Available online at www.sciencedirect.com

SCIENCE DIRECT®

Journal of Computational and Applied Mathematics 193 (2006) 474–483

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

www.elsevier.com/locate/cam

New families of orthogonal polynomials

C. Hounga[a], M.N. Hounkonnou[a,c,*], A. Ronveaux[b]

[a] *Unité de Recherche en Physique Théorique (URPT), Institut de Mathématiques et de Sciences Physiques (IMSP), 01B.P. 2628 Porto-Novo, Benin*

[b] *Département de Mathématiques, Unité d'Analyse Mathématique et de Mécanique (AMM), Université catholique de Louvain, Bat. M. de Hemptine, Chemin du Cyclotron, 2 1348 Louvain-la-Neuve, Belgium*

[c] *International Chair in Mathematical Physics and Applications (ICMPA), 072B.P. 50 Cotonou, Benin*

Received 20 April 2005

Abstract

This paper provides with a generalization of the work by Wimp and Kiesel [Non-linear recurrence relations and some derived orthogonal polynomials, Ann. Numer. Math. 2 (1995) 169–180] who generated some new orthogonal polynomials from Chebyshev polynomials of second kind. We consider a class of polynomials $\tilde{P}_n(x)$ defined by: $\tilde{P}_n(x) = (a_n x + b_n)P_{n-1}(x) + (1 - a_n)P_n(x),\ n = 0, 1, 2, \ldots, a_0 \neq 1$, where the $P_k(x)$ are monic classical orthogonal polynomials satisfying the well-known three-term recurrence relation: $P_{n+1}(x) = (x - \beta_n)P_n(x) - \gamma_n P_{n-1}(x),\ n \geqslant 1,\ P_1(x) = x - \beta_0;\ P_0(x) = 1$. We explicitly derive the sequences a_n and b_n in general and illustrate by some concrete relevant examples.

Keywords: Classical orthogonal polynomials; Recurrence relation; Second order differential equation; Structure relation

1. Introduction

Recently Wimp and Kiesel [5,4] have derived non-linear recurrence relations generating new orthogonal polynomials. This work generalizes the results of these authors and enlarges the classes of orthogonal polynomials.The problem we solve states as follows.

* Corresponding author. International Chair in Mathematical Physics and Applications (ICMPA), 072B.P. 50 Cotonou, Benin.
E-mail addresses: hounber@yahoo.fr (C. Hounga), norbert_hounkonnou@cipma.net (M.N. Hounkonnou), ronveaux@math.ucl.ac.be (A. Ronveaux).

0377-0427/$ - see front matter © 2005 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2005.06.026

Let $\{P_n\}_{n\geqslant 0}$ be a family of monic orthogonal polynomials with respect to a weight w, satisfying the three-term recurrence relations [1]

$$\begin{aligned} &P_{n+1}(x) = (x-\beta_n)P_n(x) - \gamma_n P_{n-1}(x), \quad n\geqslant 1, \\ &P_0(x) = 1, \quad P_1(x) = x-\beta_0. \end{aligned} \tag{1}$$

The coefficients γ_n and β_n are complex numbers (with $\gamma_n \neq 0$). By convention, we set $\gamma_0 = 1$. Let a_n and b_n be two given sequences of complex numbers such that a_0 is different from one and consider the new sequence $\{\tilde{P}_n\}_{n\geqslant 0}$ defined by (2)

$$\tilde{P}_n(x) = (a_n x + b_n)P_{n-1}(x) + (1-a_n)P_n(x), \quad P_{-1}(x) = 0, \ \ n = 0, 1, 2, \ldots, a_0 \neq 1. \tag{2}$$

Then, a question arises: how should one choose both sequences $\{a_n\}$ and $\{b_n\}$ so that the new polynomials $\tilde{P}_n$ are orthogonal too?

The paper is organized as follows: In Section 2, we develop the general formalism giving the recurrence relation for the new polynomials. In Section 3, we point out some relevant particular cases. In Section 4, we construct concrete examples of new orthogonal polynomials based on known classical ones. Finally in Section 5, we derive the second order differential equation satisfied by the new families of orthogonal polynomials.

2. Construction of the new non-linear recurrence relations

Let us prove the following statement.

Theorem 1. *Let $\{P_n\}_{n\geqslant 0}$ be a family of monic orthogonal polynomials satisfying Eq.* (1). *Then the sequence $\{\tilde{P}_n\}_{n\geqslant 0}$, defined by* (2), *verifies the following recurrence relation*:

$$\tilde{P}_{n+2}(x) = \eta_n^{(2)}(x)\tilde{P}_{n+1}(x) - \eta_n^{(1)}(x)\tilde{P}_n(x), \tag{3}$$

where

$$\begin{aligned} &\eta_n^{(1)}(x) = \gamma_n \frac{a_{n+1}}{a_n} + \frac{k_n^{(1)}x + k_n^{(2)}}{w_n(x)}, \quad a_n \neq 0, \ \ n\geqslant 0 \\ &\eta_n^{(2)}(x) = A_{n+2} - a_{n+1}\beta_n - b_{n+1} + x + \frac{k_n^{(3)}x + k_n^{(4)}}{w_n(x)}, \\ &w_n(x) = (a_n x + b_n)(x + A_{n+1}) + B_{n+1}, \end{aligned} \tag{4}$$

and

$$k_n^{(1)} = \gamma_n \left[(a_{n+1}A_{n+2} + b_{n+1}) - \frac{a_{n+1}}{a_n}(a_n A_{n+1} + b_n) \right], \tag{5}$$

$$k_n^{(2)} = \gamma_n \left[(A_{n+2}b_{n+1} + B_{n+2}) - \frac{a_{n+1}}{a_n}(A_{n+1}b_n + B_{n+1}) \right], \tag{6}$$

$$\begin{aligned} k_n^{(3)} = {} & a_n(a_{n+2}-1)\gamma_{n+1} - a_{n+1}(a_n - 1)\gamma_n - a_n\beta_n A_{n+2} \\ & - a_n A_{n+1}(A_{n+2} - a_{n+1}\beta_n - b_{n+1}), \end{aligned}$$

$$k_n^{(4)} = \gamma_n(1-a_n)A_{n+2} + b_n[(a_{n+2}-1)\gamma_{n+1} - \beta_n A_{n+2}] - (b_n A_{n+1} + B_{n+1})(A_{n+2} - a_{n+1}\beta_n - b_{n+1}),$$

$$A_{n+1} = (a_{n+1}-1)\beta_n + b_{n+1}, \quad B_{n+1} = \gamma_n(a_n-1)(a_{n+1}-1). \tag{7}$$

Proof. From (2) and taking into account (1) it follows that

$$\tilde{P}_{n+1}(x) = (x + A_{n+1})P_n(x) + (a_{n+1}-1)\gamma_n P_{n-1}(x). \tag{8}$$

Replacing n by $n+1$ in (8) with the use of (1), we get:

$$\tilde{P}_{n+2}(x) = [(x + A_{n+2})(x-\beta_n) + \gamma_{n+1}(a_{n+2}-1)]P_n(x) - \gamma_n(x + A_{n+2})P_{n-1}(x). \tag{9}$$

We thus form a set of three equations (2), (8), (9) with the two unknowns P_n, P_{n-1}. They will have a solution if and only if the augmented determinant is equal to zero. The computation and the transformation of this determinant give the required recurrence relation. □

We then deduce the following relations:

$$(A_{n+2}a_{n+1} + b_{n+1}) - \frac{a_{n+1}}{a_n}(A_{n+1}a_n + b_n) = 0, \tag{10}$$

$$\left[(A_{n+2}b_{n+1} + B_{n+2}) - \frac{a_{n+1}}{a_n}(A_{n+1}b_n + B_{n+1})\right] = 0, \tag{11}$$

$$a_n(a_{n+2}-1)\gamma_{n+1} - a_{n+1}(a_n-1)\gamma_n - a_n\beta_n A_{n+2} - a_n A_{n+1}(A_{n+2} - a_{n+1}\beta_n - b_{n+1}) = 0, \tag{12}$$

$$\gamma_n(1-a_n)A_{n+2} + b_n[(a_{n+2}-1)\gamma_{n+1} - \beta_n A_{n+2}] - (b_n A_{n+1} + B_{n+1})(A_{n+2} - a_{n+1}\beta_n - b_{n+1}) = 0, \tag{13}$$

where a_0, a_1, b_0 and b_1 are given arbitrary numbers. Furthermore, the following holds:

Corollary 1. *Let $a_n \neq 0, a_n \neq 1$ for all n. Then Eqs.* (10) *and* (11) *imply Eqs.* (12) *and* (13).

Proof. One can follow step by step [5,4]. □

Corollary 2. *Let $\{a_n\}, \{b_n\}$ be the sequences satisfying Eqs.* (10) *and* (11). *Then the following non-linear recurrence relations hold*:

$$b_{n+1} = C - \frac{b_n}{a_n} - (a_{n+1}-1)\beta_n, \quad n = 1, 2, 3, 4, \ldots, \tag{14}$$

$$a_{n+1} = \frac{a_n^2 M - C a_n b_n + b_n^2}{\gamma_n a_n(a_n-1)} + 1, \quad n = 1, 2, 3, \ldots, \tag{15}$$

where

$$C = \frac{a_0 A_1 + b_0}{a_0}, \quad M = \frac{b_0 A_1 + B_1}{a_0}. \tag{16}$$

Proof. Using Eq. (10) we deduce for $n = 0, 1, 2 \dots$

$$\frac{(A_{n+2}a_{n+1} + b_{n+1})}{a_{n+1}} = \frac{(A_{n+1}a_n + b_n)}{a_n}, \tag{17}$$

which implies

$$\frac{(A_{n+1}a_n + b_n)}{a_n} = C, \tag{18}$$

C being a constant (independent of n) so that (14) is achieved. Equivalently, $k_n^{(2)} = 0$ yields

$$\frac{(A_{n+2}b_{n+1} + B_{n+2})}{a_n} = (A_{n+1}b_n + B_{n+1}), \tag{19}$$

$$\frac{(A_{n+1}b_n + B_{n+1})}{a_n} = M, \tag{20}$$

where M is now a constant given by

$$M = \frac{[(a_1 - 1)\beta_0 b_0 + b_0 b_1 + (a_0 - 1)(a_1 - 1)]}{a_0}.$$

Using (14) and (20), we obtain relation (15). □

Corollary 3. *Let* $\{P_n\}_{n\geqslant 0}$, $\{a_n\}$ *and* $\{b_n\}$ *be the sequences satisfying Eqs.* (1), (10) *and* (11), *respectively. Then the sequence* $\{\tilde{P}_n\}_{n\geqslant 0}$ *verifies the following recurrence relation*:

$$\begin{aligned}\tilde{P}_{n+2}(x) &= [x + A_{n+2} - b_{n+1} - a_{n+1}\beta_n]\tilde{P}_{n+1}(x) \\ &\quad - \gamma_n \frac{a_{n+1}}{a_n}\tilde{P}_n(x), \quad a_n \neq 0, \quad n = 0, 1, 2, \dots, \\ \tilde{P}_0(x) &= 1 - a_0, \quad \tilde{P}_1(x) = x + b_1 + \beta_0(a_1 - 1).\end{aligned} \tag{21}$$

3. Particular case of $b_n \equiv 0$

Let us summarize the results of the previous section when $b_n \equiv 0$.

Theorem 2. *Let* $b_n \equiv 0$, $\beta_n \neq 0$, $a_0 \neq 1$ *and* $a_n \neq 0$. *Then the sequences* $\{\tilde{P}_n\}_{n\geqslant 0}$ *and* $\{a_n\}$ *satisfy the following recurrence relations*:

$$\begin{aligned}&\tilde{P}_{n+2}(x) = (x - \beta_n)\tilde{P}_{n+1}(x) - \gamma_n \frac{a_{n+1}}{a_n}\tilde{P}_n(x), \\ &\quad n = 0, 1, 2, \dots; \quad \tilde{P}_0(x) = 1 - a_0; \quad \tilde{P}_1(x) = x + \beta_0(a_1 - 1),\end{aligned} \tag{22}$$

$$a_n = \frac{\gamma_n}{\gamma_n - \beta_n K}, \quad K = \frac{a_0 - 1}{\beta_0 a_0}. \tag{23}$$

Proof. If $b_n \equiv 0$, $\beta_n \neq 0$ Eqs. (10), (11) and (21) become, respectively,

$$(a_{n+2}-1)\beta_{n+1} = (a_{n+1}-1)\beta_n \tag{24}$$

and

$$\frac{\gamma_{n+1}(a_{n+2}-1)}{a_{n+1}} = \frac{\gamma_n(a_n-1)}{a_n}, \tag{25}$$

$$\begin{aligned} &\tilde{P}_{n+2}(x) = [x + (a_{n+2}-1)\beta_{n+1} - a_{n+1}\beta_n]\tilde{P}_{n+1}(x) - \gamma_n \frac{a_{n+1}}{a_n}\tilde{P}_n(x),\\ &\quad n = 0, 1, 2, \ldots, \quad \tilde{P}_0(x) = 1 - a_0, \quad \tilde{P}_1(x) = x + \beta_0(a_1 - 1). \end{aligned} \tag{26}$$

Using (24), Eq. (26) rewrites

$$\begin{aligned} &\tilde{P}_{n+2}(x) = (x - \beta_n)\tilde{P}_{n+1}(x) - \gamma_n \frac{a_{n+1}}{a_n}\tilde{P}_n(x),\\ &\quad n = 0, 1, 2, \ldots; \quad \tilde{P}_0(x) = 1 - a_0; \quad \tilde{P}_1(x) = x + \beta_0(a_1 - 1). \end{aligned}$$

Combining (24) and (25) with $\beta_n \neq 0$, we obtain a first order non-linear difference equation:

$$\frac{\gamma_{n+1}(a_{n+1}-1)}{\beta_{n+1}a_{n+1}} = \frac{\gamma_n(a_n-1)}{\beta_n a_n}, \tag{27}$$

which implies

$$\frac{\gamma_n(a_n-1)}{\beta_n a_n} = K \quad \text{with } K = \frac{a_0-1}{\beta_0 a_0}. \tag{28}$$

Using Eq. (28), we obtain (23). □

Theorem 3. *Let $b_n \equiv 0$, $\beta_n = 0$, $a_n \neq 0$, and $a_n \neq 1$. Then the sequences $\{\tilde{P}_n\}_{n\geqslant 0}$ and $\{a_n\}$ satisfy the following recurrence relations*:

$$\begin{aligned} &\tilde{P}_{n+2}(x) = x\tilde{P}_{n+1}(x) - \gamma_n \frac{a_{n+1}}{a_n}\tilde{P}_n(x),\\ &\quad n = 0, 1, 2, \ldots; \quad \tilde{P}_0(x) = 1 - a_0; \quad \tilde{P}_1(x) = x, \end{aligned} \tag{29}$$

$$a_{n+1} = \frac{a_n L}{\gamma_n(a_n-1)} + 1, \quad L = \frac{(a_1-1)(a_0-1)}{a_0}. \tag{30}$$

Proof. If $b_n \equiv 0$, $\beta_n = 0$, $a_n \neq 0$, and $a_n \neq 1$ Eqs. (10), (11) and (21) become, respectively,

$$\frac{\gamma_{n+1}(a_{n+2}-1)}{a_{n+1}} = \frac{\gamma_n(a_n-1)}{a_n}, \tag{31}$$

$$\begin{aligned} &\tilde{P}_{n+2}(x) = x\tilde{P}_{n+1}(x) - \gamma_n \frac{a_{n+1}}{a_n}\tilde{P}_n(x),\\ &\quad n = 0, 1, 2, \ldots; \quad \tilde{P}_0(x) = 1 - a_0; \quad \tilde{P}_1(x) = x. \end{aligned} \tag{32}$$

Multiplying Eq. (31) by $(a_{n+1}-1)$, we obtain

$$\frac{\gamma_{n+1}(a_{n+2}-1)(a_{n+1}-1)}{a_{n+1}}=\frac{\gamma_n(a_{n+1}-1)(a_n-1)}{a_n}, \tag{33}$$

which implies

$$\frac{\gamma_n(a_{n+1}-1)(a_n-1)}{a_n}=L \quad \text{with } L=\frac{(a_1-1)(a_0-1)}{a_0}. \tag{34}$$

Using Eq. (34), we obtain (30). □

4. Examples of new families of orthogonal polynomials

In this section, let us provide data for some new families of orthogonal polynomials obtained from the classical Hermite, Laguerre and Jacobi orthogonal polynomials.

(i) Data for the modified Laguerre polynomials $\tilde{P}_n(x)$

$$C=(a_1-1)(\alpha+1)+b_1+\frac{b_0}{a_0},$$
$$M=\frac{(a_1-1)b_0(\alpha+1)+b_0b_1+(a_0-1)(a_1-1)}{a_0},$$
$$b_{n+1}=C-(a_{n+1}-1)(2n+\alpha+1)-\frac{b_n}{a_n},$$
$$a_{n+1}=\frac{Ma_n^2-Ca_nb_n+b_n^2}{n(n+\alpha)a_n(a_n-1)}+1.$$

(ii) Data for the modified Jacobi polynomials $\tilde{P}_n(x)$

$$C=\frac{(a_1-1)(\beta-\alpha)}{(\beta+\alpha+2)}+b_1+\frac{b_0}{a_0},$$
$$M=\frac{(a_1-1)(\beta-\alpha)b_0}{(\beta+\alpha+2)a_0}+\frac{b_0b_1(a_0-1)(a_1-1)}{a_0},$$
$$b_{n+1}=C-\frac{(a_{n+1}-1)(\beta^2-\alpha^2)}{(\beta+\alpha+2n+2)(\beta+\alpha+2n)}-\frac{b_n}{a_n},$$
$$a_{n+1}=\frac{(Ma_n^2-Ca_nb_n+b_n^2)(\beta+\alpha+2n)^2(\beta+\alpha+2n-1)(\beta+\alpha+2n+1)}{4na_n(a_n-1)(\alpha+n)(\beta+n)(\beta+\alpha+n)}+1.$$

(iii) Data for the modified Hermite polynomials $\tilde{P}_n(x)$

$$C=b_1+\frac{b_0}{a_0},\quad M=\frac{b_0b_1+(a_0-1)(a_1-1)}{a_0},$$
$$b_{n+1}=C-\frac{b_n}{a_n},\quad a_{n+1}=\frac{2(Ma_n^2-Ca_nb_n+b_n^2)}{na_n(a_n-1)}+1.$$

5. Second order differential equation

To derive the second order differential equation for the new orthogonal polynomials, we start with the following lemma.

Lemma 1. *Let ρ be a classical weight function satisfying the Pearson equation $(\sigma(x)\rho(x))' = \tau(x)\rho(x)$, where $\sigma(x) = \sigma_2 x^2 + \sigma_1 x + \sigma_0$, $\tau(x) = \tau_1 x + \tau_0$, $(|\tau_1|)(|\sigma_2| + |\sigma_1| + |\sigma_0|) \neq 0$. Then, the monic classical orthogonal polynomial family $\{P_n\}_{n \geqslant 0}$ satisfies the following structure relations*:

$$\sigma(x) P_n'(x) = I_n P_{n-1}(x) + (n\sigma_2 x + J_n) P_n(x), \tag{35}$$

$$\sigma(x) P_{n-1}'(x) = (C_n x + D_n) P_{n-1}(x) + E_n P_n(x), n = 0, 1, 2, \ldots, \tag{36}$$

where

$$I_n = -((2n-1)\sigma_2 + \tau_1)\gamma_n, \quad J_n = n\sigma_1 + \sigma_2 \sum_{i=0}^{n-1} \beta_i, \tag{37}$$

$$\begin{aligned} C_n = & -((n-2)\sigma_2 + \tau_1), \quad D_n = ((2n-3)\sigma_2 + \tau_1)\beta_{n-1} \\ & + (n-1)\sigma_1 + \sigma_2 \sum_{i=0}^{n-2} \beta_i, \quad E_n = ((2n-3)\sigma_2 + \tau_1). \end{aligned} \tag{38}$$

Proof. The monic classical orthogonal polynomial family $\{P_n\}_{n \geqslant 0}$ satisfies the following structure relation [2,3]:

$$\sigma(x) P_n'(x) = C_{n,n+1} P_{n+1}(x) + C_{n,n} P_n(x) + C_{n,n-1} P_{n-1}(x), \tag{39}$$

where the $C_{i,j}$ are constants. Using the relation [3] $P_n(x) = x^n + (-\sum_{i=0}^{n-1} \beta_i) x^{n-1} + \cdots$ and the orthogonality property, we obtain the following:

$$C_{n,n+1} = n\sigma_2, \quad C_{n,n-1} = -[\tau_1 + (n-1)\sigma_2]\gamma_n,$$

$$C_{n,n} = n\sigma_1 + \left(n\beta_n + \sum_{i=0}^{n-1} \beta_i \right) \sigma_2.$$

Using the recurrence relation (1), we get the structure relations (35) and (36). □

Theorem 4. *The monic orthogonal polynomial family* $\{\tilde{P}\}_{n \geqslant 0}$ *satisfies the following second order linear differential equation*:

$$
\begin{aligned}
&\sigma^2(x)[(a_n x + b_n)k_3(x;n) - (1-a_n)k_1(x;n)]\tilde{P}_n''(x)\\
&\quad - \sigma(x)[(a_n x + b_n)((n\sigma_2 x + J_n - \sigma'(x))k_3(x;n) + E_n k_1(x;n)\\
&\quad + \sigma(x)(a_n E_n + n(1-a_n)\sigma_2))\\
&\quad - (1-a_n)((C_n x + D_n - \sigma'(x))k_1(x;n)\\
&\quad + \sigma(x)k_2(x;n) + I_n k_3(x;n))]\tilde{P}_n'(x)\\
&\quad + [(n\sigma_2 x + J_n - C_n x - D_n)k_1(x;n)k_3(x;n) + E_n k_1^2(x;n)\\
&\quad + \sigma(x)(a_n E_n + n(1-a_n)\sigma_2)k_1(x;n)\\
&\quad - \sigma(x)k_2(x;n)k_3(x;n) - I_n k_3^2(x;n)]\tilde{P}_n(x) = 0,
\end{aligned}
\tag{40}
$$

where

$$
\begin{aligned}
k_1(x;n) &= a_n\sigma(x) + (a_n x + b_n)(C_n x + D_n) + (1-a_n)I_n,\\
k_2(x;n) &= a_n\sigma'(x) + a_n(C_n x + D_n) + C_n(a_n x + b_n),\\
k_3(x;n) &= E_n(a_n x + b_n) + (1-a_n)(n\sigma_2 x + J_n).
\end{aligned}
\tag{41}
$$

Proof. If we first differentiate (2), then multiply it by $\sigma(x)$ and use the above structure relations (35) and (36), we obtain:

$$
\sigma(x)\tilde{P}_n'(x) = k_1(x;n)P_{n-1}(x) + k_3(x;n)P_n(x), \quad n = 0, 1, 2, \ldots\ . \tag{42}
$$

In a similar way, we obtain:

$$
\begin{aligned}
\sigma^2(x)\tilde{P}_n''(x) = {} & [(C_n x + D_n - \sigma'(x))k_1(x;n) + \sigma(x)k_2(x;n)\\
& + I_n k_3(x;n)]P_{n-1}(x) + [(n\sigma_2 x + J_n - \sigma'(x))k_3(x;n) + E_n k_1(x;n)\\
& + \sigma(x)(a_n E_n + n(1-a_n)\sigma_2)]P_n(x), \quad n = 0, 1, 2, \ldots\ .
\end{aligned}
\tag{43}
$$

Eqs. (2), (42) and (43) are three equations with two unknowns $P_{n-1}(x)$ and $P_n(x)$. A solution will exist if and only if the augmented determinant is equal to zero.The expansion of this determinant gives the required equation. □

For the Chebyshev polynomials of the second kind $U_n(x)$, ($\sigma(x) = 1 - x^2$, $\tau(x) = -3x$, $\gamma_n = \frac{1}{4}$, $\beta_n = 0$), using Lemma 1, (41) and (40), we recover the results obtained in [5].

It is straightforward to apply the above formalism to derive new families of orthogonal polynomials $\{\tilde{P}_n\}$ from (2) for any usual classical orthogonal polynomials P_n like Laguerre, Jacobi and Hermite polynomials. For example, in the case of the modified Laguerre orthogonal polynomials, using (21), we obtain the following recurrence relation:

$$
\begin{aligned}
\tilde{P}_{n+2}(x) = {} & [x + b_{n+2} - b_{n+1} + (a_{n+2} - 1)(2n + \alpha + 3) - a_{n+1}(2n + \alpha + 1)]\tilde{P}_{n+1}(x)\\
& - n(n+\alpha)\frac{a_{n+1}}{a_n}\tilde{P}_n(x), \quad n = 0, 1, 2, \ldots, \quad a_n \neq 0,\\
\tilde{P}_0(x) = {} & 1 - a_0, \quad \tilde{P}_1(x) = x + b_1 + (\alpha + 1)(a_1 - 1).
\end{aligned}
\tag{44}
$$

The corresponding second order linear differential equation reads:

$$
\begin{aligned}
&x^2[d_2x^2 + d_1x + d_0]\tilde{P}_n''(x) - x[e_3x^3 + e_2x^2 + e_1x + e_0]\tilde{P}_n'(x)\\
&\quad [f_3x^3 + f_2x^2 + f_1x + f_0]\tilde{P}_n(x) = 0,
\end{aligned}
\tag{45}
$$

where

$$
\begin{aligned}
&d_2 = a_n, \quad d_1 = (2n+\alpha-1)(a_n-1)a_n + (a_n+1)b_n,\\
&d_0 = (n^2+n\alpha)(a_n-1)^2 + (2n+\alpha-2)(a_n+1)b_n + b_n^2,\\
&e_3 = -a_n,\\
&e_2 = (1-2n-\alpha)a_n^2 - (2-6n-2\alpha+b_n)a_n - b_n,\\
&e_1 = n^2 + n\alpha + (7n^2+5n\alpha-8n+\alpha^2-3\alpha+2)(a_n^2-a_n)\\
&\qquad + 2na_nb_n - (4-6n-2\alpha)b_n - b_n^2,\\
&e_0 = (-7n^3+6n^2-9n^2\alpha-n^3\alpha+6n\alpha-2n\alpha^2)a_n\\
&\qquad + (3n^3-3n^2+4n^2\alpha+n^3\alpha-3n\alpha+n\alpha^2)a_n^2\\
&\qquad - (-4n^3+3n^2-5n^2\alpha-n\alpha^2+3n\alpha) + (4n+\alpha-3)b_n^2\\
&\qquad + (8n^2-14n-5\alpha+6n\alpha+\alpha^2)(a_nb_n-b_n),\\
&f_2 = (6n^2+3n\alpha-n)a_n + (-2n^2+3n+\alpha-n\alpha+1)a_n^2,\\
&f_1 = -n^2(n+\alpha) + (4n^2-6n^3+n\alpha-6n^2\alpha-2n\alpha^2-n)a_n\\
&\qquad + (7n^3-4n^2+n+7n^2\alpha+2n\alpha^2-n\alpha)a_n^2 + (1-n)b_n^2\\
&\qquad + (6n^2+3n\alpha-3n)b_n + (3n+n\alpha+\alpha-3+2n^2)a_nb_n,\\
&f_0 = (n+\alpha)(4n^3+2n^2\alpha-2n^2) + (4n^2+2n\alpha-2n)b_n^2\\
&\qquad + (4n^3-8n^4+4n^2\alpha-12n^3\alpha-4n^2\alpha^2)a_n\\
&\qquad + (4n^4-2n^3-2n^2\alpha+2n^2\alpha^2+6n^3\alpha)a_n^2\\
&\qquad + (-8n^3+12n^2-4n-8n^2\alpha+6n\alpha-2n\alpha^2)b_n\\
&\qquad + (12n^3-12n^2+4n+12n^2\alpha+2n\alpha^2-6n\alpha+\alpha)a_nb_n.
\end{aligned}
$$

To conclude this paper, let us note that the complete analysis of the new orthogonal polynomials including their orthogonality measure, the location of the zeros with respect to the initial set of polynomials requires further more cumbersome work and will be thoroughly discussed in a forthcoming paper.

Acknowledgements

This work was finalized during the stay of one of the authors (M. N. H.) at the Center for Mathematical Sciences Research (Rutgers, The State University of New Jersey, USA). M. N. H. thanks Professor J. Lebowitz, Professor G. A. Goldin and all the staff of this center for their kind hospitality.

References

[1] T.S. Chihara, An Introduction to Orthogonal Polynomials, Gordon and Breach, New York, 1978.
[2] M. Foupouagnigni, Laguerre–Hahn orthogonal polynomials with respect to the Hahn operator: fourth-order difference equation for the rth associated and the Laguerre–Freud equations for the recurrence coefficients, Thése de Doctorat Unique, Université Nationale du Bénin, 1998.
[3] M. Foupouagnigni, M.N. Hounkonnou, A. Ronveaux, Laguerre–Freud equations for the recurrence coefficients of D_ω semi-classical orthogonal polynomials of class one, J. Comp. Appl. Math. 99 (1995) 143–154.
[4] J. Wimp, H. Kiesel, Some sequences arising in the creation of new orthogonal polynomials, Antwerp Conference on Nonlinear Numerical Methods and Rational Approximation, 1994, pp. 17–24.
[5] J. Wimp, H. Kiesel, Non-linear recurrence relations and some derived orthogonal polynomials, Ann. Numer. Math. 2 (1995) 169–180.