+1}(x) - ah_{n+r-m+2}(x))}{(-q)^{r-m+1} b^2x^2 - abpx^2 - a^2q} \tag{23}$$

*where* $n, m, r$ *nonnegative integers.*

**Proof.** The proof is clear by Binet's formula. ∎

Identity (23) is generalized of Cassini's, Catalan's and d'Ocagne's identities. Namely;

- If we take $n+1$ instead of $m$, $n-1$ instead of $n$ and $r = 1$ in (23), The Cassini's identity (20) is obtained.
- Taking $n$ instead of $m$ in (23), we obtain Catalan's identity (19).
- Using $m$ instead of $n$, $n+1$ instead of $m$ and $r = 1$ in (23), the d'Ocagne's identity (21) is obtained.
- If we take $x = a = b = p = q = 1$ in (23), we have

$$F_mF_n - F_{m-r}F_{n+r} = (-1)^{m-r} F_rF_{n+r-m}$$

  which is given for Fibonacci numbers by Spivey in [10].

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