Arithmeticity of vector-valued Siegel modular forms in analytic and p-adic cases

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Contents

§1. Introduction

Terminology. Modular forms = Vector-valued Siegel modular forms.

Aim. Construct p-adic counterparts of nearly holomorphic modular forms, i.e.,

	Classical modular forms	
Shimura's diff. operator ↙		↘ p-adic diff. operator
Nearly holomorphic modular forms (Eisenstein series)	same values at p-ordinary CM points $\longleftrightarrow$	p-adic and "Nearly algebraic" modular forms

Elliptic modular case (Katz). For positive integers k, l such that $k - l > 2$ is odd,

$$\frac{k!\pi^l}{2 \cdot \mathrm{Im}(z)^l} \sum_{(a,b) \in \mathbb{Z}^2 - \{(0,0)\}} \frac{(a + b\bar{z})^l}{(a + bz)^{k+1}} \longleftrightarrow \sum_{n=1}^{\infty} q^n \sum_{n=dd'} d^k (d')^l$$

which is used to construct p-adic (Hecke) L-functions.

$$(\because)\ \text{LHS} = \delta^l \left(\text{const.} + \textstyle\sum_{n=1}^{\infty} q^n \sum_{n=dd'} d^{k-l}\right); \ \delta = \frac{1}{2\pi\sqrt{-1}} \left(\frac{\mathrm{wt}}{z - \bar{z}} + \frac{\partial}{\partial z}\right)$$

$$\leftrightarrow \left(q\frac{d}{dq}\right)^l \left(\text{const.} + \textstyle\sum_{n=1}^{\infty} q^n \sum_{n=dd'} d^{k-l}\right) = \text{RHS}. \quad \square$$

§2. Classical modular forms

Notations We consider modular forms of degree $g > 1$, level $N \geq 3$, weight ρ.

- $\mathcal{H}_g = \{Z = {}^tZ \in M_g(\mathbb{C}) \mid \mathrm{Im}(Z) > 0\}$: Siegel upper half-space.
- $\Gamma(N) = \left\{ \gamma = \begin{pmatrix} A_\gamma & B_\gamma \\ C_\gamma & D_\gamma \end{pmatrix} \in Sp_{2g}(\mathbb{Z}) \;\middle|\; \gamma \equiv 1_{2g}\ (N) \right\}$: congruence subgroup
  $\Rightarrow \exists$ Shimura model of $\mathcal{H}_g/\Gamma(N)$ over $\mathbb{Z}[1/N, \zeta_N]$; $\zeta_N = e^{2\pi\sqrt{-1}/N}$.
- $\rho : GL_g \to GL_d$: representation over a sub $\mathbb{Z}[1/N, \zeta_N]$-algebra R of $\mathbb{C}$.

Classical modular forms are holomorphic maps $f : \mathcal{H}_g \to \mathbb{C}^d$ satisfying

$$f(\gamma(Z)) = \rho(C_\gamma Z + D_\gamma) \cdot f(Z) \;\; (\gamma \in \Gamma(N),\ Z \in \mathcal{H}_g).$$

q-expansion principle. $f \in \mathcal{M}_\rho(R) \overset{\text{def}}{=} \{\text{modular forms}/R \text{ of wt.}\rho\}$ if and only if

$$f(Z) = \textstyle\sum_T a(T) q^{T/N} \;\Rightarrow\; a(T) \in R^d.$$

Arithmeticity. For a field $k \supset R$, any classical modular form $f \in \mathcal{M}_\rho(k)$ satisfies

(A) $\begin{cases} \alpha \ :\ k\text{-rational CM point with basis } w_1, ..., w_g \text{ of regular } 1\text{-forms}, \\ P \doteqdot \text{(period symbols)} \in GL_g(\mathbb{C}) \text{ s.t. } {}^t(w_i) = P \cdot {}^t(du_i);\ \ (u_i) \in \mathbb{C}^g \\ \quad \Rightarrow \rho\left(P/(2\pi\sqrt{-1})\right)^{-1} \cdot f(\alpha) \in k^d. \end{cases}$

§3. Nearly holomorphic modular forms

Nearly holomorphic modular forms are analytic maps $f : \mathcal{H}_g \to \mathbb{C}^d$ satisfying

- $f(\gamma(Z)) = \rho(C_\gamma Z + D_\gamma) \cdot f(Z)$ $(\gamma \in \Gamma(N),\ Z \in \mathcal{H}_g)$.
- $f(Z) = \sum_T a(T) q^{T/N}$, where $a(T)$ consists of polynomials of the entries of $(\pi \cdot \mathrm{Im}(Z))^{-1}$.

For a subfield k of $\mathbb{C}$ containing ζ_N,

$$\mathcal{N}_\rho(k) \overset{\text{def}}{=} \{f : \text{nearly holomorphic} \mid a(T) : \text{polynomials } /k\}.$$

Arithmeticity. Any nearly holomorphic modular form $f \in \mathcal{N}_\rho(k)$ satisfies (A) when H^1_{DR} of the corresponding CM abelian variety splits over k.

(∵) Express f as the image of $\mathcal{M}_{\rho'}(k)$ by Shimura's diff. operator defined as

$$D^e_{\rho'} : \mathbb{E}_{\rho'} \xrightarrow{(1)} \mathbb{E}_{\rho'} \otimes \left(\Omega^1_{\mathcal{H}_g}\right)^{\otimes e} \xrightarrow{(2)} \mathbb{E}_{\rho'} \otimes \left(\mathrm{Sym}^2\left(\pi_*\left(\Omega^1_{\mathcal{X}/\mathcal{H}_g}\right)\right)\right)^{\otimes e}.$$

Here $\begin{cases} \mathbb{E}_{\rho'}\text{: automorphic bundle associated to } \rho', \\ (1) \Leftarrow \text{Gauss-Manin connection + Hodge decomposition}, \\ (2) \Leftarrow \text{Kodaira-Spencer map for } \mathbb{C}^g/(\mathbb{Z}^g + \mathbb{Z}^g \cdot Z)\ (Z \in \mathcal{H}_g). \end{cases}$ □

Remark. Shimura already proved the algebraicity of these CM values.

§4. p-adic modular forms

p-adic modular forms (Serre). For a prime $p \nmid N$ and a p-adic field $K \ni \zeta_N$,

$$\overline{\mathcal{M}}_\rho(K) \stackrel{\text{def}}{=} \{\lim_i f_{\rho_i} \mid f_{\rho_i} \in \mathcal{M}_{\rho_i}(K),\ \rho = \lim_i \rho_i\text{: continuous hom.}\},$$

where $\lim_i f_{\rho_i}$ is the limit as the Fourier expansions.

Theorem. If the representation ρ is defined over the integer ring of K, then

$\exists 1$ injective K-linear map $\iota_p : \mathcal{N}_\rho(K) \to \overline{\mathcal{M}}_\rho(K)$ satisfying

$$\begin{cases} \alpha & : \ p\text{-ordinary CM point with basis of regular } 1\text{-forms,} \\ P_p & \doteqdot \text{matrix of Kashio-Yoshida's } p\text{-adic period symbols} \end{cases}$$

$$\Rightarrow \rho\left(P/(2\pi\sqrt{-1})\right)^{-1} \cdot f(\alpha) = \rho(P_p)^{-1} \cdot \iota_p(f)(\alpha) \quad (f \in \mathcal{N}_\rho(K)).$$

Definition. We call elements of $\mathrm{Im}(\iota_p)$ "nearly algebraic" p-adic modular forms.

Proof of Theorem.

- Construction of ι_p : p-adic diff. operator ($\leftrightarrow$ Shimura's diff. operator D_ρ^e)

$$D_{p,\rho}^e(f) \stackrel{\text{def}}{=} \sum_{1\le i\le j\le g} q_{ij} \frac{\partial D_{p,\rho}^{e-1}(f)}{\partial q_{ij}} \quad \left(f \in \overline{\mathcal{M}}_\rho(K)\right).$$

- Definition of $\iota_p(f)$ for $f \in \mathcal{N}_\rho(k)$: Multiplying a modular form $\equiv 1$ (p),

$$\begin{aligned} &\exists g_i \in \mathcal{M}_{\rho\otimes\tau^{e_i}}(K) \text{ s.t. } f = \textstyle\sum_i \left(\theta_{e_i} \circ D_{\rho\otimes\tau^{e_i}}^{e_i}\right)(g_i) \\ &\left(\tau^{e_i} = \left(\mathrm{Sym}^2(K^g)^{\otimes e_i}\right)^\vee,\ \theta_{e_i}\text{: contraction}\right) \\ \Rightarrow\ &\iota_p(f) \stackrel{\text{def}}{=} \textstyle\sum_i \left(\theta_{e_i} \circ D_{p,\rho\otimes\tau^{e_i}}^{e_i}\right)(g_i). \end{aligned}$$

- Preserving p-ordinary CM values : For H_{DR}^1 of p-ordinary CM abelian varieties,

  Unit root space decomposition in $D_{p,\rho}^e$ = Hodge decomposition in D_ρ^e.

- Well-definedness and uniqueness of ι_p : In Serre-Tate's local moduli,

  $\exists$ nontriv. quasi-canonical lifts of ordinary abelian varieties $\longrightarrow$ canonical lift.

- Injectivity of ι_p : Hecke orbit of a point is dense in $\mathcal{H}_g/\Gamma(N)$. $\square$

§5. p-adic Siegel-Eisenstein series

Siegel-Eisenstein series. For a Dirichlet character χ modulo M, put

$$E_h(Z,s,\chi) = \sum_{\gamma \in (P\cap\Gamma_0(M))\backslash\Gamma_0(M)} \frac{\det(\mathrm{Im}(Z))^s \cdot \chi(\det(D_\gamma))}{\det(C_\gamma Z + D_\gamma)^h \cdot |\det(C_\gamma Z + D_\gamma)|^{2s}};$$

$\left\{\begin{array}{l}\text{abs. convergent, nearly holomorphic modular form of weight } h\text{, level } M \\ \text{if } s \text{ is an integer satisfying } (g+1-h)/2 < s \le 0.\end{array}\right.$

p-adic Siegel-Eisenstein series. Let

$$\left\{\begin{array}{l} N \ge 3 \text{ be a multiple of } M\text{, and } p \nmid N \text{ be a prime,} \\ h,\ s \text{ be integers such that } (g+1-h)/2 < s \le 0, \\ E_{h+2s}(Z,0,\chi) = \sum_T b_{h+2s}(T) q^T,\ \ \varepsilon_g(h) = \prod_{j=0}^{g-1}(h-j/2). \end{array}\right.$$

Then $\pi^{gs} E_h(Z,s,\chi)$ is defined over a cyclotomic field, and

$$\iota_p\left(\pi^{gs} E_h(Z,s,\chi)\right) = \prod_{i=0}^{-s-1} \varepsilon(h+2s+2i)^{-1} \sum_T b_{h+2s}(T) \det(T)^{-s} q^T.$$

$$\begin{aligned}(\because)\ \pi^{-g}\varepsilon_g(h) E_{h+2}(Z,-1,\chi) &= \left(\left(\mathrm{id}_{\mathbb{E}_\rho} \otimes \det\right) \circ D_\rho\right)\left(E_h(Z,0,\chi)\right) \\ &\leftrightarrow \theta\left(E_h(Z,0,\chi)\right);\ \theta\text{: theta operator.} \ \square\end{aligned}$$

§6. Related results and problems

Related results.

- Unitary modular case: Harris-Li-Skinner (2006), Eischen (2011–).
- Vector-valued p-adic diff. operators: Böcherer-Nagaoka (2007–).

Problems.

- Construct p-adic Siegel-Eisenstein measures and p-adic L-functions.

  $\Rightarrow \begin{cases} \text{Panchishkin (2000) gave such measures in the holomorphic case,} \\ \text{Böcherer-Schmidt (2000) gave } p\text{-adic measures for the standard } L. \end{cases}$

- Characterize nearly algebraic modular forms in the space of p-adic ones.
- $\exists$? Relation between nearly algebraic modular forms and overconvergent ones.
  $\Rightarrow$ Darmon-Rotger stated in the elliptic modular case:

$$\{\text{overconv. modular forms}\} \subsetneq \{\text{nearly overconv.} (\overset{?}{=} \text{nearly alg.}) \text{ forms}\}.$$

- $\exists$? Application of nearly algebraic modular forms to certain Selmer groups.
  $\Rightarrow$ Skinner-Urban applied unitary modular forms to Sel(elliptic curves).

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