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# Numbers Generated by the Reciprocal of $e^x - x - 1$

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**Abstract.** In this paper we examine the polynomials $A_n(z)$ and the rational numbers $A_n = A_n(0)$ defined by means of

$$e^{xz}x^2(e^x - x - 1)^{-1} = 2 \sum_{n=0}^{\infty} A_n(z)x^n/n!.$$

We prove that the numbers $A_n$ are related to the Stirling numbers and associated Stirling numbers of the second kind, and we show that this relationship appears to be a logical extension of a similar relationship involving Bernoulli and Stirling numbers. Other similarities between $A_n$ and the Bernoulli numbers are pointed out. We also reexamine and extend previous results concerning $A_n$ and $A_n(z)$. In particular, it has been conjectured that $A_n$ has the same sign as $-\cos n\theta$, where $re^{i\theta}$ is the zero of $e^x - x - 1$ with smallest absolute value. We verify this for $1 \leqslant n \leqslant 14329$ and show that if the conjecture is not true for $A_n$, then $|\cos n\theta| < 10^{-(n-1)/5}$. We also show that $A_n(z)$ has no integer roots, and in the interval $[0, 1]$, $A_n(z)$ has either two or three real roots.

1. **Introduction.** Define the rational numbers $A_0, A_1, A_2, \ldots$ by means of

$$\left(\sum_{n=0}^{\infty} \frac{2x^n}{(n+2)!}\right)^{-1} = \frac{x^2/2}{e^x - x - 1} = \sum_{n=0}^{\infty} A_n \frac{x^n}{n!}. \tag{1.1}$$

This definition is apparently due to L. Carlitz [4], who raised the question of whether a theorem like the Staudt-Clausen theorem holds for the numbers $A_n$. Because of the obvious similarity of (1.1) to the definition of the Bernoulli numbers $B_n$, i.e.

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}, \tag{1.2}$$

this seems to be a reasonable question. The writer [6] has shown, however, that evidently such a theorem does not hold: If $p$ is any odd prime, then

$$p^m A_{m(p-2)}/[m(p-2)]! \equiv 2^m \pmod p, \tag{1.3}$$

which implies that arbitrarily large powers of $p$ will divide the denominator of some $A_n$. However, for $n > 1$,

$$2A_n \equiv 1 \pmod 4, \tag{1.4}$$

so the denominator of $A_n$, for $n > 1$, is even and not divisible by 4. This last property is also true of the Bernoulli numbers $B_{2n}$.

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In the present paper we reexamine questions raised in [6] and [8] about $A_n$, and we attempt to clarify and extend the results in those papers. We also prove that the numbers $A_n$ are related to the Stirling numbers of the second kind, and we show that this relationship appears to be a logical extension of a similar relationship involving Bernoulli and Stirling numbers. The goal of the present paper is to show that (1.1) is a natural definition to make and that the $A_n$ are of interest in their own right. A summary by sections follows.

In Section 2 we examine a conjecture made in [8] about the sign of $A_n$. We prove that if $re^{i\theta}$ is the zero of $e^x - x - 1$ with smallest absolute value, then $A_n$ has the same sign as $-\cos n\theta$ if $|\cos n\theta| > 10^{-(n-1)/5}$. We show that $A_n$ does indeed have the same sign as $-\cos n\theta$ for $1 \leqslant n \leqslant 14329$.

In Section 3 we examine the polynomials $A_n(z)$, defined in [6] by means of

$$(1.5) \qquad \frac{(x^2/2)e^{xz}}{e^x - x - 1} = \sum_{n=0}^{\infty} A_n(z) \frac{x^n}{n!}.$$

We prove that if $n > 1$, $A_n(z)$ has either two or three real roots in the closed interval $[0, 1]$. We show that $A_n(z)$ has no integer roots and $A_{2n}(z)$ has no rational roots. For special values of $n$ we show $A_n(z)$ is irreducible over the rational field.

In Section 4 we prove some general theorems for numbers generated by the reciprocal of any series. We show that, in a sense, there is always an explicit formula for these numbers, and there is also a way of expressing these numbers as a linear combination of numbers which have a combinatorial interpretation.

In Section 5 we apply the theorems of Section 4 to $A_n$. We show how $A_n$ can be expressed in terms of the Stirling numbers of the second kind and the associated Stirling numbers of the second kind.

In Section 6 we prove some miscellaneous results for $A_n$ and $A_n(z)$. We show that $2|A_n| < |B_n|$, if $n$ is even; and we prove some theorems concerning possible rational roots of $A_n(z)$, if $n$ is odd. We include in this section a table of values of $\cos n\theta$, $1 \leqslant n \leqslant 46$, where $re^{i\theta}$ is the zero of $e^x - x - 1$ with smallest absolute value. We also include a table indicating how the sign of $A_n$ changes for $1 \leqslant n \leqslant 14329$.

All calculations in this paper were performed on a Texas Instruments SR-50A calculator. This machine computes to thirteen significant digits and rounds off to ten significant digits.

We note that a listing of the first 15 numbers $A_n$ can be found in [6].

2. **Sign of $A_n$.** It is pointed out in [8] that by using Hadamard's factorization theorem [17, p. 250], we can write $2A_n = -n! \sum_{s=1}^{\infty} (x_s)^{-n}$, where $x_1, x_2, \ldots$ are the zeros of $e^x - x - 1$. Using

$$x_s = r_s e^{i\theta_s}, \quad \overline{x_s} = r_s e^{-i\theta_s},$$

we can write

$$A_n = -n! \sum_{s=1}^{\infty} r_s^{-n} \cos n\theta_s. \tag{2.1}$$

If we let $x_1$ be the zero with smallest absolute value, the following conjecture was made in [8].

*Conjecture.* For $n > 0$, $A_n$ has the same sign as $-\cos n\theta_1$.

We shall refer to this as "the sign conjecture", and we shall show that it is true at least for $n \leqslant 14329$. In [8] the conjecture was verified for $n \leqslant 37$.

It is not too difficult to find approximations for $x_s$. If we set $e^x - x - 1 = 0$ and let $x = a + bi$, we see that

$$a = b \cot b - 1 = \ln b - \ln(\sin b), \quad (\sin b)\exp(b \cot b) = eb;$$

and by examining the graphs of $e^x$ and $(\sin x)\exp(x \cot x)$, we see that

$$(2n + 1/4)\pi < b < (2n + 1/2)\pi, \qquad n = 1, 2, \ldots . \tag{2.2}$$

We can compute the following approximations: $x_1 = a + bi$, with

$$\begin{aligned} 2.08884300 &< a < 2.08884302, \\ 7.461489270 &< b < 7.461489300, \\ 74.360416^\circ &< \theta_1 < 74.360417^\circ, \\ 7.748360 &< r_1 < 7.748361. \end{aligned} \tag{2.3}$$

From (2.1) we see that $A_n$ has the same sign as $-\cos n\theta_1$ if

$$|\cos n\theta_1| > \left| \sum_{s=2}^{\infty} \left(\frac{r_1}{r_s}\right)^n \cos n\theta_s \right| .$$

Since, by (2.2) and (2.3),

$$\sum_{s=2}^{\infty} \left(\frac{r_1}{r_s}\right)^n < \sum_{s=2}^{\infty} \left(\frac{7.75}{2\pi s}\right)^n < \left(\frac{5}{4}\right)^n \sum_{s=2}^{\infty} s^{-n},$$

we have the following theorem.

THEOREM 2.1. *If* $|\cos n\theta_1| > (5/4)^n \Sigma_{s=2}^{\infty} s^{-n}$, *then* $A_n$ *has the same sign as* $-\cos n\theta_1$.

The sum in Theorem 2.1 is very small for large $n$. In fact, it is not difficult to show it is less than $(5/8)^{n-1}$ and hence less than $10^{-(n-1)/5}$.

COROLLARY. *If* $|\cos n\theta_1| > 10^{-(n-1)/5}$, *then* $A_n$ *has the same sign as* $-\cos n\theta_1$.

The values of $\cos n\theta_1$ have been computed for $1 \leqslant n \leqslant 46$ and are included in Section 6. We see that the sign conjecture holds for $1 \leqslant n \leqslant 46$, the smallest value of $\cos n\theta_1$, being .005 when $n = 23$. We have the following approximations modulo 360 degrees.

$$23\theta_1 = 270.289^\circ, \quad 46\theta_1 = 180.579^\circ,$$

$$69\theta_1 = 90.869^\circ, \quad 92\theta_1 = 1.158^\circ.$$

We see that if $n = 46 + k$, $0 < k < 46$, then $A_n$ and $A_k$ have different signs; the exact opposite of the original pattern of signs occurs for $46 < n < 92$. ($A_0$ is a special case for which the sign conjecture is not true.) In fact, we expect $A_{46+k}/(46+k)!$ to be approximately $-(r_1)^{-46}A_k/k!$. Also, the pattern of signs for $0 < n \leqslant 92$ will be repeated for $92 < n \leqslant 184$; that is, $A_{92+k}$ and $A_k$ will have the same sign for $k = 1, 2, \ldots, 92$. The following theorem tells exactly what the signs are for $1 \leqslant n \leqslant 327$.

THEOREM 2.2. *For positive n, let* $n = 46k + s$, $0 \leqslant k \leqslant 6$, $0 \leqslant s < 46$. *Let* $s = 12m + t$, $0 \leqslant t < 12$. *If* $t = 0, 1, 4, 5, 6, 9$ *or* $10$, *then* $(-1)^{k+m+1}A_n > 0$. *If* $t = 2, 3, 7$ *or* $8$, *then* $(-1)^{k+m}A_n > 0$. *If* $s = 11$ *or* $23$, *then* $(-1)^{k+m}A_n > 0$. *If* $s = 35$, *then* $(-1)^{k+1}A_n > 0$.

As $n$ gets larger, the discrepancy between $46\theta_1$ and 180 degrees begins to make a difference. Using Table 1 in Section 6, we see that the first change in the pattern of Theorem 2.2 occurs at $n = 328 = 46 \cdot 7 + 6$. That is, as $k$ increases from 0 to 7, the angle $(46k + 6)\theta_1$ changes in the following way (approximately): 86°, 267°, 87°, 268°, 88°, 269°, 89.637°, 270.2166°. If $n = 46k + s$, $7 \leqslant k \leqslant 12$, the pattern of Theorem 2.2 holds with one exception: if $n = 46k + 6$, then $(-1)^kA_n > 0$. As $n$ gets larger, the pattern will continue to change. Table 2 in Section 6 indicates when the pattern of Theorem 2.2 changes for various values of $s$. When $n = 633 = 46 \cdot 13 + 35$, for example, the pattern changes for numbers of the form $46k + 35$; i.e. $(-1)^kA_{46k+35} > 0$. By checking the value of $\cos n\theta_1$ at the numbers given in Table 2, and also at $n = 46(k - 1) + s$, we see, by the corollary to Theorem 2.1, that $A_n$ has the same sign as $-\cos n\theta_1$ for $1 \leqslant n \leqslant 14329$. The smallest value of $\cos n\theta_1$ for $1 \leqslant n \leqslant 14329$ occurs when $n = 1243$ and is about .00004. We have used the approximation $74.360416 < \theta_1 < 74.360417$ in these calculations. We see by the corollary to Theorem 2.1 that if the sign conjecture is not true for $A_n$, then $|\cos n\theta_1| < 10^{-2865}$.

THEOREM 2.3. *For* $n > 0$, *we never have* $A_n > 0, A_{n+1} < 0, A_{n+2} > 0$ *or* $A_n < 0, A_{n+1} > 0, A_{n+2} < 0$.

*Proof.* Suppose $A_n > 0, A_{n+1} < 0, A_{n+2} > 0$. Since $\theta_1$ is about 74 degrees, it is clear the sign conjecture does not hold for at least one of $n$, $n + 1$ or $n + 2$. Suppose $A_n$ does not have the same sign as $-\cos n\theta_1$. Then by the corollary to Theorem 2.1, $n\theta_1$ is within one degree (modulo 360 degrees) of either 90 or 270 degrees. It is then clear that the sign conjecture does hold for $A_{n+1}$ and $A_{n+2}$, and, in fact, they both must have the same sign, which is a contradiction. If the sign conjecture does not holds for $A_{n+1}$, we see that $A_n$ and $A_{n+2}$ must have opposite signs, and if the sign conjecture is not true for $A_{n+2}$, we see that $A_n$ and $A_{n+1}$ must have the same sign. The reasoning is similar if $A_n < 0, A_{n+1} > 0, A_{n+2} < 0$.

Using the same kind of reasoning, we have the following theorem.

THEOREM 2.4. *For* $n \geqslant 0$, *we never have four consecutive numbers* $A_n$, $A_{n+1}$, $A_{n+2}$, $A_{n+3}$ *with the same sign.*

Because of (2.1) and the fact that

$$\sum_{s=2}^{\infty} (r_1/r_s)^n < (5/8)^{n-1},$$

we see that, for $n \geqslant 20$, if $|\cos(n+1)\theta_1| - r_1|\cos n\theta_1| > .001$, then $|A_{n+1}| > (n+1)|A_n|$. On the other hand, if $r_1|\cos n\theta_1| > 1.001$, then $(n+1)|A_n| > |A_{n+1}|$. Thus we have the following theorem, which actually holds for all $n \geqslant 0$.

THEOREM 2.5. *If* $|\cos n\theta_1| \leqslant .118$, *then* $|A_{n+1}| > (n+1)|A_n|$. *If* $|\cos n\theta_1| \geqslant .1292$, *then* $(n+1)|A_n| > |A_{n+1}|$.

Usually $(n+1)|A_n| > |A_{n+1}|$, but this is not true for many values of $n$ including

$$n = 46k + 6, \qquad 0 \leqslant k \leqslant 6,$$
$$n = 46k + 35, \qquad 2 \leqslant k \leqslant 12,$$
$$n = 46k + 18, \qquad 9 \leqslant k \leqslant 19.$$

For these particular values of $n$, $A_n$ and $A_{n+1}$ have opposite signs, a fact that is important when we are examining the real roots of $A_{n+1}(z)$. Of course there are cases, like $n = 23$, when $A_n$ and $A_{n+1}$ have the same sign and $(n+1)|A_n| < |A_{n+1}|$.

**3. The Polynomials** $A_n(z)$. It was proved in [8] that the polynomial $A_n(z)$ defined by (1.5) has at least one real root in the closed interval $[0, 1]$ for $n > 0$. In this section we show that $A_n(z)$ has either two or three real roots in $[0, 1]$, and in addition we prove that $A_{2n}(z)$ has no rational roots for $n \geqslant 0$. For a few specific values of $n$, we show that $A_n(z)$ is irreducible over the rational field. These results can be compared to similar properties of the Bernoulli and Euler polynomials [1], [2], [9], [10], [15].

In [6] the following formulas were proved.

$$A_n(z) = \sum_{r=0}^{n} \binom{n}{r} A_r z^{n-r}, \tag{3.1}$$

$$A_n'(z) = nA_{n-1}(z), \tag{3.2}$$

$$A_n(z+1) - A_n(z) - A_n'(z) = \binom{n}{2} z^{n-2} \quad \text{for } n > 1. \tag{3.3}$$

It follows from (3.2) and (3.3) that

$$\int_0^1 A_n(z)\, dz = A_n, \tag{3.4}$$

and more generally

$$\int_y^{y+1} A_n(z)\, dz = A_n(y) + ny^{n-1}/2. \tag{3.5}$$

In the theorems that follow, we assume $u/b$ is a rational number reduced to its lowest terms. Also, we note that

$$A_0(z) = 1, \quad A_1(z) = z - 1/3,$$

so $A_1(z)$ does have the rational root $1/3$.

THEOREM 3.1. *If* $A_n(u/b) = 0$, *then* $b = 3$ *and* $u \equiv n \equiv 1 \pmod 3$.

*Proof.* By (3.1) we have

$$\frac{3^n}{n!} A_n(z) = \sum_{r=0}^{n} \frac{3^r}{r!} A_r \frac{3^{n-r}}{(n-r)!} z^{n-r},$$

and since $3^{n-r}/(n-r)! \equiv 0 \pmod 3$, unless $r = n$, we have, by (1.3),

$$3^n A_n(z)/n! \equiv (-1)^n \pmod 3.$$

It follows that if $u/b$ is a root then $b \equiv 0 \pmod 3$. Otherwise we have $(-1)^n \equiv 0$

(mod 3). (See also Lemma 2.3 in [7].) We have, from (3.1),

$$(3.6) \qquad 0 = \frac{u^n}{b} - \frac{nu^{n-1}}{3} + \binom{n}{2}\frac{u^{n-2}b}{18} + \sum_{r=3}^{n}\binom{n}{r}A_r u^{n-r}b^{r-1}.$$

In [6] it is shown that if $m = [n/(p-2)] + 1$, $p$ an odd prime, then $p^m A_n/n! \equiv 0 \pmod{p}$. Thus $b^{r-1}A_r$ is integral (mod $b$) for $r > 2$, and we see that

$$(3.7) \qquad \frac{u^n}{b} - \frac{nu^{n-1}}{3} + \binom{n}{2}\frac{u^{n-2}b}{18}$$

must be integral (mod 3); i.e., the above sum is a rational number with denomination not divisible by 3. For any prime $p \neq 3$, let $p^s$ be the highest power of $p$ dividing $b$. Then if $s > 0$,

$$0 \equiv p^s u^n/b \not\equiv 0 \pmod{p},$$

by (3.6), which is impossible. Now suppose $b = 3^s$. If $s > 1$, we see from (3.6) that $0 \equiv u^n \pmod 3$, a contradiction since g.c.d. $(u, 3) = 1$. Hence $b = 3$, and since (3.7) must be integral (mod 3), we must have $u \equiv n \equiv 1 \pmod 3$.

Theorem 3.1 shows that no polynomial $A_n(z)$ has an integer root.

THEOREM 3.2. *For $n \geqslant 0$, $A_{2n}(z)$ has no rational roots.*

*Proof.* By (1.4) and (3.1), we have, for any $k \geqslant 2$,

$$2A_k(z) \equiv \sum_{r=2}^{k}\binom{k}{r}z^{k-r} + 2z^k + 2kz^{k-1}$$

$$\equiv (1+z)^k + z^k + kz^{k-1} \pmod 4.$$

If $k = 2n$, we see that $2A_{2n}(u/3) \equiv 1 \pmod 2$, so $u/3$ cannot be a root of $A_{2n}(z)$.

Unfortunately, it is not clear whether or not $A_{2n+1}(z)$ can have rational roots. If we let $k = 2n + 1$ in the proof of Theorem 3.2, the only conclusion we can draw is that $u$ is odd and $u \equiv 2n + 1 \pmod 4$. We do know by Theorems 3.1 and 3.2 that if $A_n(u/3) = 0$, then $n \equiv 1 \pmod 6$. Furthermore, it can be proved that if $p - 2$ divides $n$, where $p$ is any prime number larger than 3, then $A_n(z)$ does not have a rational root. Also, if $A_n(1/3) = 0$, $n > 1$, then $n \equiv 1 \pmod{36}$. These last two results are proved in Section 6.

Next we examine the real roots of $A_n(z)$ on the closed interval $[0, 1]$.

LEMMA 3.1. *If $n > 1$, then $A_n(z)$ has at least two real roots in* $[0, 1]$.

*Proof.* We shall consider four different cases, using (3.2), (3.3), (3.4).

*Case* 1. $A_n > 0$, $A_{n+1} > 0$. We see that $A_{n+1}(z)$ is an increasing function at $z = 0$ and that $A_{n+1}(1) > A_{n+1}(0)$. It follows from (3.4) that the area bounded by $A_{n+1}(z)$, the $x$-axis and the lines $x = 0$, $x = 1$ is exactly $A_{n+1} = A_{n+1}(0)$. Thus for some values of $z$ we must have $A_{n+1}(z) < A_{n+1}$, and we see there must be at least two "critical points" on the graph, i.e., there are two real numbers $a$ and $b$, $0 < a < b < 1$, such that $0 = A'_{n+1}(a) = A'_{n+1}(b)$. Thus $A_n(a) = 0 = A_n(b)$. The case $A_n < 0$, $A_{n+1} < 0$ is similar.

*Case* 2. $A_n < 0$, $A_{n+1} > 0$. In this case $A_{n+1}(1) < A_{n+1}(0)$ and $A_{n+1}(z)$ is a decreasing function at $z = 0$. As in Case 1, we see there must be at least two real numbers $a$ and $b$ such that $A'_{n+1}(a) = 0 = A'_{n+1}(b)$. The case $A_n > 0, A_{n+1} < 0$ is similar.

LEMMA 3.2. *If $n \geqslant 0$, then $A_n(z)$ has no more than three real roots in* $[0, 1]$.

*Proof.* Suppose $n$ is the smallest positive integer such that $A_n(z)$ has more than three real roots in $[0, 1]$. Then $n > 3$.

*Case* 1. $A_n > 0, A_{n-1} > 0$. Since $A_n(z)$ is increasing at $z = 0$, we see that there must be at least four critical points on the graph of $A_n(z)$. This implies that $A_{n-1}(z)$ has at least four real roots in $[0, 1]$, a contradiction. The case $A_n < 0, A_{n-1} < 0$ is similar. It is clear that if the lemma is true for $A_n(z)$, and $A_n$ and $A_{n-1}$ have the same sign, then $A_n(z)$ has exactly two real roots in $[0, 1]$.

*Case* 2. $A_n > 0, A_{n-1} < 0, A_n(1) < 0$. If $A_n(z)$ has at least four real roots in $[0, 1]$, it is clear there are at least four critical points on the graph of $A_n(z)$. This implies $A_{n-1}(z)$ has at least four real roots in $[0, 1]$, a contradiction. The case $A_n < 0, A_{n-1} > 0, A_n(1) > 0$ is similar.

*Case* 3. $A_n > 0, A_{n-1} < 0, A_n(1) > 0$. By Theorem 2.3 we know $A_{n-2} < 0$, and from Case 1 we know $A_{n-1}(z)$ has exactly two real roots in $[0, 1]$. If $A_n(z)$ has at least four real roots in $[0, 1]$, there are at least three critical points on the graph of $A_n(z)$, which is impossible. The case $A_n < 0, A_{n-1} > 0, A_n(1) < 0$ is similar.

LEMMA 3.3. *If $n \geqslant 0$, $A_n(z)$ has no multiple real roots in* $[0, 1]$.

*Proof.* Suppose $n$ is the smallest positive integer such that $A_n(z)$ has a multiple root. By (3.2) it must be a double root.

*Case* 1. $A_n > 0, A_{n-1} > 0$. We know $A_n(z)$ is increasing at $z = 0$; $A_n(1) > A_n(0)$, and $A_n(z)$ has exactly two distinct real roots in $[0, 1]$. We see, then, that a double root implies four critical points on the graph of $A_n(z)$, a contradiction. The case $A_n < 0, A_{n-1} < 0$ is similar.

*Case* 2. $A_n > 0, A_{n-1} < 0, A_n(1) < 0$. The only possibility is that $A_n(z)$ has exactly two real roots in $[0, 1]$, one of them a double root. By Theorem 2.3, we know $A_{n+1} > 0$, so $A_{n+1}(z)$ has exactly two real roots in $[0, 1]$. Also, $A_{n+1}(z)$ is decreasing at $z = 1$, since $A_n(1) < 0$, and is increasing at $z = 0$. This implies there are at least three critical points on the graph of $A_{n+1}(z)$, a contradiction. The case $A_n < 0, A_{n-1} > 0, A_n(1) > 0$ is similar.

*Case* 3. $A_n > 0, A_{n-1} < 0, A_n(1) > 0$. Since $A_n(z)$ has at least two distinct real roots in $[0, 1]$, a double root implies at least three critical points on the graph of $A_n(z)$. We know, however, that $A_{n-1}$ has exactly two real roots in $[0, 1]$ since $A_{n-2} < 0$. The case $A_n < 0, A_{n-1} > 0, A_n(1) < 0$ is similar.

By Lemmas 3.1, 3.2 and 3.3, we have the following theorem.

THEOREM 3.3. *Suppose $n > 1$. Then $A_n(z)$ has no multiple real roots in* $[0, 1]$, *and*

(a) *if $A_n$ and $A_{n-1}$ have the same sign, then $A_n(z)$ has exactly two real roots in* $[0, 1]$.

(b) *if $A_n$ and $A_{n-1}$ have opposite signs, and if $n|A_{n-1}| > |A_n|$, then $A_n(z)$ has exactly three real roots in* $[0, 1]$.

(c) *if* $A_n$ *and* $A_{n-1}$ *have opposite signs, and if* $n|A_{n-1}| < |A_n|$, *then* $A_n(z)$ *has exactly two real roots in* $[0, 1]$.

By (3.3), the condition $n|A_{n-1}| > |A_n|$ is equivalent to $A_n(1)$ having the same sign as $A_{n-1}$, if $A_n$ and $A_{n-1}$ have different signs. Similarly, the condition $n|A_{n-1}| < |A_n|$ is equivalent to $A_n(1)$ having the same sign as $A_n$. By Theorem 2.5 and the remarks following it, we see that usually $A_n(1)$ has the same sign as $A_{n-1}$. However, this is not the case for many values of $n$, such as $n = 46k + 6$, $0 \leqslant k \leqslant 6$.

It is not clear how the roots of $A_n(z)$ are distributed outside the interval $[0, 1]$. If $y > 0$ and $A_n(y) < 0$, it follows from (3.5) that $A_n(y)$ has at least one real root between $y$ and $y + 1$. This is because $A_{n+1}(z)$ is decreasing at $z = y$ and

$$\int_y^{y+1} A_n(z)\, dz > A_{n+1}(y),$$

so there must be at least one real number $a$, $y < a < y + 1$, such that $A'_{n+1}(a) = 0 = A_n(a)$. By the same type of reasoning, if $y < 0$ and $A_{2n}(z) < 0$, then $A_{2n}(z)$ has at least one real root between $y - 1$ and $y$. If $y < 0$ and $A_{2n+1}(y) > 0$, then $A_{2n+1}(z)$ has at least one real root between $y - 1$ and $y$. The distributions of the real roots of the Bernoulli and Euler polynomials can be found in [10] and [9] respectively.

Eisenstein's irreducibility criterion has been used to show that certain Bernoulli, Euler and van der Pol polynomials are irreducible over the rational field. The same method can be used on $A_n(z)$.

THEOREM 3.4. *If* $n = 2^k$, $k \geqslant 0$, *or* $n = m(p - 2)$ *where* $p$ *is an odd prime*, $2m < p$, *then* $A_n(z)$ *is irreducible over the rational field.*

*Proof.* If $n = 2^k$, we have

$$2A_n(z) = 2 \sum_{r=0}^{n} \binom{n}{r} A_r z^{n-r} \equiv 2A_n \equiv 1 \pmod 2,$$

and furthermore $2A_0 \not\equiv 0 \pmod 4$. Thus $2A_n(z)$ is an Eisenstein polynomial and is irreducible over the rational field. Suppose $2m < p$. From a theorem in [6], we know that if $r$ is in any of the intervals $[0, p - 2)$, $[p, 2(p - 2))$, $\ldots$, $[(m - 1)p, m(p - 2))$, then $A_r$ is integral (mod $p$), and also $p^2 A_r \equiv 0 \pmod p$ for $0 \leqslant r \leqslant m(p - 2)$. We see, by (1.3), that if $n = m(p - 2)$ then $pA_n$ is an Eisenstein polynomial.

**4. The Reciprocal of a Series.** In this section we prove some theorems that are true for the reciprocal of any power series. Some of our results can be proved by using generalized chain rule differentiation formulas; instead we shall generalize methods used by Jordan [12] and Riordan [16]. We do not claim these results are new, though references are somewhat hard to find. Perhaps [14] is a good general reference. The goal of this and the subsequent section is to show how the numbers $A_n$ are related to the Stirling numbers, and associated Stirling numbers, of the second kind.

Suppose $a_0 + a_1 x + a_2 x^2 + \cdots$ is a given power series, $a_0 \neq 0$. We shall assume that the series has a positive radius of convergence, though this condition is not really necessary for the theorems of this section. Define the numbers $c_n$ by means of

$$\left(\sum_{r=0}^{\infty} a_r x^r\right)^{-1} = \sum_{n=0}^{\infty} c_n x^n. \tag{4.1}$$

Then $c_0 = 1/a_0$ and $\Sigma_{i=0}^n a_i c_{n-i} = 0$. By Cramer's rule, we have the following theorem [13, p. 116]:

THEOREM 4.1. *If $c_n$ is defined by* (4.1), *then*

$$c_n = \frac{(-1)^n}{(a_0)^{n+1}} \begin{vmatrix} a_1 & a_0 & 0 \cdots 0 \\ a_2 & a_1 & a_0 \cdots 0 \\ & \cdots & \\ a_n & a_{n-1} & a_{n-2} \cdots a_1 \end{vmatrix}.$$

An alternate approach is the following:

$$\begin{aligned} a_0 \sum_{n=0}^{\infty} c_n x^n &= \left( \sum_{n=0}^{\infty} \frac{a_n}{a_0} x^n \right)^{-1} \\ &= \left( 1 + \sum_{n=1}^{\infty} \frac{a_n}{a_0} x^n \right)^{-1} = \sum_{j=0}^{\infty} (-1)^j \left( \sum_{n=1}^{\infty} \frac{a_n}{a_0} x^n \right)^j. \end{aligned}$$

By comparing coefficients of $x$, we have the next theorem.

THEOREM 4.2. *If $c_n$ is defined by* (4.1), *then for $n > 0$,*

$$c_n = \sum_{j=1}^{n} (-1)^j a_{k_1} \cdots a_{k_j} / (a_0)^{j+1}$$

*where for each $j$ the sum is over all compositions (ordered partitions) $k_1 + \cdots + k_j = n$, each $k_i \geqslant 1$.*

In Theorem 4.2 the order of the numbers $k_1, \ldots, k_j$ is important. For example, $1 + 3$ is not considered the same composition of 4 as $3 + 1$.

Define $F(n, j)$ by means of

$$\left( \sum_{r=1}^{\infty} a_r x^r \right)^j = \sum_{n=j}^{\infty} j! F(n, j) \frac{x^n}{n!}. \tag{4.2}$$

Then

$$j! F(n, j) = n! \sum a_{k_1} \cdots a_{k_j}, \tag{4.3}$$

where the sum is over all compositions $k_1 + \cdots + k_j = n$, each $k_i \geqslant 1$. Comparing (4.3) with Theorem 4.2, we have the next theorem.

THEOREM 4.3. *If $c_n$ is defined by* (4.1) *and $F(n, j)$ is defined by* (4.2), *then*

$$n! c_n = \sum_{j=1}^{n} (-1)^j j! (a_0)^{-j-1} F(n, j).$$

The number $F(n, j)$ has the following interpretation [5], [16, pp. 74–78]: Consider all the partitions of the set $\{1, 2, \ldots, n\}$ into $j$ nonempty subsets (called *blocks* of the set partition). Assign a "weight" of $k! a_k$ to each block which has exactly $k$ elements. For each set partition there is a weight, found by multiplying the weights of the $j$ blocks making up the partition. Then $F(n, j)$ is the sum of the weights of all the set partitions of $\{1, 2, \ldots, n\}$ consisting of $j$ blocks.

For example, to compute $F(4, 2)$, we see there are three set partitions with weight $4a_2^2$ and four set partitions with weight $6a_1a_3$. Thus $F(4, 2) = 12a_2^2 + 24a_1a_3$.

If we define $F_n(s)$ by means of

$$\sum_{n=0}^{\infty} F_n(s)\frac{x^n}{n!} = \exp\left(s\sum_{r=1}^{\infty} a_r x^r\right), \tag{4.4}$$

we see that

$$F_n(s) = \sum_{j=1}^{n} F(n, j)s^j. \tag{4.5}$$

If a generating function is written in the form

$$a_m x^m\left(\sum_{r=m}^{\infty} a_r x^r\right)^{-1} = \sum_{n=0}^{\infty} d_n x^n, \tag{4.6}$$

where $m$ is a fixed nonnegative integer, $a_m \neq 0$, it is perhaps more convenient to proceed as follows. We have $d_0 = 1$, and for $n > 0$ we have, by Theorem 4.2,

$$d_n = \sum_{j=1}^{n} (-1)^j a_{k_1+m} \cdots a_{k_j+m}/(a_m)^j, \tag{4.7}$$

where the sum is over all compositions $k_1 + \cdots + k_j = n$, each $k_i \geqslant 1$. For $t \geqslant 0$ define $G_{t,n}(s)$ and $G(t; n, j)$ by means of

$$\sum_{n=0}^{\infty} G_{t,n}(s)\frac{x^n}{n!} = \exp\left(s\sum_{r=t+1}^{\infty} a_r x^r\right), \tag{4.8}$$

$$G_{t,n}(s) = \sum_{j=1}^{[n/t+1]} G(t; n, j)s^j. \tag{4.9}$$

Then

$$j!G(t; n, j) = \sum n!a_{k_1} \cdots a_{k_j}, \tag{4.10}$$

where the sum is over all compositions $k_1 + \cdots + k_j = n$, each $k_i \geqslant t + 1$. The number $G(t; n, j)$ has the same interpretation as $F(n, j)$, except each block used in a set partition of $\{1, \ldots, n\}$ must contain at least $t + 1$ elements. For example, $G(1; 4, 2) = 12a_2^2$ and $G(2; 4, 2) = 0$. By (4.7) and (4.10) we have

$$d_n = \sum_{j=1}^{n} (-1)^j j!(a_m)^{-j} G(m; n + mj, j)/(n + mj)!. \tag{4.11}$$

By using the principle of inclusion-exclusion and the identity

$$\sum_{j=r}^{n}\binom{j}{r} = \binom{n+1}{r+1}$$

(see also the derivation of formula 18 in [12, p. 598]), we can derive the formula

$$d_n = \sum_{j=1}^{n} (-1)^j j!(a_m)^{-j}\binom{n+1}{j+1} G(m-1; n + mj, j)/(n + mj)!. \tag{4.12}$$

So if $c_n$ is defined by (4.1) and $d_n$ by (4.6), it is always possible to write "explicit" formulas for $c_n$ and $d_n$, as shown by Theorem 4.2 and (4.7). It is also possible to write $c_n$ and $d_n$ as linear combinations of numbers which have a combinatorial interpretation, as shown by Theorem 4.3, (4.11) and (4.12). The next theorem shows it is always possible to find an application for the numbers $c_n$ and $d_n$ (see [12, pp. 587–599]).

THEOREM 4.4. *If $c_n$ is defined by* (4.1) *and $f(x)$, $h(x)$ are functions defined for positive integers $x$, then*

$$h(n) = \sum_{i=0}^{n-1} a_i f(n-i) \tag{4.13}$$

*if and only if*

$$f(n) = \sum_{m=0}^{n-1} c_m h(n-m). \tag{4.14}$$

*Proof.* Suppose (4.13) holds. Then

$$\begin{aligned}\sum_{n=1}^{\infty} h(n)x^{n-1} &= \sum_{n=1}^{\infty} x^{n-1} \sum_{i=0}^{n-1} a_i f(n-i) \\ &= \sum_{i=0}^{\infty} a_i x^i \sum_{n=i+1}^{\infty} f(n-i)x^{n-i-1} \\ &= \left(\sum_{i=0}^{\infty} c_i x^i\right)^{-1} \sum_{n=i+1}^{\infty} f(n-i)x^{n-i-1}.\end{aligned}$$

This implies

$$\left(\sum_{n=0}^{\infty} c_i x^i\right)\left(\sum_{n=1}^{\infty} h(n)x^{n-1}\right) = \sum_{n=1}^{\infty} f(n)x^{n-1}$$

and (4.14) follows. If we assume (4.14), we use a similar method to prove (4.13).

We note that several formulas in [12, pp. 219, 247, 599] involving the Bernoulli numbers are special cases of the theorems of this section.

**5. Relationship of $A_n$ to the Stirling Numbers.** We now apply the results of Section 4 to the numbers $A_n$. From (1.1) and (4.7) we have, for $n > 0$,

$$A_n = n! \sum_{j=1}^{n} \frac{(-2)^j}{(k_1+2)! \cdots (k_j+2)!}, \tag{5.1}$$

the sum being over all compositions $k_1 + \cdots + k_j = n$, each $k_i \geqslant 1$. This can be compared to a similar formula for the Bernoulli numbers [12, p. 247]:

$$B_n = n! \sum_{j=1}^{n} \frac{(-1)^j}{(k_1+1)! \cdots (k_j+1)!}.$$

To find formulas corresponding to (4.11) and (4.12), we define $b_{t,n}(s)$ and $b(t; n, j)$ by means of

$$\sum_{n=0}^{\infty} b_{t,n}(s)\frac{x^n}{n!} = \exp(s(e^x - 1 - \cdots - x^t/t!)) \tag{5.2}$$

and

$$b_{t,n}(s) = \sum_{j=1}^{[n/t+1]} b(t; n, j)s^j. \tag{5.3}$$

Then (5.2) and (5.3) imply

$$(e^x - 1 - x - \cdots - x^t/t!)^j = \sum_{n=tj}^{\infty} j!b(t; n, j)\frac{x^n}{n!}. \tag{5.4}$$

Using a different notation, these definitions were made by Riordan [16, p. 102, problem 7]. The numbers $b(0; n, j)$ are the Stirling numbers of the second kind, which are very important in combinatorial analysis and finite differences. See [12] and [16] for applications. We shall use the notation

$$b(0; n, j) = S(n, j). \tag{5.5}$$

The numbers $b(1; n, j)$, called the associated Stirling numbers of the second kind, have also been studied [16, p. 77], [12, pp. 171–173], [3]. Following Riordan, we shall use the notation

$$b(1; n, j) = b(n, j). \tag{5.6}$$

We shall also write

$$b(2; n, j) = g(n, j). \tag{5.7}$$

The numbers $b(t; n, j)$ have the following interpretations (see the remarks following Theorem 4.3): $b(t; n, j)$ is the number of set partitions of $\{1, \ldots, n\}$ consisting of exactly $j$ blocks, where each block contains at least $t + 1$ elements. Another interpretation is that $b(t; n, j)$ is the number of ways of placing $n$ distinct objects into $j$ nondistinct cells, where each cell must contain at least $t + 1$ objects.

By (4.11) and (4.12), we have the following formulas:

$$A_n = \sum_{j=1}^{n} (-1)^j \binom{n+2j}{n}^{-1} [1 \cdot 3 \cdots (2j-1)]^{-1} g(n+2j, j), \tag{5.8}$$

$$A_n = \sum_{j=1}^{n} (-1)^j \binom{n+1}{j+1}\binom{n+2j}{n}^{-1} [1 \cdot 3 \cdots (2j-1)]^{-1} b(n+2j, j). \tag{5.9}$$

We can compare (5.8) and (5.9) to similar formulas for the Bernoulli numbers [12, pp. 219, 599]. Since [16, p. 77]

$$b(n, j) = \sum_{k=0}^{j} (-1)^k \binom{n}{k} S(n-k, j-k),$$

we have, from (5.9),

$$A_n = \sum_{j=1}^{n}\sum_{k=1}^{j} (-1)^k \binom{n+1}{j+1}\binom{n+2j}{j-k}\binom{n+2j}{n}^{-1} \cdot [1 \cdot 3 \cdots (2j-1)]^{-1} S(n+j+k, k). \tag{5.10}$$

The integers $g(n, j)$ defined by (5.4) and (5.7) have properties similar to those of the Stirling numbers and associated Stirling numbers of the second kind. In particular, with $g(0, 0) = 1$, we have

$$g(n+1, j) = jg(n, j) + \binom{n}{2} g(n-2, j-1), \tag{5.11}$$

and we can easily compute a few values of $g(n, j)$:

| $j \backslash n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 0 | 0 | 0 | 0 | 0 | 0 | 35 | 91 | 210 | 456 |
| 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 280 | 2100 |

We also have

$$g(n, j) = \sum_{k=0}^{j} (-1)^k \binom{n}{2k} [1 \cdot 3 \cdots 2k - 1] b(n - 2k, j - k), \tag{5.12}$$

$$b(n, j) = \sum_{k=0}^{j} \binom{n}{2k} [1 \cdot 3 \cdots 2k - 1] g(n - 2k, j - k). \tag{5.13}$$

Formulas (5.11), (5.12) and (5.13) can be proved in a more general setting. Following Riordan [16, pp. 76–78], we see that

$$b_{t,n+1}(s) = s \sum_{r=0}^{n-t} \binom{n}{r} b_{t,n}(s), \tag{5.14}$$

$$b_{t,n}(s) = \sum_{r=0}^{n} \frac{n!(t!)^{-r}(-s)^r}{r!(n-tr)!} b_{t-1,n-tr}(s), \tag{5.15}$$

$$b_{t,n-1}(s) = \sum_{r=0}^{n} \frac{n!(t!)^{-r}(s)^r}{r!(n-tr)!} b_{t,n-tr}(s). \tag{5.16}$$

By differentiating (5.2) with respect to $u$ and subtracting $s$ times the derivative of (5.2) with respect to $s$, we derive

$$b(t; n+1, j) = jb(t; n, j) + \binom{n}{t} b(t; n-t, j-1), \tag{5.17}$$

with $b(t; 0, 0) = 1$. Also, from (5.2) and (5.3),

$$b(t; n, j) = \sum \frac{n!}{j! k_1! \cdots k_j!}, \tag{5.18}$$

the sum being over all compositions $k_1 + \cdots + k_j = n$, each $k_i \geqslant t + 1$.

A natural generalization of (1.1) is

$$\frac{x^m/m!}{e^x - 1 - x - \cdots - x^{m-1}/(m-1)!} = \sum_{n=0}^{\infty} A_{m,n} \frac{x^n}{n!}. \tag{5.19}$$

Definition (5.19) was made in [8], and arithmetic properties of the rational numbers $A_{m,n}$ were discussed in that paper. It follows that

$$A_{m,n} = \sum_{j=1}^{n} (-m!)^j j! n! b(m; n + mj, j)/(n + mj)! \tag{5.20}$$

and

$$A_{m,n} = n! \sum_{j=1}^{n} \frac{(-m!)^j}{(m+k_1)! \cdots (m+k_j)!}, \tag{5.21}$$

the sum being over all compositions $k_1 + \cdots + k_j = n$, each $k_i \geq 1$. Applying Theorem 4.4, we see that if

$$h(n) = \sum_{i=0}^{n-1} \binom{n-1}{i} \frac{m!}{(i+1)\cdots(i+m)} f(n-i),$$

then

$$f(n) = \sum_{i=0}^{n-1} \binom{n-1}{i} A_{m,i} h(n-i).$$

From (5.19) we have $A_{1,n} = B_n$ and $A_{2,n} = A_n$.

**6. Miscellaneous Results.** From (1.1) and (2.3) we see that

$$(e-2)^{-1} = 2 \sum_{n=0}^{\infty} A_n/n!; \tag{6.1}$$

and the convergence appears to be very rapid since

$$(e-2)^{-1} = 1.392211191 \cdots \quad \text{and} \quad 2 \sum_{n=0}^{5} A_n/n! = 1.392210464 \cdots .$$

By letting $x = -1$ in (1.1), we have

$$e = 2 \sum_{n=0}^{\infty} (-1)^n A_n/n!, \tag{6.2}$$

and again the convergence is rapid. More generally, from (1.5) we have for all $z$

$$e^{1-z} = 2 \sum_{n=0}^{\infty} (-1)^n A_n(z)/n!. \tag{6.3}$$

We can compare the sizes of $A_n$ and the Bernoulli numbers. From (2.1) and (2.2) we see that

$$|A_n| < n! \sum_{s=1}^{\infty} (2\pi s)^{-n}, \tag{6.4}$$

and since [12, p. 244]

$$2(n!) \sum_{s=1}^{\infty} (2\pi s)^{-n} = |B_n|,$$

for $n$ even, we see that for $n = 2m$, $m > 0$,

$$2|A_{2m}| < |B_{2m}|, \tag{6.5}$$

and it follows [12, p. 245] that for $m > 0$

(6.6) $$24|A_{2m}| < (2m)!(2\pi)^{2-2m}.$$

Generally, using the approximation

$$|A_n| = n!(\cos n\theta_1)r_1^{-n},$$

we conjecture that for all $n > 0$

(6.7) $$|A_n| < n!7^{-n}.$$

It was proved in [8] that the numbers $A_n$ are not bounded.

As we saw in Section 3, there is still a question of whether or not $A_n(z)$ can have rational roots when $n$ is odd. The following theorems shed a little light on this situation.

THEOREM 6.1. *If $p$ is a prime number, $p > 3$, and if $p - 2$ divides $n$, then $A_n(z)$ has no rational roots.*

*Proof.* By the proof of Theorem 6.2 in [6], we have

$$\frac{p^m A_{m(p-2)}(u/3)}{[m(p-2)]!} \equiv \frac{p^m}{[m(p-2)]!} A_{m(p-2)} \not\equiv 0 \pmod{p}.$$

It follows that $u/3$ cannot be a root of $A_{m(p-2)}(z)$.

THEOREM 6.2. *Suppose $u/3$ is a rational root of $A_n(z)$ and $n = 1 + 3^t k$, $k \not\equiv 0 \pmod 3$. If $t = 1$, then $u \equiv 1 \pmod 9$. If $t > 1$, then $u \equiv 1 \pmod{3^{t+2}}$.*

*Proof.* We know from Theorem 3.1 that $u \equiv n \equiv 1 \pmod 3$. Note that

$$\binom{n}{r} 3^r A_r = n(n-1) \cdots (n-r+1) 3^r A_r / r!,$$

so

$$\sum_{r=3m+2}^{n} \binom{n}{r} 3^{r-1} A_r u^{n-r} \equiv 0 \pmod{3^{t+m-1}}.$$

From (3.6) we have

$$\begin{aligned} 0 &\equiv \sum_{r=0}^{4} \binom{n}{r} 3^{r-1} A_r u^{n-r} \\ &\equiv u^{n-1}(u-1)/3 + 3^{t-1} k u^{n-4}(-1 - 2u + 10u^2 - 40u^3)/40 \\ &\equiv u^{n-1}(u-1)/3 \pmod{3^t}, \end{aligned}$$

which implies $u \equiv 1 \pmod{3^{t+1}}$. In fact, if $t > 1$,

$$0 \equiv u^{n-1}(u-1)/3 - 3^t k \cdot 11/40 - 3^t k \cdot 47/1400 \pmod{3^{t+1}},$$

which implies $u \equiv 1 \pmod{3^{t+2}}$.

We can use the method of Theorem 6.2 to get more information about $u$, if $u/b$ is a rational root of $A_n(z)$. Suppose $n = 1 + 3^t k$, $t > 2$, $k \equiv 0 \pmod 3$ and suppose $u = 1 + 3^{t+2} m$. Then we have

$$0 \equiv \sum_{r=0}^{10} \binom{n}{r} 3^{r-1} u^{n-r} \equiv 3^{t+1} m + 3^t k(-11/40 - 47/1400) + 3^{t+1} k(5120)$$

$$\equiv 3^{t+1} m - 3^{t+1} k \pmod{3^{t+2}}.$$

For $r = 8, 9, 10$ we have used (1.3). Thus we see that in this case we must have $m \equiv k \pmod 3$.

If $n = 4 + 9k$ or $7 + 9k$, $k \not\equiv 0 \pmod 3$, we can use this method to show that $u \equiv 19 \pmod{27}$. If $n = 1 + 9k$, $k \not\equiv 0 \pmod 3$, we can use this method to show that $u \equiv 1 \pmod{243}$.

By these results and the remarks following Theorem 3.2, we see that if $A_n(1/3) = 0$, then $n \equiv 1 \pmod{36}$.

Returning to definitions (5.2) and (5.3), we can find a relationship between $b_{2,n}(s)$ and the Hermite polynomials. Let

$$g_n(s) = b_{2,n}(s), \quad a_n(s) = b_{0,n}(s).$$

From (5.2) we have

$$\sum_{n=0}^{\infty} g_n(s) \frac{u^n}{n!} = \exp[s(e^u - 1)] \exp[-s(u + u^2/2)]. \tag{6.8}$$

In [11, p. 181] the Hermite polynomial $H_n(x)$ is defined by means of

$$\exp(xu - u^2/2) = \sum_{n=0}^{\infty} H_n(x) \frac{u^n}{n!}. \tag{6.9}$$

Thus by (6.8) and (6.9) we have

$$g_n(1) = \sum_{r=0}^{n} \binom{n}{r} a_r(1) H_{n-r}(-1),$$

where $H_0(-1) = 1$, $H_1(-1) = -1$ and

$$H_{n+1}(-1) = -H_n(-1) - nH_{n-1}(-1).$$

It follows that

$$g_n(1) = \sum_{r=0}^{n} \sum_{j=0}^{r} \binom{n}{r} S(n-r, j) H_r(-1).$$

The number $g_n(1)$ is the number of ways of putting $n$ different objects into $n$ like cells, where each nonempty cell must contain at least three objects.

We conclude with two tables. Table 1 gives the value of $n\theta_1$ (modulo 360°), rounded off to the nearest degree, and also the values of $\cos n\theta_1$ rounded off at the third place. This is done for $1 \leqslant n \leqslant 46$. Table 2 indicates when the pattern of Theorem 2.2 changes for $A_n$ when $n = 46k + s$.

TABLE 1

| | | |
|---|---|---|
| (1) 74°, .270 | (17) 184°, −.997 | (32) 220°, −.771 |
| (2) 149°, −.855 | (18) 258°, −.288 | (33) 294°, .405 |
| (3) 223°, −.730 | (19) 333°, .890 | (34) 8°, .990 |
| (4) 297°, .461 | (20) 47°, .679 | (35) 83°, .129 |
| (5) 12°, .979 | (21) 122°, −.524 | (36) 157°, −.920 |
| (6) 86°, .067 | (22) 196°, −.962 | (37) 231°, −.625 |
| (7) 161°, −.943 | (23) 270.3°, .005 | (38) 306°, .583 |
| (8) 235°, −.575 | (24) 345°, .964 | (39) 20°, .939 |
| (9) 309°, .633 | (25) 59°, .515 | (40) 94°, −.077 |
| (10) 24°, .916 | (26) 133°, −.689 | (41) 169°, −.981 |
| (11) 98°, −.139 | (27) 208°, −.885 | (42) 243°, −.452 |
| (12) 172°, −.991 | (28) 282°, .209 | (43) 317°, .737 |
| (13) 247°, −.396 | (29) 356°, .998 | (44) 32°, .849 |
| (14) 321°, .778 | (30) 71°, .329 | (45) 106°, −.279 |
| (15) 35°, .815 | (31) 145°, −.821 | (46) 180.6°, −.9999 |
| (16) 110°, −.338 | | |

TABLE 2

| $s$ | 6 | 35 | 18 | 1 | 30 | 13 | 42 | 25 | 8 | 27 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $k$ | 7 | 13 | 20 | 28 | 34 | 41 | 47 | 54 | 61 | 67 | 74 |

| $s$ | 3 | 32 | 15 | 44 | 27 | 10 | 39 | 22 | 5 | 34 |
|---|---|---|---|---|---|---|---|---|---|---|
| $k$ | 82 | 88 | 95 | 101 | 108 | 115 | 121 | 128 | 136 | 142 |

| $s$ | 17 | 46 | 29 | 12 | 41 | 24 | 7 | 36 | 19 |
|---|---|---|---|---|---|---|---|---|---|
| $k$ | 149 | 155 | 162 | 169 | 175 | 182 | 190 | 196 | 203 |

| $s$ | 2 | 31 | 14 | 43 | 26 | 9 | 38 | 21 | 4 |
|---|---|---|---|---|---|---|---|---|---|
| $k$ | 210 | 216 | 223 | 229 | 236 | 244 | 250 | 257 | 264 |

| $s$ | 33 | 16 | 45 | 28 | 11 | 40 | 23 |
|---|---|---|---|---|---|---|---|
| $k$ | 270 | 277 | 283 | 290 | 298 | 304 | 311 |

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$$a = 2.0888430156130,$$
$$b = 7.46148928565425,$$
$$\theta_1 = 74.36041657449774°,$$
$$r_1 = 7.74836031065984.$$

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