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A GENERAL LACUNARY RECURRENCE FORMULA

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1. INTRODUCTION

The Bernoulli numbers B_n may be defined by means of the generating function

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}. \tag{1.1}$$

An example of a "lacunary" recurrence for these numbers is

$$\sum_{j=0}^{n} \binom{6n+3}{6j} B_{6j} = 2n + 1. \tag{1.2}$$

This recurrence has lacunae, or gaps, or length 6. That is, to compute B_{6n}, it is not necessary to know the values of B_j for all $j < 6n$; we need only know the values of B_{6j} for $j = 0, 1, \cdots, n-1$.

The purpose of this paper is to prove a general lacunary recurrence, for arbitrary gaps, that is applicable to the Bernoulli numbers, the Genocchi numbers, the Eulerian numbers, the Fibonacci numbers, and many other special sequences. The writer believes that the method used in this paper is new and that the formulas are especially easy to use. It is interesting that some of the formulas with gaps of 5 involve the Lucas numbers.

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The problem of finding lacunary recurrences for the Bernoulli numbers has a long history, with the motivation being to find quick ways of computing the numbers. Using different methods, Ramanujan [4], [11], Lehmer [10], Riordan [12, pp. 138–140], Chellali [2], Yalavigi [13], and Berndt [1] have all worked out formulas. References to the nineteenth century work of van den Berg and Haussner, and other historical information, can be found in [10].

2. A GENERAL FORMULA

Let $F(x)$ be a function, not identically 0, that can be represented by a power series with a positive radius of convergence:

$$F(x) = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!}. \tag{2.1}$$

If $n < 0$, we define $f_n = 0$. Define the numbers a_n by means of the generating function

$$\frac{hx^t}{F(x)} = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} \tag{2.2}$$

where t is an arbitrary nonnegative integer and h is an arbitrary rational number. The following lemma [5] is essential, and for completeness we include the proof.

Lemma 2.1: Let m be a positive integer, and let $\theta = e^{2\pi i/m}$, a primitive m^{th} root of unity. Let

$$F(x)F(\theta x)\cdots F(\theta^{m-1}x) = \sum_{n=0}^{\infty} b_n \frac{x^n}{n!}.$$

Then $b_n = 0$ unless m divides n. That is,

$$F(x)F(\theta x)\cdots F(\theta^{m-1}x) = \sum_{n=0}^{\infty} b_{mn} \frac{x^{mn}}{(mn)!}. \tag{2.3}$$

Proof: Let $H(x) = F(x)F(\theta x)\cdots F(\theta^{m-1}x)$. Clearly $H(x) = H(\theta x)$, so $b_n = \theta^n b_n$ for $n = 0, 1, 2, \cdots$. Since $\theta^n = 1$ only when m divides n, we see that the lemma is valid. This completes the proof. □

For our main result, Theorem 2.1 below, we also need the numbers c_n, defined by

$$F(\theta x)F(\theta^2 x)\cdots F(\theta^{m-1}x) = \sum_{n=0}^{\infty} c_n \frac{x^n}{n!}, \tag{2.4}$$

with $c_n = 0$ for $n < 0$. Also, throughout the paper we use the notation

$$(x)_t = x(x-1)(x-2)\cdots(x-t+1). \tag{2.5}$$

Theorem 2.1: Let m be a positive integer and let θ be a primitive m^{th} root of unity. Let a_n, b_n, c_n be defined by (2.2), (2.3) and (2.4), respectively. Then for $0 \le r < m$,

$$\sum_{j=0}^{n} \binom{mn+r}{mj+r} b_{m(n-j)} a_{mj+r} = h(mn+r)_t c_{mn+r-t}. \tag{2.6}$$

Proof: In (2.3) divide both sides by $F(x)$ to obtain

$$F(\theta x)\cdots F(\theta^{m-1}x) = \frac{1}{F(x)} \sum_{n=0}^{\infty} b_{mn} \frac{x^{mn}}{(mn)!}. \tag{2.7}$$

Now multiply both sides of (2.6) by hx^t, and use (2.4), to get

$$hx^t \sum_{n=0}^{\infty} c_n \frac{x^n}{n!} = \frac{hx^t}{F(x)} \sum_{n=0}^{\infty} b_{mn} \frac{x^{mn}}{(mn)!}. \tag{2.8}$$

Substitute (2.2) into (2.8), and compare coefficients of $x^{mn+r}/(mn+r)!$ to complete the proof. □

We now look at two simple special cases. If $m = 1$, then $\theta = 1$, $b_n = f_n$, $c_0 = 1$ and $c_n = 0$ if $n \neq 0$. Theorem 2.1 gives the recurrence

$$\sum_{j=0}^{n} \binom{n}{j} f_{n-j} a_j = h(n)_t c_{n-t}. \tag{2.9}$$

For example, for the Bernoulli numbers, $f_0 = 0$ and $f_n = 1$ for $n > 0$. Thus (2.9) gives us $B_0 = 1$, and for $n > 1$:

$$\sum_{j=0}^{n-1} \binom{n}{j} B_j = 0.$$

If $m = 2$, then $\theta = -1$, $b_{2n} = \sum_{j=0}^{2n} \binom{2n}{j} (-1)^j f_j f_{2n-j}$, and $c_n = (-1)^n f_n$. Thus for $r = 0$ or $r = 1$,

$$\sum_{j=0}^{n} \binom{2n+r}{2j+r} b_{2(n-j)} a_{2j+r} = h(2n+r)_t (-1)^{r-t} f_{2n+r-t}. \tag{2.10}$$

For the Bernoulli numbers, $b_0 = 0$ and $b_{2n} = -2$, so (2.10) gives us: for $n \geq 1$

$$\sum_{j=0}^{n-1} \binom{2n}{2j} B_{2j} = n.$$

3. BERNOULLI, GENOCCHI AND EULERIAN NUMBERS

In this section, and section 4, we show how Theorem 2.1 can be applied to the Bernoulli, Genocchi and Eulerian numbers for $m = 3$, 4 and 5. More generally, the results of these sections are for numbers a_n defined by the following type of generating function: Let h, v and q be nonzero rational numbers, let w be an arbitrary rational number, and let t be a nonnegative integer. Let

$$F(x) = ve^{qx} + w, \tag{3.1}$$

in definition (2.2). Note that if $v + w \neq 0$, then $a_n = 0$ for $n < t$ and $a_t = t!h/(v+w)$. If $v + w = 0$, then $a_t = 0$ for $n < (t-1)$ and $a_{t-1} = (t-1)!h/(qv)$. It follows from (2.9), with $m = 1$, that

$$(v+w)a_n + v\sum_{j=0}^{n-1}\binom{n}{j} a_j q^{n-j} = \begin{cases} 0 \text{ if } n \neq t \\ h(n!) \text{ if } n = t. \end{cases}$$

When $h = t = v = q = 1$ and $w = -1$, we have $a_n = B_n$, the Bernoulli number defined by (1.1). When $h = 2$, $t = v = q = 1$ and $w = 1$, then $a_n = G_n$, the Genocchi number:

$$\frac{2x}{e^x + 1} = \sum_{n=0}^{\infty} G_n \frac{x^n}{n!}.$$

When $t = 0$, $h = u - 1$, $v = u$, $q = 1 - u$ and $w = -1$, then $a_n = A_n(u)$, the Eulerian number:

$$\frac{u-1}{ue^{(1-u)x} - 1} = \sum_{n=0}^{\infty} A_n(u)\frac{x^n}{n!}.$$

A good reference for all of these special numbers is [3, pp. 48-50].

Using the notation of Section 2, let $m = 3$ and let θ be a primitive third root of unity. Let $F(x)$ be defined by (3.1), and note that $1 + \theta + \theta^2 = 0$. Then

$$\sum_{n=0}^{\infty} b_n \frac{x^n}{n!} = F(x)F(\theta x)F(\theta^2 x) = (v^3 + w^3) + v^2 w \sum_{j=0}^{2} e^{-\theta^j qx} + vw^2 \sum_{j=0}^{2} e^{\theta^j qx}.$$

Thus $b_0 = (v+w)^3$, and for $n > 0$

$$\begin{aligned} b_{3n} &= v^2w(-q)^{3n}(1+\theta^{3n}+\theta^{6n}) + vw^2q^{3n}(1+\theta^{3n}+\theta^{6n}) \\ &= 3vwq^{3n}[(-1)^n v + w]. \end{aligned} \tag{3.2}$$

We now compute c_n. We have

$$\sum_{n=0}^{\infty} c_n \frac{x^n}{n!} = F(\theta x)F(\theta^2 x) = v^2 e^{-qx} + vw(e^{\theta qx} + e^{\theta^2 qx}) + w^2.$$

Thus $c_0 = (v+w)^2$, and for $n > 0$

$$c_n = v^2(-q)^n + vwq^n(\theta^n + \theta^{2n}) = \begin{cases} v^2(-q)^n + 2vwq^n, & \text{if } n \equiv 0 \pmod 3 \\ v^2(-q)^n - vwq^n, & \text{if } n \equiv 1 \text{ or } 2 \pmod 3 \end{cases}. \tag{3.3}$$

By Theorem 2.1, we have for $r = 0$, 1, and 2:

$$\sum_{j=0}^{n} \binom{3n+r}{3j+r} b_{3(n-j)} a_{3j+r} = h(3n+r)_t c_{3n+r-t}, \tag{3.4}$$

with $b_{3(n-j)}$ and c_{3n+r-t} given by (3.2) and (3.3), respectively. For the Bernoulli numbers, the case $r = 0$ gives formula (1.2), and we get similar formulas for $r = 1$ and $r = 2$. We get gaps of 6, instead of 3, because $B_n = 0$ if n is odd, $n > 1$. That is, in (3.4) we can assume $3j + r$ is even for the Bernoulli numbers.

We proceed in the same way when $m = 4$ and $\theta = i$, a primitive fourth root of unity. We have

$$\begin{aligned} \sum_{n=0}^{\infty} b_n \frac{x^n}{n!} &= F(x)F(\theta x)F(\theta^2 x)F(\theta^3 x) \\ &= (v^4 + w^4) + v^3 w \sum_{j=0}^{3} e^{-\theta^j qx} + v^2 w^2 \sum_{0 \le j < s \le 3} e^{(\theta^j + \theta^s)qx} + vw^3 \sum_{j=0}^{3} e^{\theta^j qx}. \end{aligned} \tag{3.5}$$

Now since

$$1 + i^2 = i + i^3 = 0,$$

$$i^2 + i^3 = -(i+1),\ i + i^2 = i - 1,\ 1 + i^3 = 1 - i,$$

$$(1+i)^4 = (1-i)^4 = -4,$$

By (2.9) we have the simple recurrence: $B_{k,0} = 1$ and for $n > k$

$$\sum_{j=0}^{n-k} \binom{n}{j} B_{k,j} = 0.$$

By (2.10) we have for $r = 0$, $r = 1$, and $n \geq k$:

$$\sum_{j=0}^{n-k} \binom{2n+r}{2j+r} b_{2(n-j)} B_{k,2j+r} = \binom{2n+r}{k} (-1)^{r-k},$$

where

$$b_{2(n-j)} = \sum_{s=k}^{2n-2j-k} (-1)^s \binom{2n-2j}{s} = 2\sum_{s=0}^{k-1} (-1)^{s+1} \binom{2n-2j}{s}.$$

Let $m = 3$, let θ be a primitive third root of unity and let $F_k(x)$ be defined by (5.1). Define $b_{k,n}$, $c_{k,n}$, $G_k(x)$ and $H_k(x)$ in the following way.

$$G_k(x) = F_k(x)F_k(\theta x)F_k(\theta^2 x) = \sum_{n=0}^{\infty} b_{k,n} \frac{x^n}{n!}, \tag{5.3}$$

$$H_k(x) = F_k(\theta x)F_k(\theta^2 x) = \sum_{n=0}^{\infty} c_{k,n} \frac{x^n}{n!}. \tag{5.4}$$

Thus

$$\begin{cases} b_{0,0} = 1, \text{ and } b_{0,n} = 0 \text{ for } n > 0, \\ c_{0,n} = (-1)^n. \end{cases} \tag{5.5}$$

Our goal here is to use an inductive method to find formulas for $b_{k,n}$ and $c_{k,n}$ for general k. Since

$$G_{k+1}(x) = \prod_{j=0}^{2} \left(F_k(\theta^j x) - \frac{(\theta^j x)^k}{k!} \right),$$

4. GAPS OF LENGTH 5

For the numbers a_n defined by (3.1), the lacunary formulas with gaps of length 5 involve Lucas numbers. This happens because of the following relationships: Let $\theta = e^{2\pi i/5}$, a primitive fifth root of unity. Then

$$\theta + \theta^4 = \frac{-1+\sqrt{5}}{2}, \quad \theta^2 + \theta^3 = \frac{-1-\sqrt{5}}{2}.$$

so

$$L_n = (-1)^n[(\theta+\theta^4)^n + (\theta^2+\theta^3)^n], \tag{4.1}$$

where L_n is the n^{th} Lucas number. Since θ is a fifth root of unity, the following equations are obvious:

$$\left.\begin{aligned} \theta^3+\theta^4 &= \theta(\theta^2+\theta^3), & \theta^2+1 &= \theta(\theta+\theta^4) \\ \theta^4+1 &= \theta^2(\theta^2+\theta^3), & \theta^3+\theta &= \theta^2(\theta+\theta^4) \\ 1+\theta &= \theta^3(\theta^2+\theta^3), & \theta^4+\theta^2 &= \theta^3(\theta+\theta^4) \\ \theta+\theta^2 &= \theta^4(\theta^2+\theta^3), & 1+\theta^3 &= \theta^4(\theta+\theta^4) \end{aligned}\right\} \tag{4.2}$$

We also note that

$$\theta^5 = 1 \text{ and } 1+\theta+\theta^2+\theta^3+\theta^4 = 0. \tag{4.3}$$

Using the notation of section 3, with $m = 5$, we have

$$\begin{aligned} \sum_{n=0}^{\infty} b_n \frac{x^n}{n!} &= F(x)F(\theta x)F(\theta^2 x)F(\theta^3 x)F(\theta^4 x) \\ &= (v^5+w^5) + v^4 w \sum_{j=0}^{4} e^{-\theta^j qx} + v^3 w^2 \sum_{0\le j<s\le 4} e^{-(\theta^j+\theta^s)qx} \\ &\quad + v^2 w^3 \sum_{0\le j<s\le 4} e^{(\theta^j+\theta^s)qx} + v w^4 \sum_{j=0}^{4} e^{\theta^j qx}. \end{aligned} \tag{4.4}$$

By (4.1), (4.2), (4.3) and (4.4), we have $b_0 = (v+w)^5$ and for $n > 0$:

$$\begin{aligned} b_{5n} &= v^4 w q^{5n}[5(-1)^n] + v^3 w^2 q^{5n}[5(-1)^n\{\theta^2+\theta^3)^{5n} + (\theta+\theta^4)^{5n}\}] \\ &\quad + v^2 w^3 q^{5n}[5\{(\theta^2+\theta^3)^{5n} + (\theta+\theta^4)^{5n}\}] + 5vw^4 q^{5n} \\ &= 5vwq^{5n}[(-1)^n v^3 + v^2 w L_{5n} + vw^2(-1)^n L_{5n} + w^3] \end{aligned} \tag{4.5}$$

To compute c_n, we will use (4.1)–(4.3) and the fact that

$$(\theta+\theta^4)(\theta^2+\theta^3) = -1. \tag{4.6}$$

We have

$$\sum_{n=0}^{\infty} c_n \frac{x^n}{n!} = F(\theta x)F(\theta^2 x)F(\theta^3 x)F(\theta^4 x)$$

$$= w^4 + v^4 e^{-qx} + v^3 w \sum_{s=1}^{4} e^{-(1+\theta^s)qx} + v^2 w^2 \sum_{1\le j<s\le 4} e^{(\theta^j+\theta^s)qx}$$

$$+ vw^3 \sum_{j=1}^{4} e^{\theta^j qx}. \tag{4.7}$$

Now we observe that, by (4.2),

$$\sum_{s=1}^{4} e^{-(1+\theta^s)qx} = \sum_{n=0}^{\infty} y_n \frac{x^n}{n!},$$

where

$$y_n = (-q)^n[(\theta^{2n}+\theta^{3n})(\theta^2+\theta^3)^n + (\theta^n+\theta^{4n})(\theta+\theta^4)^n]$$

$$= \begin{cases} 2q^n L_n & \text{if } n \equiv 0 \pmod 5, \\ -q^n L_{n+1} & \text{if } n \equiv 1 \text{ or } 4 \pmod 5, \\ q^n L_{n-1} & \text{if } n \equiv 2 \text{ or } 3 \pmod 5. \end{cases} \tag{4.8}$$

Also by (4.2),

$$\sum_{1\le j<s\le 4} e^{(\theta^j+\theta^s)qx} = \sum_{n=0}^{\infty} p_n \frac{x^n}{n!},$$

where

$$p_n = q^n[(-1)^n L_n + (\theta^n+\theta^{4n})(\theta^2+\theta^3)^n + (\theta^{2n}+\theta^{3n})(\theta+\theta^4)^n]$$

$$= \begin{cases} 3(-q)^n L_n & \text{if } n \equiv 0 \pmod 5, \\ (-q)^n L_{n+1} & \text{if } n \equiv 1 \text{ or } 4 \pmod 5, \\ -(-q)^n L_{n-1} & \text{if } n \equiv 2 \text{ or } 3 \pmod 5. \end{cases} \tag{4.9}$$

By (4.7), (4.8) and (4.9), $c_0 = (v+w)^4$ and

$$c_n = \begin{cases} q^n[(-1)^n v^4 + 4vw^3 + 2v^3wL_n + 3(-1)^n v^2w^2L_n], & \text{if } n \equiv 0 \pmod 5, \\ q^n[(-1)^n v^4 - vw^3 - v^3wL_{n+1} + (-1)^n v^2w^2L_{n+1}], & \text{if } n \equiv 1 \text{ or } 4 \pmod 5, \\ q^n[(-1)^n v^4 - vw^3 + v^3wL_{n-1} + (-1)^{n+1} v^2w^2L_{n-1}], & \text{if } n \equiv 2 \text{ or } 3 \pmod 5. \end{cases} \tag{4.10}$$

Thus for $m = 5$ and $r = 0,1,2,3$, or 4:

$$\sum_{j=0}^{n} \binom{5n+r}{5j+r} b_{5(n-j)} a_{5j+r} = h(5n+r)_t c_{5n+r-t},$$

where $b_{5(n-j)}$ and c_{5n+r-t} are given by (4.5) and (4.10), respectively. For example, for the Bernoulli numbers with $r = 0$, we have

$$5\sum_{j=0}^{n} \binom{10n+5}{10j+5} (1 + L_{10(n-j)+5}) B_{10j} = (10n+5)(1 + L_{10n+5}).$$

5. THE RECIPROCAL OF $e^x - 1 - x - \cdots - \frac{x^{k-1}}{(k-1)!}$

Let $k \geq 0$, and define

$$F_k(x) = e^x - 1 - x - \cdots - \frac{x^{k-1}}{(k-1)!}. \tag{5.1}$$

In this section we show how Theorem 2.1 can be applied to the numbers $B_{k,n}$ defined by

$$\frac{x^k/k!}{F_k(x)} = \sum_{n=0}^{\infty} B_{k,n} \frac{x^n}{n!}. \tag{5.2}$$

We first observe that $B_{0,n} = (-1)^n$, and $B_{1,n} = B_n$, the Bernoulli number. The numbers $B_{2,n}$ have been examined in some detail [7], [9], and $B_{k,n}$, for general k, has also been studied [8]. To avoid confusion with the Eulerian numbers, in the present paper we have changed the notation of [7]–[9] from A_n to $B_{2,n}$ and from $A_{k,n}$ to $B_{k,n}$.

from (3.5) we have for $n > 0$

$$\begin{aligned} b_{4n} &= v^3w[4q^{4n}] + v^2w^2[4(-4)^n q^{4n}] + vw^3[4q^{4n}] \\ &= 4vwq^{4n}[vw(-4)^n + w^2 + v^2], \\ \textit{and } b_0 &= (v+w)^4. \end{aligned} \tag{3.6}$$

To compute c_n, we examine

$$\sum_{n=0}^{\infty} c_n \frac{x^n}{n!} = F(\theta x)F(\theta^2 x)F(\theta^3 x)$$

$$= w^3 + v^3 e^{(\theta+\theta^2+\theta^3)qx} + v^2 w \sum_{0<j<s\leq 3} e^{(\theta^j+\theta^s)qx} + vw^2 \sum_{j=1}^{3} e^{\theta^j qx}.$$

Thus we have $c_0 = (v+w)^3$, and for $n > 0$:

$$c_n = q^n\{v^3(-1)^n + v^2w[(i-1)^n + (-i-1)^n] + vw^2[(-1)^n + i^n + (-i)^n]\}.$$

This gives

$$c_{4n} = q^{4n}v[v^2 + 2vw(-4)^n + 3w^2], \tag{3.7}$$
$$c_{4n+1} = q^{4n+1}v[-v^2 - 2vw(-4)^n - w^2], \tag{3.8}$$
$$c_{4n+2} = q^{4n+2}v[v^2 - w^2], \tag{3.9}$$
$$c_{4n+3} = q^{4n+3}v[-v^2 + 4v^2w(-4)^n - w^2]. \tag{3.10}$$

Thus when $m = 4$, for $r = 0,\ 1,\ 2,\ 3$ we have:

$$\sum_{j=0}^{n} \binom{4n+r}{4j+r} b_{4(n-j)} a_{4j+r} = h(4n+r)_t c_{4n+r-t},$$

where $b_{4(n-j)}$ and c_{4n+r-t} are given by (3.6) and (3.7) - (3.10), respectively. When $r = 0$, for example, we get the following formula for the Bernoulli numbers:

$$\sum_{j=0}^{n-1} \binom{4n}{4j} [4(-4)^{n-j} - 8] B_{4j} = 4n[(-4)^n - 2],$$

and there are similar formulas for $r = 1,\ 2$ and 3.

we have

$$G_{k+1}(x) =$$

$$G_k(x) - \frac{x^k}{k!}\sum_{j=0}^{2}\left[\theta^{jk}H_k(\theta^j x)\right] + \frac{x^{2k}}{(k!)(k!)}\sum_{j=0}^{2}\left[\theta^{(3-j)k}F_k(\theta^j x)\right] - \frac{x^{3k}}{(k!)(k!)(k!)}. \quad (5.6)$$

Now

$$x^k\sum_{j=0}^{2}\left[\theta^{jk}H_k(\theta^j x)\right] = \sum_{n=2k}^{\infty}\left[1+\theta^{n+k}+\theta^{2n+2k}\right]c_{k,n}\frac{x^{n+k}}{n!}$$

$$= \sum_{n=3k}^{\infty}\left[1+\theta^{n}+\theta^{2n}\right](n)_k c_{k,n-k}\frac{x^n}{n!}, \quad (5.7)$$

and

$$x^{2k}\sum_{j=0}^{2}\left[\theta^{(3-j)k}F_k(\theta^j x)\right] = \sum_{n=k}^{\infty}\left[1+\theta^{n+2k}+\theta^{2n+k}\right]\frac{x^{n+2k}}{n!}$$

$$= \sum_{n=3k}^{\infty}\left[1+\theta^{n}+\theta^{2n}\right](n)_{2k}\frac{x^n}{n!}. \quad (5.8)$$

By (5.6), (5.7) and (5.8) we have, for $n > k$:

$$b_{k+1,3n} = b_{k,3n} - 3\binom{3n}{k}c_{k,3n-k} + 3\binom{3n}{2k}\binom{2k}{k}. \quad (5.9)$$

If $n = k$, then $(3k)!/(k!)^3$ must be subtracted from (5.9).
Next we find a recurrence for $c_{k,n}$. Since

$$H_{k+1}(x) = \prod_{j=1}^{2}\left(F_k(\theta^j x) - \frac{(\theta^j x)^k}{k!}\right),$$

it is clear that

$$H_{k+1}(x) = H_k(x) - \frac{x^k}{k!}\sum_{j=1}^{2}\left[\theta^{(3-j)k}F_k(\theta^j x)\right] + \frac{x^{2k}}{(k!)(k!)}.$$

Thus for $n > k$:

$$c_{k+1,n} = c_{k,n} - \binom{n}{k}(\theta^{n+k} + \theta^{2n-k})$$

$$= \begin{cases} c_{k,n} - 2\binom{n}{k} \text{ if } n+k \equiv 0 \pmod 3, \\ c_{k,n} + \binom{n}{k} \text{ if } n+k \equiv 1 \text{ or } 2 \pmod 3. \end{cases} \tag{5.10}$$

If $n = 2k$, then $\binom{2k}{k}$ must be added to (5.10).

Thus we can say: For $r = 0, 1, 2$

$$\sum_{j=0}^{n}\binom{3n+r}{3j+r} b_{k,(n-j)} B_{k,3j+r} = \binom{3n+r}{k} c_{k,3n+r-k}, \tag{5.11}$$

where $b_{k,(n-j)}$ and $c_{k,3n+r-k}$ are given recursively by (5.9) and (5.10). For example, using (5.5) as a starting point, we have

$$b_{1,3n} = -3(-1)^n + 3 = \begin{cases} 0 \text{ if } n \text{ is even}, \\ 6 \text{ if } n \text{ is odd}, \end{cases}$$

and

$$c_{1,n} = \begin{cases} (-1)^n - 2 \text{ if } n \equiv 0 \pmod 3, \\ (-1)^n + 1 \text{ if } n \equiv 1 \text{ or } 2 \pmod 3, \end{cases}$$

and these values agree with (3.2) and (3.3). For $k = 2$, we obtain

$$b_{2,3n} = 3(3n-1)\left[3n - 1 + (-1)^n\right], \tag{5.12}$$

$$c_{2,n} = \begin{cases} (-1)^n - 2 + n \text{ if } n \equiv 0 \pmod 3, \\ (-1)^n + 1 + n \text{ if } n \equiv 1 \pmod 3, \\ (-1)^n + 1 - 2n \text{ if } n \equiv 2 \pmod 3. \end{cases} \tag{5.13}$$

For example, let $k = 2, r = 0$ and define $g(n) = (3n-1)[3n-1+(-1)^n]$. By (5.11), (5.12) and (5.13), we have for $n > 0$:

$$\sum_{j=0}^{n} \binom{3n+6}{3j} g(n+2-j) B_{2,3j} = \frac{1}{2}(n+2)g(n+2). \tag{5.14}$$

To illustrate how (5.14) can be used to compute $B_{2,3n}$, we first note that $g(2) = 30$, $g(3) = 56$ and $g(4) = 132$. Then by (5.14), for $n = 0$ we get $B_{2,0} = 1$. For $n = 1$, we get

$$\binom{9}{3} g(2) B_{2,3} + g(3) B_{2,0} = \frac{3}{2} g(3),$$

so

$$B_{2,3} = \frac{1}{90}.$$

For $n = 2$, we have

$$\binom{12}{6} g(2) B_{2,6} + \binom{12}{3} g(3) B_{2,3} + g(4) B_{2,0} = 2g(4),$$

which gives

$$B_{2,6} = \frac{-1}{5670}.$$

The recursive method used in this section can also be used for $m = 4$ and $m = 5$, but the formulas become complicated and cumbersome. For example, here is the formula (without proof) for $b_{2,5n}$.
For n odd, $n > 1$: $b_{2,5n} = 5(5n-1)(5n-2)(25n^2 - 20n + 1 + L_{5n} - 5nL_{5n-3})$;
For n even, $n > 0$: $b_{2,5n} = 25n(5n-1)(5n-3)(5n-3+L_{5n-3})$.
Also, $c_{2,5n} = c_{1,5n} + 5n[(1+2(-1)^{n-1})L_{5n} + (5n-1)(-1)^n L_{5n-1} + (25n^2 - 25n + 7)]$ where the value of $c_{1,5n}$ can be computed from (4.10). There are similar formulas for $c_{2,5n+r}$, $(r = 1, 2, 3, 4)$.

6. FINAL COMMENTS

The generating function (2.1) can be generalized by defining polynomials $a_n(z)$ by means of

$$\frac{hx^t e^{xz}}{F(x)} = \sum_{n=0}^{\infty} a_n(z) \frac{x^n}{n!}. \tag{6.1}$$

Thus

$$a_n(z) = \sum_{j=0}^{n} \binom{n}{j} a_j z^{n-j}. \tag{6.2}$$

One well known example is the Bernoulli polynomial $B_n(z)$, defined by

$$\frac{xe^{xz}}{e^x - 1} = \sum_{n=0}^{\infty} B_n(z) \frac{x^n}{n!}.$$

Another example is the polynomial $B_{2,n}(z)$, studied in [7] and [9]:

$$\frac{(x^2/2)e^{xz}}{e^x - 1 - x} = \sum_{n=0}^{\infty} B_{2,n}(z) \frac{x^n}{n!}.$$

Define $c_n(z)$ by

$$c_n(z) = \sum_{j=0}^{n} \binom{n}{j} c_j z^{n-j}. \tag{6.3}$$

It is easy to see that for the polynomials defined by (6.1) and (6.3), we can extend Theorem 2.1 in the following way.

Theorem 6.1: Let m be a positive integer and let θ be a primitive m^{th} root of unity. Let a_n, $b_n, c_n, a_n(z), c_n(z)$ be defined by (2.2), (2.3), (2.4) (6.1), (6.3), respectively. Then for $0 \leq r < n$,

$$\sum_{j=0}^{n} \binom{mn+r}{mj+r} b_{m(n-j)} a_{mj+r}(z) = h(mn+r)_t c_{mn+r-t}(z).$$

We also observe that the Genocchi polynomial $G_n(z)$ defined by

$$\frac{2xe^{xz}}{e^x + 1} = \sum_{n=0}^{\infty} G_n(z) \frac{x^n}{n!}$$

is closely related to the Euler number E_n [3, p. 48]:

$$\frac{2e^x}{e^{2x} + 1} = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!}.$$

It is clear that we have the relationship

$$(n+1)E_n = 2^n G_{n+1}\left(\frac{1}{2}\right),$$

and Theorem 6.1 can be used to find a lacunary recurrence for the Euler numbers.

In a letter to the writer, A. Granville made the following observation. If $F(x)$ in the present paper is replaced by $G(x, e^x)$, where $G(x, y)$ is a polynomial in two variables with integer coefficients, then b_n and c_n are both linear combinations of elements of linear recurrence sequences of order dividing $\phi(n)$. We reserve this topic for a later paper.

Finally, we note that the general method of this paper was used by the writer [5], [6] to find lacunary recurrences for the Tribonacci numbers and generalized Fibonacci numbers. To the writer's knowledge, Theorem 2.1 and Theorem 6.1 are new, and the lacunary recurrences for the Eulerian numbers and the numbers $B_{k,n}$ have not appeared in the literature.

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