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MIXED GENERATING RELATIONS FOR POLYNOMIALS RELATED TO KONHAUSER BIORTHOGONAL POLYNOMIALS

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ABSTRACT. Recently, Carlitz has derived general results on generating functions by making use of the Lagrange expansion. In the present paper, results of this kind are derived for the polynomials related to Konhauser biorthogonal polynomials, which were studied by Karande and Thakare.

1. Introduction

Konhauser [6] studied the biorthogonal sets $Z_n^\alpha(x;k)$ and $Y_n^\alpha(x;k)$ which satisfy the following condition:

$$\int_0^\infty x^\alpha \exp(-x) Y_i^\alpha(x;k)\, Z_j^\alpha(x;k)\, dx \begin{cases} = 0, & i \neq j\,, \\ \neq 0, & i = j \end{cases}$$

$$(i,j = 0,1,2,\ldots),$$

where k is a positive integer and $\alpha > -1$.

Carlitz [2] stated that

$$\sum_{n=0}^\infty Y_n^c(x;k)t^n = (1-t)^{-\frac{(c+1)}{k}} \exp\{-x[(1-t)^{-1/k}-1]\}\,. \qquad (1.1)$$

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For $k = 1$, both the polynomials reduce to Laguerre polynomials. We designate $Z_n^\alpha(x;k)$ and $Y_n^\alpha(x;k)$ as the Konhauser biorthogonal polynomial sets of the first and second kinds, respectively. Further, Karande and Thakare [5] studied the polynomials $B_n^c(x;k)$, $S_n^{(\alpha,\beta)}(x,y;k)$ and $T_n^{(\alpha,\beta)}(x,y;k)$ related to the polynomials $Y_n^c(x;k)$ having generating relations

$$\sum_{n=0}^{\infty} B_n^c(x;k)t^n = (1+t)^{-\frac{(c+1)}{k}} \exp\{x[(1+t)^{1/k}-1]\}, \tag{1.2}$$

$$\sum_{n=0}^{\infty} S_n^{(\alpha,\beta)}(x,y;k)t^n = (1+t)^{\frac{\alpha-\beta}{k}} \exp\{(x-y)[(1+t)^{1/k}-1]\} \tag{1.3}$$

and

$$\sum_{n=0}^{\infty} T_n^{(\alpha,\beta)}(x,y;k)t^n = (1-t)^{\frac{\alpha-\beta}{k}} \exp\{-(y-x)[(1-t)^{-1/k}-1]\} \tag{1.4}$$

Clearly, $B_n^c(x;k)$ is uniquely determined, and for $k = 1$, it reduces to $A_n^c(x)$ defined by Srivastava [7].

The aim of the present paper is to derive some mixed generating relations for $Y_n^\alpha(x;k)$, $B_n^c(x;k)$, $S_n^{(\alpha,\beta)}(x,y;k)$ and $T_n^{(\alpha,\beta)}(x,y;k)$. The method used is mainly due to Carlitz [4].

2. By Taylor's theorem, relation (1.1) can be written as

$$Y_n^c(x;k) = \frac{1}{n!}[D_t^n[(1-t)^{-\frac{(c+1)}{k}} \exp\{-x[(1-t)^{-1/k}-1]\}]]_{t=0} \tag{2.1}$$

Replacing x by $x+ny$, this becomes

$$Y_n^c(x+ny;k) = \frac{1}{n!}[D_t^n[(1-t)^{-\frac{(c+1)}{k}} \exp\{-(x+ny)[(1-t)^{-1/k} - 1]\}]]_{t=0} \tag{2.2}$$

Let

$$f(t) = (1-t)^{-\frac{(c+1)}{k}} \exp\{-x[(1-t)^{-1/k}-1]\}$$

and

$$\phi(t) = \exp\{-y[(1-t)^{-1/k}-1]\},$$

where $f(t)$ and $\phi(t)$ are analytic about the origin and $f(o) = \phi(0) = 1$.

Now, following Carlitz's approach [4], we obtain

$$\sum_{n=0}^{\infty} Y_n^c(x+ny;k)u^n = \frac{(1-t)^{-\frac{(c+1)}{k}} \exp\{-x[(1-t)^{-1/k}-1]\}}{1+\frac{y}{k}\left(\frac{t}{1-t}\right)(1-t)^{-1/k}},$$

where (2.3)

$$u = t\exp\{y[(1-t)^{-1/k}-1]\}. \tag{2.4}$$

The result is presumably new.

If we put $v = \frac{t}{1-t}$, (2.3) becomes

$$\sum_{n=0}^{\infty} Y_n^c(x+nt;k)u^n = \frac{(1+v)^{-\frac{(c+1)}{k}} \exp\{-x[(1+v)^{1/k}-1]\}}{1+\frac{y}{k}v(1+v)^{1/k}},$$

where (2.5)

$$u = v(1+v)^{-1}\exp\{y[(1+v)^{1/k}-1]\}. \tag{2.6}$$

Next, by (2.1), we have

$$Y_n^{(c+Dn)}(x;k) = \frac{1}{n!}[D_t^n(1-t)^{-\frac{(c+Dn+1)}{k}} \exp\{-x[(1-t)^{-1/k} -1]\}]]_{t=0}.$$

We, therefore, take

$$f(t) = (1-t)^{-\frac{(c+1)}{k}} \exp\{-x[(1-t)^{-1/k}-1]\}$$

and

$$\phi(t) = (1-t)^{-D/k},$$

where $f(t)$ and $\phi(t)$ are analytic about the origin and such that $f(o) = \phi(o) = 1$.

Now making use of a result due to Carlitz [4, p. 521, Theorem 1], we get

$$\sum_{n=0}^{\infty} Y_n^{(c+Dn)}(x;k)u^n = \frac{(1-t)^{-\frac{(c+1)}{k}} \exp\{-x[(1-t)^{-1/k}-1]\}}{1-\frac{D}{k}\left(\frac{t}{1-t}\right)}, \tag{2.7}$$

where

$$u = t(1-t)^{D/k}. \tag{2.8}$$

Making the same substitution, as before, (2.7) becomes

$$\sum_{n=0}^{\infty} Y_n^{(c+Dn)}(x;k)u^n = \frac{(1+v)^{\frac{(c+1)}{k}} \exp\{-x[(1+v)^{1/k}-1]\}}{1-\frac{D}{k}v}, \tag{2.9}$$

where

$$u = v(1+v)^{-1-D/k}. \tag{2.10}$$

It may be of interest to note that (2.7) and (2.9) were obtained earlier by Calvez and Génin [1] by a different method.

On taking $k = 1$, the Calvez-Génin formula (2.7) or (2.9) gives the corresponding result due to Carlitz [3, p. 826, (8)] for Laguerre polynomials.

Indeed one can now get a mixed generating function for $Y_n^{(c+Dn)}(x+ny;k)$ as

$$\sum_{n=0}^{\infty} Y_n^{(c+Dn)}(x+ny;k)u^n = \frac{(1-t)^{-\frac{(c+1)}{k}} \exp\{-x[(1-t)^{-1/k}-1]\}}{1-\frac{D}{k}\left(\frac{t}{1-t}\right)+\frac{y}{k}\, t(1-t)^{-1-1/k}},$$

where (2.11)

$$u = t(1-t)^{D/k}\exp\{y[(1-t)^{-1/k}-1]\}^{-1/k}. \qquad (2.12)$$

For $y = o$, (2.11) reduces to (2.7) and for $D = o$, it reduces *to* (2.3).

If we put $v = \frac{t}{1-t}$, (2.11) becomes

$$\sum_{n=0}^{\infty} Y_n^{(c+Dn)}(x+ny;k)u^n = \frac{(1+v)^{\frac{(c+1)}{k}} \exp\{-x[(1+v)^{1/k}-1]\}}{1-\frac{Dv}{k}+\frac{y}{k}\, v(1+v)^{1/k}},$$

where (2.13)

$$u = v(1+v)^{-1-D/k}\exp\{y[(1+v)^{1/k}-1]\}. \qquad (2.14)$$

For $k = 1$, both (2.11) and (2.13) were given earlier by Carlitz [4, p. 525, (5.2) and (5.5)].

3. General generating functions like (2.11) can also be obtained for

$$B_n^c(x;k),\; S_n^{(\alpha,\beta)}(x,y;k) \quad \text{and} \quad T_n^{(\alpha,\beta)}(x,y;k),$$

We omit the proof which runs parallel to that of the above results. Here we mention the results directly.

$$\sum_{n=0}^{\infty} B_u^{(c+Dn)}(x+ny;k)u^n = \frac{(1+t)^{-\frac{(c+1)}{k}} \exp\{x[(1+t)^{1/k}-1]\}}{1+\frac{D}{k}\left(\frac{t}{1+t}\right)-\frac{y}{k}\, t(1+t)^{-1+1/k}},$$

where (3.1)

$$u = t(1+t)^{D/k}\exp\{-y[(1+t)^{1/k}-1]\}, \qquad (3.2)$$

$$\sum_{n=0}^{\infty} S_n^{(\alpha+vn,\,\beta+\delta n)}(x+nz_1, y+nz_2; k)u^n$$

$$= \frac{(1+t)^{\frac{\alpha-\beta}{k}} \exp\{(x-y)[(1+t)^{1/k}-1]\}}{1-\left(\frac{\nu+\delta}{k}\right)t(1+t)^{-1}-\frac{(z_1-z_2)}{k}\,t(1+t)^{-1+1/k}}, \tag{3.3}$$

where

$$u = t(1+t)^{-\frac{(v+\delta)}{k}} \exp\{-(z_1-z_2)[(1+t)^{1/k}-1]\}, \tag{3.4}$$

$$\sum_{n=0}^{\infty} T_n^{(\alpha+vn,\beta+\delta n)}(x+nz_1, y+nz_2; k)u^n$$

$$= \frac{(1-t)^{\frac{\alpha-\beta}{k}} \exp\{-(y-x)[(1-t)^{-1/k}-1]\}}{1+\frac{\nu+\delta}{k}t(1-t)^{-1}-\frac{(z_1-z_2)}{k}t(1-t)^{-1-1/k}}, \tag{3.5}$$

where

$$u = t(1-t)^{-\frac{(v+\delta)}{k}} \exp\{-(z_1-z_2)[(1-t)^{-1/k}-1]\} \tag{3.6}$$

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