JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **84**, 178–207 (1981)

A q-Umbral Calculus*

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An algebraic setting for the Roman–Rota umbral calculus is introduced. It is shown how many of the umbral calculus results follow simply by introducing a comultiplication map and requiring it to be an algebra map. The same approach is used to construct a q-umbral calculus. Our umbral calculus yields some of Andrews recent results on Eulerian families of polynomials as corollaries. The homogeneous Eulerian families are studied. Operator and functional expansions are also included.

1. INTRODUCTION

A sequence of polynomials $\{p_n(x)\}$ is of binomial type if $p_n(x)$ is of exact degree n for all non-negative integers n and $\{p_n(x)\}$ satisfies the binomial type theorem

$$p_n(x+y)=\sum_{k=0}^{n}\binom{n}{k}p_k(x)\,p_{n-k}(y). \tag{1.1}$$

A sequence of polynomials $\{p_n(x)\}$ is a polynomial set if $p_0=1$, and $p_n(x)$ is a polynomial of precise degree n, $n=0, 1, 2,\ldots$. The combinatorial theory of polynomials of binomial type was developed by Mullin and Rota [13], and in later papers by several authors. We refer the interested reader to the extensive bibliographies in Rota *et al.* [16] and in Roman and Rota [14]. The analytic theory of these polynomials is much older, see Sheffer [17]. Guinand's work [8] contains an interesting review of the classical umbral method. Goldman and Rota [7] suggested the importance of a similar study for polynomials related to enumeration problems in vector spaces over finite fields. Andrews develioped this theory in [1]. He introduced the concept of an Eulerian family of polynomials.

* This research was partially supported by a summer research fellowship from Arizona State University and NSF Contract MCS 7903518.

0022-247X/81/110178-30$02.00/0

DEFINITION 1.1. A polynomial set $\{p_n(x)\}$ is an Eulerian family if its members satisfy the functional relationship

$$p_n(xy) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q p_k(x)\, y^k p_{n-k}(y). \tag{1.2}$$

The Gaussian binomial coefficient $[{n \atop k}]_q$ is given by

$$\begin{aligned} \begin{bmatrix} n \\ k \end{bmatrix}_q &= 0 & &\text{if} \quad k > n \\ &= \frac{(q;q)_n}{(q;q)_k\,(q;q)_{n-k}} & &\text{if} \quad k \leqslant n, \end{aligned} \tag{1.3}$$

with

$$(a;q)_0 = 1 \quad \text{and} \quad (a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}), \quad n > 0. \tag{1.4}$$

The Gaussian binomial coefficient $[{n \atop k}]_q$ counts the number of k-dimensional subspace of an n-dimensional vector space over a field with q-elements, $GF(q)$. An Eulerian family is $\{\theta_n(x)\}$, where

$$\theta_0(x) = 1, \qquad \theta_n(x) = (x-1)(x-q)\cdots(x-q^{n-1}), \qquad n > 0. \tag{1.5}$$

Another Eulerian family is $((q;q)_n/n!)(x-1)^n$. We shall use the notation

$$(a;q)_\infty = \prod_{n=0}^{\infty} (1-aq^n). \tag{1.6}$$

A related important set is $\{\theta_n(x, y)\}$, with

$$\theta_0(x,y) = 1, \qquad \theta_n(x,y) = (x-y)(x-qy)\cdots(x-q^{n-1}y), \qquad n > 0. \tag{1.7}$$

Roman and Rota [14] introduced an umbral calculus by introducing a product on P^*, the dual of the algebra of polynomials over a field of characteristic zero. They defined the product of two functionals L and M by

$$\langle LM \mid x^n \rangle = \sum_{0}^{n} \binom{n}{j} \langle L \mid x^j \rangle \langle M \mid x^{n-j} \rangle, \tag{1.8}$$

and showed that under the usual addition and above multiplication P^* is a topological algebra. The topology is defined in the following manner.

DEFINITION 1.2. A sequence $\{L_n\}$ of linear functionals converges to $L \in P^*$ if and only if for every $p \in P$, there is an n_0 such that

$$\langle L_n \mid p(x) \rangle = \langle L \mid p(x) \rangle, \qquad \text{for} \quad n > n_0. \tag{1.9}$$

Roman and Rota refer to the algebra P^* as the umbral algebra. We shall invariably use P and $K[x]$ to denote the same algebra, namely, the polynomials over a commutative integral domain K.

One of the purposes of the present work is to construct an umbral calculus that plays in the theory of enumeration of vector spaces over $GF(q)$ the role played by the Roman–Rota umbral calculus in the theory of binomial enumeration. In this setting we replace the product (1.8) on P^* by

$$\langle LM \mid x^n\rangle = \langle L \mid x^n\rangle\langle M \mid x^n\rangle. \tag{1.10}$$

Section 2 contains the algebraic setting for the polynomials of binomial type. Although the results of Section 2 are not new, the presentation is certainly new. Section 3 contains the umbral lemma and its applications. This is one of the main results of the present paper. The umbral lemma tells us when two polynomials sets are "similar" in the sense that properties of one of them can be deduced from the other. Usually we have a model polynomial set and we would like to identify all the other "similar" polynomial sets. In the Roman–Rota umbral calculus the model set is $\{x^n\}$ and the class of similar polynomials is the class of polynomials of binomial type. Section 4 illustrates the intricate interplay of comultiplications in bialgebras, functional relationships like (1.1) and (1.2) and the product (of functionals) on the umbral algebra P^*. This is applied to derive some properties of the Eulerian families of polynomials. Section 5 contains further results on these polynomials. Some of the results of the present paper were announced in [10].

We attempted to include in the present work only the basic part of a q-umbral calculus with very few applications. Several related important topics, for example, the Lagrange inversion, are still under investigation.

In the remaining part of the introduction we include some standard definitions. K shall always denote a commutative integral domain.

DEFINITION 1.3. (Tensor Products). Let V_1 and V_2 be two modules. A tensor product is a pair (ϕ, G) such that

(i) ϕ is a bilinear mapping f of $V_1 \times V_2$ into the module G.

(ii) The range of ϕ spans G.

(iii) For very bilinear map f of $V_1 \times V_2$ into a module H there is a map g that maps G into H such that $f = g\phi$.

Usually $\phi(x, y)$ is denoted by $x \otimes y$ and G is denoted by $V_1 \otimes V_2$. If V_1 and V_2 are algebras then $V_1 \otimes V_2$ equipped with the product

(iv) $(x_1 \otimes y_1)(x_2 \otimes y_2) = x_1 x_2 \otimes y_1 y_2$, is also an algebra.

DEFINITION 1.4. (Tensor Product of Functionals). Let V_1, V_2 be two modules over a K. The tensor product $L_1 \otimes L_2$ of the linear functionals L_1 and L_2 maps $V_1 \otimes V_2$ into K via

$$\langle L_1 \otimes L_2 \mid v_1 \otimes v_2 \rangle = \langle L_1 \mid v_1 \rangle \langle L_2 \mid v_2 \rangle. \tag{1.11}$$

Similarly, the tensor product of mappings μ and ν is $(\mu \otimes \nu)(v_1 \otimes v_2) = \mu(v_1) \otimes \nu(v_2)$.

Throughout the present paper we shall use "functional" to mean "linear functional." We now introduce the concept of a bialgebra.

DEFINITION 1.5. (Bialgebras). A bialgebra V is an algebra over K equipped with a multiplication $\mu: V \otimes V \to V$, a unit $e: K \to V$, a comultiplication $\Delta: V \to V \otimes V$ and a counit $E: V \to K$ satisfying

(i) $\mu(\mu \otimes I) = \mu(I \otimes \mu)$ (associativity),

(ii) $\mu(I \otimes e) j_1 = \mu(e \otimes I) j_2 = I$ (e is a unit),

(iii) $(\Delta \otimes I)\Delta = (I \otimes \Delta)\Delta$ (coassociativity),

(iv) $j_2^{-1}(E \otimes I)\Delta = j_1^{-1}(I \otimes E)\Delta = I$ (E is a counit),

where I is the identity mapping of V into itself, $j_1, j_2, j_1^{-1}, j_2^{-1}$ are defined by

$$j_1(v) = v \otimes 1, \qquad j_2(v) = 1 \otimes v, \qquad j_1^{-1}(v \otimes 1) = v, \qquad j_2^{-1}(1 \otimes v) = v$$

for all $v \in V$ and 1 is the multiplicative unit in K.

Finally we define symmatric maps. Let V be a vector space over a field K and let J be the map:

$$J(v_1 \otimes v_2) = v_2 \otimes v_1, \qquad v_1, v_2 \in V, \tag{1.12}$$

defined on $V \otimes V$.

DEFINITION 1.6. (Symmetric Maps). A map I of V into $V \otimes V$ is symmetric if and only if

$$J \circ I = I, \tag{1.13}$$

where J is the map defined in (1.12).

DEFINITION 1.7. (ε_a and η^a). The functional ε_a and the operator η^a are defined on $K[x]$ by

$$\langle \varepsilon_a \mid p(x) \rangle = p(a) \tag{1.14}$$

and

$$(\eta^a p)(x) = p(ax), \tag{1.15}$$

respectively. In two variables the defining formulas are

$$\langle \varepsilon_{a,b} \mid p(x, y) \rangle = p(a, b) \tag{1.16}$$

and

$$(\eta^{a,b} p)(x, y) = p(ax, by). \tag{1.17}$$

2. An Algebraic Setting for Polynomials of Binomial Type

It has been generally felt that certain families of polynomials $\{p_n(x)\}$ may be formally manipulated as if $p_n(x)$ was actually x^n. Thus results that are easy to show for $p_n(x) = x^n$ would follow in an identical formal way for the other $p_n(x)$'s yielding less obvious results. Roman and Rota [14] made this process precise for polynomials of binomial type $\{p_n(x)\}$ by formulating what they termed "the umbral calculus." This formulation works because polynomials of binomial type are modelled after the monomials $\{x^n\}$ and the monomials satisfy the binomial theorem. This formulation then can be carried out in the same way for other polynomials. So, one may show results for these model polynomials and get as a corollary results for any "similar" sets of polynomials. The important contribution of this procedure is not providing results that are necessarily new or deep, but rather unifying seemingly different results in a simple way. In Section 3 we will illustrate what is involved in this procedure by testing the case of Eulerian families of polynomials in which the model polynomials are the θ_n's of (1.5). We hope that this will illustrate some aspects of the procedure which go unnoticed in the umbral calculus of Roman and Rota [14] because of the specific form of the model polynomials in that case. As we already mentioned in the Introduction, we give a general outline of the method by first reviewing the ideas of the umbral calculus in the present section. The model polynomials will have to satisfy functional relationships like (1.1) and (1.2). Our approach can be extended to treat model polynomials satisfying the relationships

$$P_n(\alpha(x, y)) = \sum_{j=0}^{n} a_j(y)\, p_j(x)\, p_{n-j}(y) \tag{2.1}$$

for some polynomial $\alpha(x, y)$.

We want to consider $p_n(x)$ to be x^n, $n = 0, 1, \ldots$, in some precise manner. This is not too difficult to do because $\{p_n(x)\}$ forms a basis for the algebra $K[x]$, the algebra of polynomials over K. $K[x]$ is endowed with the usual addition and multiplication. We now introduce a new product "$*$" on the set $K[x]$. The star product is defined relative to a polynomial set $\{p_n(x)\}$.

DEFINITION 2.1. ${}_*K[x]$ will denote the algebra of polynomials equipped with the usual addition, the usual multiplication by scalars and the star product "$*$" defined on the given set of polynomials $\{p_n(x)\}$ by

$$p_n * p_m = p_{m+n}, \tag{2.2}$$

and is extended to ${}_*K[x]$ by linearity, since $\{p_n(x)\}$ is a basis.

It is clear that the star product is well defined. We shall adopt the convention of adding a $*$ to a formula involving products when the star product is used instead of the ordinary product. Thus

$$p^{n*} := \underbrace{p * p * \cdots * p}_{n \text{ times}}. \tag{2.3}$$

Clearly

$$p_n = p_1^{n*}. \tag{2.4}$$

Observe that the functional equation (1.1) satisfied by polynomials of binomial type $\{p_n(x)\}$ implies $p_0(x) = 1$ and $p_n(0) = \delta_{n,0}$. So, at this stage we restrict ourselves to polynomials satisfying

$$p_0(x) = 1, \qquad p_1(0) = 0. \tag{2.5}$$

Combining this assumption with (2.3) we arrive at the crucial relationship

$$p_n(x) = (p_1(1))^n x^{n*}. \tag{2.6}$$

So, $p_n(x)$ is $(cx)^{n*}$, in a suitable multiplication. We must now relate this star multiplication to the usual multiplication in $K[x]$ since a concrete theorem involving the star multiplication will not be very useful without a translation to the language of $K[x]$. Of course it is obvious that the star multiplication coincides with the ordinary multiplication if and only if $p_n(x) = (cx)^n$, the trivial case. Luckily for us $K[x]$ is equipped with another very natural structure, the comultiplication Δ mapping $K[x]$ into $K[x] \otimes K[x]$ in the following manner:

$$\Delta(x) = x \otimes 1 + 1 \otimes x \tag{2.7}$$

and

$$\Delta(p(x)) = p(\Delta(x)), \qquad \text{for all} \quad p \in K[x]. \tag{2.8}$$

Recall that $K[x] \otimes K[x]$ is an algebra with

$$(q_1 \otimes q_2)(r_1 \otimes r_2) = q_1 r_1 \otimes q_2 r_2, \qquad q_1 q_2, r_1 r_2 \in K[x]. \tag{2.9}$$

Condition (2.8) just says that Δ is an algebra map. So, Δ is completely specified once we require it to be an algebra map and define it symmetrically on a generator. Both properties, i.e., being an algebra map and symmetry, are essential in what follows. Observe that $K[x]$ also happens to be a graded algebra and the comultiplication (2.7), (2.8) is actually grade preserving. This is just a coincidence and is not necessary. In fact the comultiplication $\Delta' : x \to x \otimes x$ used in the case of Eulerian families of polynomials in Section 3 is not grade preserving.

The principal use of the comultiplication is that it can be used to define a product of linear functionals on $K[x]$.

DEFINITION 2.2. Let L and M be two linear functionals on $K[x]$. The product functional LM is defined as

$$\langle LM \mid p(x) \rangle = \langle L \otimes M \mid \Delta(p(x)) \rangle, \qquad \text{for all} \quad p \in K[x]. \tag{2.10}$$

The above product of linear functionals is commutative since Δ is symmetric.

DEFINITION 2.3. The $*$ product of two linear functionals, say, L and M is defined by

$$\langle L * M \mid p(x) \rangle = \langle L \otimes M \mid \Delta^* P(x) \rangle, \quad \text{for all polynomials } p(x) \in {}_*K[x], \tag{2.11}$$

where Δ^* is the comultiplication on ${}_*K[x]$,

$$\Delta^* x = x \otimes 1 + 1 \otimes x, \tag{2.12}$$

and Δ^* is an algebra map.

We now observe that although the linear functionals on $K[x]$ do not depend on the product of members of $K[x]$, the products of such functionals, namely (2.10) and (2.11), seem to depend on the product used. So we next ask ourselves the question, When will the products (2.10) and (2.11) be equal? It is clear that this will happen if and only if $\Delta = \Delta^*$. When this holds we find ourselves in a very good position because any results that can be formulated using only the products of functionals (2.10) will hold for the star multiplication if they hold for the usual multiplication. This will accomplish the desired result.

PROPOSITION 2.4. *Let* $\{p_n(x)\}$ *be a polynomial set satisfying* (2.5) *and defining a star product. The comultiplications* Δ *and* Δ^* *are equal if and only if the sequence of polynomials* $\{p_n(x)\}$ *is of binomial type.*

Proof. Since $\Delta(x) = \Delta^*(x) = 1 \otimes x + x \otimes 1$ we obtain

$$\begin{aligned}\Delta^*(p_1(x)) &= \Delta^*(p_1(1)x) \\ &= (1 \otimes (p_1(1)x + (p_1(1)x) \otimes 1) = 1 \otimes p_1(x) + p_1(x) \otimes 1,\end{aligned}$$

hence

$$\begin{aligned}\Delta^*(p_n(x)) &= \Delta^*((p_1(x))^{n*}) = (\Delta^*(p_1(x)))^{n*} \\ &= (p_1(x) \otimes 1 + 1 \otimes p_1(x))^{n*} \\ &= \sum_{j=0}^{n} \binom{n}{j} (p_1(x) \otimes 1)^{j*} (1 \otimes p_1(x))^{(n-j)*} \\ &= \sum_{j=0}^{n} \binom{n}{j} (p_1(x))^{j*} \otimes (p_1(x))^{(n-j)*},\end{aligned}$$

since $(a \otimes 1)(1 \otimes b) = a \otimes b$. Therefore

$$\Delta^*(p_n(x)) = \sum_{j=0}^{n} \binom{n}{j} p_j(x) \otimes p_{n-j}(x).$$

On the other hand, we know that $\Delta(p_n(x))$ is $p_n(\Delta(x))$. Thus

$$\Delta(p_n(x)) = p_n(x \otimes 1 + 1 \otimes x).$$

Let us denote $x \otimes 1$ and $1 \otimes x$ by y and z, respectively. Clearly $\Delta = \Delta^*$ if and only if

$$\begin{aligned}p_n(y+z) &= \sum_{j=0}^{n} \binom{n}{j} p_j(x) \otimes p_{n-j}(x) \\ &= \sum_{j=0}^{n} \binom{n}{j} p_j(x \otimes 1)\, p_{n-j}(1 \otimes x) \\ &= \sum_{j=0}^{n} \binom{n}{j} p_j(y)\, p_{n-j}(z),\end{aligned}$$

where we again used $(a \otimes 1)(1 \otimes b) = a \otimes b$. This completes the proof.

Note that in the process of proving the above theorem we actually proved

COROLLARY 2.5. *Any sequence of polynomials* $\{p_n(x)\}$ *that satisfies* (2.5) *also satisfies*

$$\Delta^*(p_n(x)) = \sum_{j=0}^{n} \binom{n}{j} p_j(x) \otimes p_{n-j}(x). \tag{2.13}$$

We combine (2.13) and the definition of product of functionals to establish

COROLLARY 2.6. *When a sequence of polynomials $\{p_n(x)\}$ satisfies (2.5), the product of functionals L and M is given by*

$$\langle LM \mid p_n(x)\rangle = \sum_{j=0}^{n} \binom{n}{j} \langle L \mid p_j(x)\rangle\langle M \mid p_{n-j}(x)\rangle, \tag{2.14}$$

then (2.14) *holds whenever* $\{p_n(x)\}$ *is a sequence of polynomials of binomial type.*

COROLLARY 2.7. (Roman and Rota [14]). *If the product of functionals on $K[x]$ is defined by*

$$\langle LM \mid x^n\rangle = \sum_{j=0}^{n} \binom{n}{j} \langle L \mid x^j\rangle\langle M \mid x^{n-j}\rangle, \tag{2.15}$$

then (2.14) *holds whenever* $\{p_n(x)\}$ *is a sequence of polynomials of binomial type.*

Proof. Construct the star multiplication corresponding to the sequence $\{x^n\}$. Since $\{x^n\}$ is of binomial type, $\Delta = \Delta^*$ and (2.15) must give rise to the same product defined in (2.10). Finally the validity of (2.14) follows from Corollary 2.6.

Let us now see how Proposition 2.4 will enable us to find a result involving polynomials of binomial type. We shall prove

THEOREM 2.8. (Expansion Theorem). *Let $\{p_n(x)\}$ be a polynomial set of binomial type. Then*

$$p(x) = \sum_{0}^{\infty} \frac{1}{j!} \langle \tilde{L}^j \mid p(x)\rangle\, p_j(x), \tag{2.16}$$

where

$$\langle \tilde{L} \mid p_n(x)\rangle = \delta_{n,1}.$$

Proof. We start with the Taylor series expansion for polynomials, namely,

$$p(x) = \sum_{0}^{\infty} \frac{x^j}{j!} \langle M_j \mid p(x)\rangle, \tag{2.17}$$

M_j being the functional

$$\langle M_j \mid p(x)\rangle = \left.\frac{d^j}{dx^j}\, p(x)\right|_{x=0}. \tag{2.18}$$

The next step is to attempt to express M_j as a product of a functional by itself j times. Set

$$\langle M \mid p(x)\rangle = \frac{d}{dx} p(x)\Big|_{x=0};$$

so

$$\langle M \mid x^n\rangle = \delta_{n,0}.$$

Making use of (2.14) with $p_n(x) = x^n$ and straightforward induction we obtain

$$\langle M^j \mid x^n\rangle = n!\,\delta_{n,j}, \tag{2.19}$$

which identifies M^j as M_j (see (2.18)). This enables us to write (2.17) in the form

$$p(x) = \sum_0^\infty \frac{x^{j*}}{j!} \langle M^{j*} \mid p(x)\rangle$$

Therefore

$$p(x) = \sum_0^\infty \frac{x^{j*}}{j!} \langle M^{j*} \mid p(x)\rangle \tag{2.20}$$

follows from the fact that the starring map S: $x^n \to x^{n*}$ is an algebra isomorphism from $K[x]$ to ${}_*K[x]$. The ordinary and star products coincide because $p_n(x)$ is binomial type. Furthermore $p_n(x)$ is $(p_1(1)\,x)^{n*}$. Hence (2.20) becomes

$$p(x) = \sum_0^\infty \frac{p_j(x)}{j!} \left\langle \left(\frac{M}{p_1(1)}\right)^j \mid p(x)\right\rangle,$$

which is (2.16) with $\tilde{L} = (p_1(1))^{-1} M$.

Let us take a closer look at (2.16) and (2.17). The relationship (2.16) gives an expansion formula that uses only functionals and their products while (2.17) involves the differential operator followed by ε_0. Although both expansions are easy to establish for polynomials of binomial type, usually functional expansions like (2.16) are harder to prove. Operator expansions like (2.17) are usually trivial. We hope the subsequent section will clarify this point.

Professor Jack Freeman of Florida Atlantic University kindly pointed out to us that, in an unpublished manuscript [4], he had used the star product (2.2) to treat general polynomial sequences like sequences of monomials. However, he did not explore the connection with the comultiplication or the algebraic setting for the umbral calculus. Joni and Rota [12] treated some combinatorial problems using bialgebras.

3. The Umbral Lemma and Applications

We now generalize the results of Section 2 to almost any family of polynomials. The umbral lemma (see below) says, roughly speaking, that almost any sequence of polynomials may be considered the same as any other sequence as long as we are allowed to alter the multiplication. This will allow us to alter formulas involving one sequence of polynomials to find formulas for other sequences.

Definition. The polynomial $p^*(x)$ associated with a polynomial $p(x)$ and a multiplication $*$ is defined by

$$p^*(x) = \sum_j c_j x^{j*} \qquad \text{iff} \quad p(x) = \sum_j c_j x^j. \tag{3.1}$$

We now state and prove the umbral lemma

Theorem 3.1. (The Umbral Lemma). *Let $\{p_n(x)\}$ and $\{b_n(x)\}$ be two sequences of polynomials such that the* nth *polynomials is of precise degree n. Then* (a) *and* (b) *are equivalent.*

(a) *There is a unique multiplication $*$ on $K[x]$ such that*

(i) $p_n(x) = b_n^*(x)$, $n = 0, 1, \ldots$.

(ii) *There exists a mapping S: $(K[x], \cdot) \to (K[x], *)$, called the starring map, such that S is an isomorphism and $S(x) = x$, $S(1) = 1$.*

(b) $p_0(x) = b_0(x)$ *and* $p_1(x) = b_1(x)$.

Proof of (a) $\Rightarrow$ (b). Observe that $x^{1*} = x$ and $x^{0*} = 1^*$, the identity in ${}_*K[x]$. But $S(1) = 1$ implies $1 = 1^*$, that is, $x^{0*} = 1$. Hence $b_1^* = b_1$ and $b_0^* = b_0$.

Proof of (b) $\Rightarrow$ (a). The existence of the star product and the map S can be proved as follows. Define c_{mn}^k by

$$b_m(x)\, b_n(x) = \sum_k c_{mn}^k\, p_k(x)$$

and defined the $*$ operation by

$$p_m * p_n = \sum_k c_{mn}^k\, p_k(x),$$

and extend it to a multiplication on $K[x]$ by

$$\Big(\sum_j a_j p_j(x)\Big) * \Big(\sum c_k p_k(x)\Big) = \sum_{j,k} a_j c_k\, p_j * p_k.$$

where $a_j, c_k \in K$. This multiplication is clearly well defined. We define S by

$$S\left(\sum_j a_j b_j(x)\right) = \sum_j a_j p_j(x).$$

S is clearly a module isomorphism since $\{b_n(x)\}$ and $\{p_n(x)\}$ are bases. By the definition of $*$ S is an algebra map from $(K[x], \cdot)$ to $(K[x], *)$, so S is an algebra isomorphism. Note that

$$S(b_0(x)) = p_0(x) \qquad \text{and} \qquad b_0(x) = p_0(x) = a \cdot 1, \qquad a \neq 0.$$

imply $a(S(1) - 1) = 0$, hence $S(1) = 1$. Let

$$p_1(x) = b_1(x) = \alpha x + \beta \cdot 1,$$

so

$$S(\alpha x + \beta \cdot 1) = \alpha S(x) + \beta S(1) = \alpha S(x) + \beta \cdot 1 = \alpha x + \beta \cdot 1,$$

hence $\alpha(S(x) - x) = 0$ and $\alpha \neq 0$; that is, $S(x) = x$. Thus S satisfies condition (ii) of (a) and it remains to show that a(i) holds. Set

$$b_n(x) = \sum_j A_{nj} x^j \qquad \text{and} \qquad p_n(x) = S(b_n(x)).$$

Clearly

$$p_n(x) = \sum_j A_{nj} S(x^j) = \sum A_{nj}(S(x))^{j*} = \sum A_{nj} x^{j*} = b_n^*(x).$$

Thus we have shown that the proper $*$ exists and we now show that it is unique. If $*_1, *_2$ are two $*$ products on $K[x]$ and S_1, S_2 are their respective isomorphism then

$$S = S_2 S_1^{-1} : (K[x], *_1) \to (K[x], *_2),$$

is an isomorphism with $S(x) = x$, $S(1) = 1$. Since it is an algebra map we have

$$S(b_n^{*_1}(x)) = b_n^{*_2}(x),$$

that is, $S(p_n(x)) = p_n(x)$ by a(i), hence S must be the identify map. This concludes the proof of the umbral lemma.

The map S of the umbral lemma will be called the starring map and its inverse will be called the star erasing map. Next we investigate consequences of the umbral lemma. Our first result is

THEOREM 3.2. *Let $\{p_n(x)\}$ be any polynomial set. Then there exists a product $*$ such that*

$$p_n(x) = (c(x-a))^{n*}, \tag{3.2}$$

where $p_1(x) = c(x-a)$.

Proof. Define $\tilde{p}_n(x) = c^{-n} p_n(x+a)$, $b_n(x) = x^n$. Clearly $\tilde{p}_n(x)$ and $b_n(x)$ satisfy (b) in the umbral lemma and the result then follows from (a) in Theorem 3.1.

Note that for any polynomial set $\{p_n(x)\}$ there is no loss of generality in assuming $p_0(x) = 1$. When $a = 0$ in Theorem 3.2 we are lead to polynomials more general than the polynomials of binomial type. What forces these polynomials to be of binomial type is the comultiplication structures associated with Δ and Δ^*, as we saw in Section 2. Our next result covers the Eulerian families of polynomials, but unfortunately we have to go to two variables in order to handle them.

THEOREM 3.3. *Let $\{p_n(x, y)\}$ be a sequence of polynomials such that $p_n(x, y)$ is homogeneous of degree n and*

$$p_0(x, y) = 1, \qquad p_1(x, y) = c(x-y), \tag{3.3}$$

then there is a product $*$ on $\mathbb{R}[x, y]$ so that

$$p_n(x, y) = c^n (x-y) * (x-qy) * \cdots * (x-q^{n-1}y), \qquad n \geqslant 1. \tag{3.4}$$

Proof. Let $K = \mathbb{R}[y]$ and apply (b) in the unbral lemma to $b_n(x) = \theta_n(x, y)$ and $\tilde{p}_n(x) = c^{-n} p_n(x, y)$.

We shall adopt the notation

$$\theta_0^*(x, y) = 1, \ \theta_n^*(x, y) = (x-y) * (x-qy) * \cdots * (x-q^{n-1}y), \ n > 0. \tag{3.5}$$

We now come to the operator expansion and show how it follows easily from the umbral lemma. Let us introduce the necessary notations

DEFINITIONS 3.4. By an operator U we mean a mapping of $K[x]$ to $K[x]$. An operator U is a degree reducing operator if the degree of Ux^n is $n-1$. The functionals ε_a and ε_a^* are defined via

$$\langle \varepsilon_a \mid p(x) \rangle = \sum_j c_j a^j, \qquad \langle \varepsilon_a^* \mid p(x) \rangle = \sum_j c_j a^{j*}, \tag{3.6}$$

when $p(x) = \sum_j c_j x^j \in K[x]$.

The degree reducing operator expansion theorem is

THEOREM 3.5. (Degree Reducing Operator Expansion). *Let $\{p_n(x)\}$ be a polynomial set. Then there is a degree reducing operator U and a functional L such that the following expansion holds*:

$$p(x) = \sum_0^\infty \langle L \mid U^n p(x)\rangle\, p_n(x)/n!, \qquad p(x) \in K[x]. \tag{3.7}$$

Furthermore if $\langle \varepsilon_a \mid p_n(x)\rangle$ vanishes for all positive n then $L = \varepsilon_a$.

Proof. The Taylor series is

$$p(x) = \sum_{n=0}^\infty \langle \varepsilon_a \mid D^n p(x)\rangle\, (x-a)^n/n!, \tag{3.8}$$

where D is the differentiation operator d/dx. Let $p_1(x) = c(x-a)$ and use the $*$ product of Theorem 3.2. The operator D maps x^n to nx^{n-1}. The starring operator is an isomorphism mapping 1 to 1 and x to x. S maps (3.8) to

$$p^*(x) = \sum_{n=0}^\infty \langle \varepsilon_a^* \mid (D^*)^n\, p^*(x)\rangle\, (n-a)^{n*}/n!, \tag{3.9}$$

where D^* is defined on ${}_*K[x]$ by $D^* x^{n*} = nx^{(n-1)*}$, $n \geqslant 0$. Let U be the operator

$$Up(x) = S^{-1}\left(\frac{1}{c} D^* p^*(x)\right).$$

U is a degree reducing operator because

$$SUp_n(x) = \frac{D^*}{c}\, [c(x-a)]^{n*} = n[c(x-a)]^{(n-1)*} = np^*_{n-1}(x),$$

implies $Up_n(x) = np_{n-1}(x)$, hence the degree of Ux^n is $n-1$. The functional L is defined by $\langle L \mid p(x)\rangle = \langle \varepsilon_a^* \mid p^*(x)\rangle$.

When $\langle \varepsilon_a \mid p_n(x)\rangle = 0$, $n > 0$, then $L = \varepsilon_a$ because

$$\langle \varepsilon_a^* \mid p_n^*(x)\rangle = \langle \varepsilon_a \mid c^n (x-a)^n\rangle = 0.$$

This completes the proof.

For polynomials of binomial type $\{p_n(x)\}$ the functional L is ε_0 since $p_n(0) = \delta_{n,0}$. When $\{p_n(x)\}$ is an Eulerian family of polynomials $p_n(1)$ is $\delta_{n,0}$, hence L coincides with ε_1.

Theorem 3.5 implies another expansion theorem, namely,

COROLLARY 3.6. *For every polynomial set $\{p_n(x)\}$ with $p_0(x) = 1$ there* *exists a functional L and a degree reducing operator U such that any functional or operator A has the expansion*

$$A \cdot = \sum_{0}^{\infty} \frac{Ap_n(x)}{n!} \langle L \mid U^n \cdot \rangle. \tag{3.10}$$

Proof. Apply A to (3.7).

In Theorem 3.5 we discovered that a polynomial set $\{p_n(x)\}$ determines a degree reducing operator U and an expansion (3.7). We now show that the converse is also true, that is, U and (3.7) essentially determine the p_n's.

THEOREM 3.7. *Given a degree reducing operator U and a sequence of scalars* $\{a_n\}$ *there exists a unique polynomial set* $\{p_n(x)\}$ *and a unique functional L such that*

$$p_n(a_n) = 0, \qquad n > 0, \tag{3.11}$$

and (3.7) *hold.*

Proof. Let $\phi_{n-1}(x) = Ux^n$, $n > 0$. Since the degree of $\phi_n(x)$ is precisely n they span $\mathbb{R}[x]$. Define the polynomial set $\{p_n(x)\}$ inductively by

$$p_0(x) = 1, \qquad Up_n(x) = np_{n-1}(x), \qquad p_n(a_n) = \delta_{n,0}. \tag{3.12}$$

These polynomials are well defined and unique because U is onto, the null space of U is the constant polynomials. We now start with the p_n's and can easily show that the degree reducing operator U constructed in the proof of Theorem 3.5 agrees with the one we started with.

Next we prove L is unique. Set $p(x) = p_k(x)$, $k = 0, 1,...$ in (3.7) to obtain

$$\langle L \mid p_k(x) \rangle = \delta_{k,0},$$

which defines L uniquely on the p_n's, hence on all polynomials because $\{p_n(x)\}$ is a basis for $\mathbb{R}[x]$. This completes the proof.

DEFINITION 3.8. A polynomial set $\{p_n(x)\}$ is associated with a degree reducing operator U if

$$Up_n(x) = np_{n-1}(x). \tag{3.13}$$

Let us reexamine the role played by the sequence $\{a_n\}$ in Theorem 3.7 and its proof. The sequence was used to exhibit a functional M defined on the p_n's by

$$\langle M \mid p_n(x) \rangle = \delta_{n,0}. \tag{3.14}$$

and extended to all of $K[x]$ by linearity. This functional enables us to compute the constant term in $p_n(x)$ once all the other terms have been already computed from (3.13). The converse is also true in the sense that every polynomial set uniquely determines a degree reducing U satisfying (3.13) and generates a functional M via (3.14). This proves

THEOREM 3.9. *There is a one-to-one correspondence between polynomial sets* $\{p_n(x)\}$ *and pairs* $\{U, M\}$, *U is a degree reducing operator and M is a functional. Furthermore* (3.13) *and* (3.14) *hold.*

When the polynomials $p_1(x), p_2(x), \ldots$ *have a common zero a, M is* ε_a. *For polynomials of binomial type* $a = 0$ *while* $a = 1$ *for Eulerian families of polynomials. Our next result is an expansion theorem in polynomials that resemble the* θ_n*'s.*

THEOREM 3.10. (Operator Expansion Theorem). *Let* $\{p_n(x)\}$ *be a polynomial set with* $p_1(x) = x - 1$. *There exists an operator U and a functional L so that*

$$p(x) = \sum_0^{\infty} \frac{q^{-n(n-1)/2}}{(q;q)_n} (-1)^n \langle L \mid \theta_n(U)\, p(x)\rangle\, p_n(x). \tag{3.15}$$

Furthermore if $p_n(1) = \delta_{n,0}$ *then* $L = \varepsilon_1$.

Proof. We first consider the case $p_n(x) = \theta_n(x)$. In this case we claim that U is η^q. Set

$$\theta_n(x) = \sum_{k=0}^{n} c_{nk} x^k.$$

Clearly

$$\begin{aligned}\langle \varepsilon_1 \mid \theta_n(\eta^q)\, \theta_m(x)\rangle &= \sum_0^n c_{nk} \langle \varepsilon_1 \mid \theta_m(q^k x)\rangle = \sum_0^n c_{nk}\, \theta_m(q^k) \\ &= \sum_0^n c_{nk} \langle e_{q^k} \mid \theta_m(x)\rangle = \sum_0^n c_{nk} \langle \varepsilon_q^k \mid \theta_m(x)\rangle,\end{aligned}$$

where the functional product is given by (4.2). With that functional product ε_{q^k} is ε_q^k. Therefore

$$\begin{aligned}\langle \varepsilon_1 \mid \theta_n(\eta^q)\, \theta_m(x)\rangle &= \left\langle \sum_0^n c_{nk} \varepsilon_q^k \;\middle|\; \theta_m(x)\right\rangle = \langle \theta_n(\varepsilon_q) \mid \theta_m(x)\rangle \\ &= (-1)^n q^{n(n-1)/2} (q;q)_n\, \delta_{m,n},\end{aligned}$$

by (4.22). This establishes (3.15) when $p_n(x)$ is $\theta_n(x)$. We now use the umbral lemma to obtain the general result snce $p_n(x)$ and $\theta_n(x)$ coincide for $n = 0, 1$.

Note that the operator expansion (3.15) is easy to prove because all it uses is (3.12). What is nontrivial is finding a functional expansion where the coefficients of $p_n(x)$ are scalar multiples of $\langle r_n(L) \mid p(x) \rangle$ for some sequence of polynomials $\{r_n(L)\}$ and a certain functional L. In the next section we shall see that it is impossible to do so for Eulerian families of one variable. This situation can be remedied if we go to homogeneous polynomials in two variables.

Another treatment of polynomial sequences using general degree reducing operators appears in Freeman [4–6].

4. Eulerian Families of Polnomials

In the present section we explore certain properties of Eulerian families of polynomials via the comultiplication

$$\Delta: x \to x \otimes x. \tag{4.1}$$

As we pointed out in Section 2 the comultiplication induces a product on P^*, the dual of $K[x]$. The product of functionals is

$$\langle LM \mid p(x) \rangle = \langle L \otimes M \mid p(\Delta x) \rangle,$$

hence

$$\langle LM \mid x^n \rangle = \langle L \otimes M \mid (x \otimes x)^n \rangle = \langle L \otimes M \mid x^n \otimes x^n \rangle.$$

Consequently

$$\langle LM \mid x^n \rangle = \langle L \mid x^n \rangle \langle M \mid x^n \rangle. \tag{4.2}$$

We now characterize Eulerian families of polynomials in terms of functional products. Recall that a polynomial set $\{p_n(x)\}$ is an Eulerian family if and only if

$$p_n(xy) = \sum_0^n \begin{bmatrix} n \\ k \end{bmatrix}_q p_k(x)\, y^k p_{n-k}(y). \tag{4.3}$$

THEOREM 4.1. *A polynomial set $\{p_n(x)\}$ is an Eulerian family if and only if*

$$\langle LM \mid p_n(x) \rangle = \sum_0^n \begin{bmatrix} n \\ k \end{bmatrix}_q \langle L \mid p_k(x) \rangle \langle M \mid x^k p_{n-k}(x) \rangle. \tag{4.4}$$

Proof. Let $\{p_n(x)\}$ be an Eulerian family of polynomials. Clearly

$$\begin{aligned}
\langle LM \mid p_n(x)\rangle &= \langle L\otimes M \mid p_n(\Delta x)\rangle = \langle L\otimes M \mid p_n(x\otimes x)\rangle \\
&= \langle L\otimes M \mid p_n((x\otimes 1)(1\otimes x))\rangle \\
&= \left\langle L\otimes M \,\middle|\, \sum_0^n \begin{bmatrix} n \\ k \end{bmatrix}_q p_k(x\otimes 1)(1\otimes x)^k p_{n-k}(1\otimes x)\right\rangle \\
&= \sum_0^n \begin{bmatrix} n \\ k \end{bmatrix} \langle L\otimes M \mid p_k(x)\otimes x^k p_{n-k}(x)\rangle \\
&= \sum_0^n \begin{bmatrix} n \\ k \end{bmatrix} \langle L \mid p_k(x)\rangle\langle M \mid x^k p_{n-k}(x)\rangle.
\end{aligned}$$

and (4.4) follows. Conversely if (4.4) holds then

$$\begin{aligned}
\langle L\otimes M \mid p_n(\Delta x)\rangle &= \langle LM \mid p_n(x)\rangle \\
&= \sum_0^n \begin{bmatrix} n \\ k \end{bmatrix}_q \langle L \mid p_k(x)\rangle\langle M \mid x^k p_{n-k}(x)\rangle \\
&= \sum_0^n \begin{bmatrix} n \\ k \end{bmatrix}_q \langle L\otimes M \mid p_k(x)\otimes x^k p_{n-k}(x)\rangle \\
&= \left\langle L\otimes M \,\middle|\, \sum_0^n \begin{bmatrix} n \\ k \end{bmatrix}_q p_k(x)\otimes x^k p_{n-k}(x)\right\rangle.
\end{aligned}$$

This shows that

$$p_n(\Delta x) = \sum_0^n \begin{bmatrix} n \\ k \end{bmatrix}_q p_k(x)\otimes x^k p_{n-k}(x),$$

that is, since $\Delta x = x\otimes x = (x\otimes 1)(1\otimes x)$,

$$p_n((x\otimes 1)(1\otimes x)) = \sum \begin{bmatrix} n \\ k \end{bmatrix}_q p_k(x\otimes 1)(x\otimes 1)^k p_{n-k}(x\otimes 1),$$

which is a disguised form of (4.3).

Theorem 4.1 illustrates the connection between the functional relationships (4.3) and the product of functionals (4.2). For a given Eulerian family $\{p_n(x)\}$ set

$$p_n(x) = \sum_0^n c_{n,m} x^m, \qquad n = 0, 1,... \tag{4.5}$$

and

$$x^n = \sum_0^n d_{n,m} p_m(x), \qquad n = 0, 1, \ldots. \tag{4.6}$$

Our next result will eventually enable us to prove that the sequence of leading coefficients $\{c_{n,m}\}$ uniquely determines the Eulerian family $\{p_n(x)\}$. Clearly

$$d_{n,n} = 1/c_{n,n}. \tag{4.7}$$

THEOREM 4.2. *We have*

$$c_{n,m} = \begin{bmatrix} n \\ m \end{bmatrix}_q c_{m,m} p_{n-m}(0). \tag{4.8}$$

Proof. Define the sequence of functionals L_n, M_n via

$$\langle L_n \mid p_m(x) \rangle = \langle M_n \mid x^m \rangle = \delta_{m,n}. \tag{4.9}$$

We now compute $\langle L_n M_m \mid p_j(x) \rangle$ in two different ways and equate the results. Recalling (4.2) and (4.5) we get

$$\begin{aligned}\langle L_n M_m \mid p_j(x) \rangle &= \left\langle L_n M_m \,\middle|\, \sum_l c_{j,l} x^l \right\rangle = \sum_l c_{j,l} \langle L_n \mid x^l \rangle \langle M_m \mid x^l \rangle \\ &= c_{j,m} \langle L_n \mid x^m \rangle = c_{j,m} \left\langle L_n \,\middle|\, \sum_k d_{m,k} p_k(x) \right\rangle = c_{j,m} d_{m,n}.\end{aligned}$$

On the other hand (4.4) and (4.9) yield

$$\begin{aligned}\langle L_n M_m \mid p_j(x) \rangle &= \sum_k \begin{bmatrix} j \\ k \end{bmatrix}_q \langle L_n \mid p_k(x) \rangle \langle M_m \mid x^k p_{j-k}(x) \rangle \\ &= \begin{bmatrix} j \\ n \end{bmatrix}_q \left\langle M_m \,\middle|\, \sum_l c_{j-n,l} x^{n+l} \right\rangle = \begin{bmatrix} j \\ n \end{bmatrix}_q c_{j-n,m-n}.\end{aligned}$$

Therefore

$$c_{j,m} d_{m,n} = \begin{bmatrix} j \\ n \end{bmatrix}_q c_{j-n,m-n}. \tag{4.10}$$

The result now follows by letting $n = m$ in (4.10) and the proof is complete. The special case $m = j$ of (4.10) establishes the following result

COROLLARY 4.3.

$$d_{m,n} = \begin{bmatrix} m \\ n \end{bmatrix}_q c_{m-n,m-n}/c_{m,m}. \tag{4.11}$$

Observe that the relationship (4.11) determines d_{mn} uniquely when the sequence $\{c_{n,n}\}$ of leading terms in the p's is given. The uniqueness of the polynomials then follows from the uniqueness of the d's and simple induction since (recall (4.6) and (4.11))

$$c_{m,m}x^m = \sum_{n=0}^{m} \begin{bmatrix} m \\ n \end{bmatrix}_q c_{m-n,m-n}\, p_n(x). \tag{4.12}$$

Relationship (4.12) solves the connection coefficient problem expressing the monomials in terms of members of an Eulerian family of polynomials. The connection coefficients $\left[{n \atop m}\right]_q c_{m-n,m-n}$ have a very simple form indeed. The inverse problem, namely, expressing the p's in terms of the monomials, seems to be much harder. Luckily generating functions come to the rescue because a generating function is really an infinite family of identities. The next result characterizes Eulerian families of polynomials in terms of their generating functions.

THEOREM 4.4. (Andrews [1]). *A polynomial sequence* $\{p_n(x)\}$ *with* $p_0(x) = 1$ *is an Eulerian family if and only if it has a generating function*

$$\sum_{0}^{\infty} p_n(x) \frac{t^n}{(q;q)_n} = \frac{f(xt)}{f(t)}, \tag{4.13}$$

where

$$f(t) = \sum_{0}^{\infty} \gamma_n t^n/(q;q)_n, \qquad \gamma_0 = 1, \quad \gamma_n \neq 0, \quad n = 1, 2, \ldots, \tag{4.14}$$

and the leading term $c_{n,n}$ *in* $p_n(x)$ *is* γ_n.

Proof. It is easy to see that if the p_n's are generated by (4.13) then $c_{n,n}$ is γ_n. The functional equation (4.3) follows from (4.13), the observation

$$\frac{f(xyt)}{f(t)} = \frac{f(xyt)}{f(yt)} \frac{f(yt)}{f(t)}$$

and formal power series multiplication. To prove the converse we start with an Eulerian family of polynomials $\{p_n(x)\}$, pick γ_n to be $c_{n,n}$ and define $f(t)$ by (4.14). At this stage we go back to (4.12) multiply it by $t^m/(q;q)_n$ and sum over the values $m = 0, 1, \ldots$. This process leads to

$$f(xt) = f(t) \sum_{0}^{\infty} p_n(x)\, t^n/(q;q)_n,$$

after some easy manipulation. This completes the proof.

Andrews [1] proved Theorem 4.4 in a different way. As a matter of fact, our approach is completely different. We now investigate the Eulerian family $\{\theta_n(x)\}$. Let X, Y, Z be vector subspaces of V_n, the n-dimensional vector space over $GF(q)$. Assume that X, Y, Z contain x, y, z vectors, respectively. Goldman and Rota [7] showed that $\theta_n(x, y)$ counts the number of one-to-one linear transformations f of V_n into X such that $f(x) \cap Y = \{0\}$, and Y is a subspace of X. By a clever counting argument that squeezes a third subspace Z between the subspaces X, Y Goldman and Rota proved that

$$\theta_n(x, y) = \sum_0^n \begin{bmatrix} n \\ k \end{bmatrix}_q \theta_k(x, z)\, \theta_{n-k}(z, y). \tag{4.15}$$

From this one can easily use the obvious identity

$$\theta_n(x, y) = y^n \theta_n(x/y)$$

to show that the polynomials $\theta_n(x)$ satisfy (4.3), hence $\{\theta_n(x)\}$ is an Eulerian family. Andrews [1] gave a combinational argument, different from Goldman and Rota, to show that $\{\theta_n(x)\}$ is an Eulerian family. The polynomials $\{\theta_n(x)\}$ are monic, thus (4.12) reduces to

$$x^m = \sum_{n=0}^m \begin{bmatrix} m \\ n \end{bmatrix}_q \theta_n(x). \tag{4.16}$$

We now determine the inverse relation to (4.16), namely, the following

THEOREM 4.5. (Gauss' Binomial Theorem). *The polynomials $\theta_n(x)$ are given explicitly by*

$$\theta_n(x) = \sum_{m=0}^n \begin{bmatrix} n \\ m \end{bmatrix}_q (-1)^{n-m} q^{(n-m)(n-m-1)/2} x^m. \tag{4.17}$$

Proof. Use (4.5), (4.8) and

$$\theta_n(0) = (-1)^n q^{1+\cdots+n-1} = (-1)^n q^{n(n-1)/2}. \qquad \text{Q.E.D.}$$

Formula (4.17) may also be proved from (4.16) by using the Möbius inversion on the lattice of vector spaces over $GF(q)$ (see Rota [15]). The Möbius function of this lattice was first computed by Weisner. Note that (4.15) actually comprises (4.16) and (4.17) because (4.16) is (4.15) with $y = 0$ and $z = 1$, while (4.17) is (4.15) with $z = 0$ and $y = 1$.

As an application of Theorem 4.4 we prove a nonterminating version of the Gaussian binomial theorem.

THEOREM 4.5. (Heine's Binomial Theorem). *We have*

$$\sum_0^{\infty} \frac{(a;q)_n}{(q;q)_n} x^n = \frac{(ax;q)_\infty}{(x;q)_\infty}. \tag{4.18}$$

Proof. When we replace $p_n(x)$ in (4.13) by $\theta_n(x)$ and apply Euler's formula

$$\sum_0^{\infty} \frac{x^n}{(q;q)_n} = \frac{1}{(x;q)_\infty} \tag{4.19}$$

we get

$$\sum_0^{\infty} \theta_n(y) \frac{t^n}{(q;q)_n} = \frac{(t;q)_\infty}{(yt;q)_\infty}. \tag{4.20}$$

which is equivalent to (4.18) since $\theta_n(y) = y^n(1/y;q)_n$. This completes the proof.

Euler's formula (4.19) can be proved analytically by showing that both sides satisfy the functional equation

$$\frac{f(x) - f(qx)}{x} = f(x), \qquad \text{that is,} \quad f(x) = \frac{f(qx)}{1-x}$$

and are continuous at $x = 0$. A truly combinational proof using Ferrer's diagram for partitions of integers is in Hardy and Wright [9] (see also Andrews [1]). The sum (4.18) is an important result in special functions. Ismail [11] showed how to use it to give an incredibly simple proof of the Ramanujan ${}_1\Psi_1$ sum. Recently the Ramanujan sum has been useful in evaluating a new extension of the β integral (Andrews and Askey [3]). We next prove an expansion theorem that uses only products of functions and characterize the θ_n's in terms of such expansions.

THEOREM 4.7. (Expansion in terms of $\theta_n(x)$). *The expansion of an arbitrary polynomial $p(x)$ in terms of the θ's is*

$$p(x) = \sum_n \frac{q^{-n(n-1)/2}}{(q;q)_n} (-1)^n \langle \theta_n(\varepsilon_q) \mid p(x) \rangle \, \theta_n(x), \tag{4.21}$$

and $\theta_0(\varepsilon_q)$ is interpreted as ε_1.

Proof. It is sufficient to show

$$\langle \theta_n(\varepsilon_q) \mid \theta_m(x) \rangle = q^{n(n-1)/2} (q;q)_n \, (-1)^n \, \delta_{m,n}. \tag{4.22}$$

We use induction on n. Clearly (4.22) holds for $n=0$ because $\langle \varepsilon_1 \mid \theta_n(x)\rangle$ is indeed $\delta_{n,0}$. Next observe that

$$\theta_n(\varepsilon_q) = \theta_{n-1}(\varepsilon_q)(\varepsilon_q - q^{n-1}\varepsilon_1)$$

so

$$\langle \theta_n(\varepsilon_q) \mid \theta_m(x)\rangle = \sum_j \begin{bmatrix} m \\ j \end{bmatrix}_q \langle \theta_{n-1}(\varepsilon_q) \mid \theta_j(x)\rangle \langle \varepsilon_q - q^{n-1}\varepsilon_1 \mid x^j \theta_{m-j}(x)\rangle.$$

that is,

$$(-1)^{n-1} \langle \theta_n(\varepsilon_q) \mid \theta_m(x)\rangle$$
$$= \begin{bmatrix} m \\ n-1 \end{bmatrix}_q q^{(n-1)(n-2)/2}(q;q)_{n-1} \langle \varepsilon_q - q^{n-1}\varepsilon_1 \mid x^{n-1}\theta_{m-n+1}(x)\rangle. \quad (4.23)$$

Note that $\langle \varepsilon_q | x^l \theta_r(x)\rangle = 0$ unless $r = 0, 1$ while $\langle \varepsilon_1 | x^l \theta_r(x)\rangle = \delta_{r,0}$. Thus the right side of (4.23) vanishes unless $m = n$ or $m = n-1$. When $m = n$ we have

$$\langle \varepsilon_q - q^{n-1}\varepsilon_1 \mid x^{n-1}\theta_1(x)\rangle = q^{n-1}(q-1),$$

and when $m = n-1$ we have

$$\langle \varepsilon_q - q^{n-1}\varepsilon_1 \mid x^{n-1}\theta_0(x)\rangle = q^{n-1} - q^{n-1} = 0.$$

Combining these observations with (4.23) we obtain (4.22) and the proof is complete.

Since our model polynomials are $\{\theta_n(x)\}$, the expansion formula (4.21) suggests replacing $\{\theta_n(x)\}$ by any Eulerian family of polynomials $\{p_n(x)\}$ and replacing ε_q by some other functional L so $\theta_n(\varepsilon_q)$ becomes $\theta_n(L)$. Unfortunately this is not the case, and (4.21) or (4.22) in effect characterize the $\theta_n(x)$s up to scaling factors.

THEOREM 4.8. *The only functional L and Eulerian family $\{p_n(x)\}$ that satisfy*

$$\langle \theta_n(L) \mid p_m(x)\rangle = a_n \delta_{m,n} \quad (4.24)$$

for some sequence of nonzero constants $\{a_n\}$ are $L = \varepsilon_q$ and $p_n(x)) = k^n \theta_n(x)$, where $k = a_1/(q-1)$. Furthermore a_n is given by

$$a_n = (q;q)_n q^{n(n-1)/2} a_1^n/(1-q)^n.$$

Proof. Clearly

$$\begin{aligned}\langle\theta_{n+1}(L)\mid p_m\rangle &= \langle(L-q^n\varepsilon_1)\,\theta_n(L)\mid p_m\rangle\\ &= \sum \begin{bmatrix} m \\ j \end{bmatrix}_q \langle\theta_n(L)\mid p_j(x)\rangle\langle L-q^n\varepsilon_1\mid x^j p_{m-j}(x)\rangle\\ &= a_n \begin{bmatrix} m \\ n \end{bmatrix}_q \{\langle L\mid x^n p_{m-n}(x)\rangle - q^n\delta_{m,n}\}\end{aligned}$$

follows from the assumption (4.24). When $m=n$ one gets

$$0=\langle\theta_{n+1}(L)\mid p_n(x)\rangle = a_n\{\langle L\mid x^n\rangle - q^n\}.$$

Thus L is ε_q. At this stage we apply the functional L to (4.12) and obtain

$$c_{m,m}q^m = \sum_{n=0}^{m} \begin{bmatrix} m \\ n \end{bmatrix} c_{m-n,m-n}\langle L\mid p_n(x)\rangle. \tag{4.25}$$

Recall that $\theta_1(x)$ is $x-1$, so $\langle L\mid p_n(x)\rangle$ is $\langle\theta_1(L)+\varepsilon_1\mid p_n(x)\rangle$, that is,

$$\langle L\mid p_n(x)\rangle = a_1\delta_{n,1}+\delta_{n,0}. \tag{4.26}$$

The relationships (4.25) and (4.26) imply

$$c_{m,m}q^m = c_{m,m} + a_1\frac{(1-q^m)}{1-q}c_{m-1,m-1},$$

so $c_{m,m}=kc_{m-1,m-1}$, $m>0$, and a simple iteration identifies $c_{m,m}$ as k^m. With these values for $c_{m,m}$ we go back to (4.12) and find that

$$x^m = \sum_0^m \begin{bmatrix} m \\ n \end{bmatrix}_q p_n(x)\left(\frac{(q-1)}{a_1}\right)^n. \tag{4.27}$$

Comparing (4.27) and (4.16) identifies $((q-1)/a_1)^n\, p_n(x)$ as $\theta_n(x)$ for all n. This completes the proof.

In the next section we will extend the functional expansion (4.21) to expansion in terms of Eulerian families of polynomials by constructing a two variable umbral calculus.

5. Homogeneous Eulerian Families

We start this section by developing a two variable umbral calculus that will yields the functional expansion for Eulerian families of polynomials.

DEFINITION 5.1. Define the comultiplication Δ on the generators x and y of $\mathbb{R}[x, y]$ by

$$\Delta x = qx \otimes 1, \qquad \Delta y = 1 \otimes y \tag{5.1}$$

and extend it to $\mathbb{R}[x, y]$ using

$$\Delta p(x, y) = p(\Delta x, \Delta y). \tag{5.2}$$

The above definition induces the functional product

$$\langle LM \mid p(x, y)\rangle = L \otimes M \mid p(\Delta x, \Delta y)\rangle. \tag{5.3}$$

Our model polynomials are the two variable polynomials $\theta_n(x, y)$ introduced in (1.7). Let us compute the action of a product of two functionals on $\theta_n(x, y)$. Clearly

$$\begin{aligned}\langle LM \mid \theta_n(x, y)\rangle &= \langle L \otimes M \mid \theta_n(\Delta x, \Delta y)\rangle \\ &= \left\langle L \otimes M \,\middle|\, \sum_0^n \begin{bmatrix} n \\ j \end{bmatrix}_q \theta_j(qx \otimes 1, 1 \otimes 1)\, \theta_{n-j}(1 \otimes 1, 1 \otimes y)\right\rangle \\ &= \left\langle L \otimes M \,\middle|\, \sum_0^n \begin{bmatrix} n \\ j \end{bmatrix}_q \theta_j(qx, 1) \otimes \theta_{n-j}(1, y)\right\rangle.\end{aligned}$$

Therefore

$$\langle LM \mid \theta_n(x, y)\rangle = \sum_0^n \begin{bmatrix} n \\ j \end{bmatrix}_q \langle L \mid \theta_j(qx, 1)\rangle \langle M \mid \theta_{n-j}(1, y)\rangle. \tag{5.4}$$

The above product of functionals is not commutative. Set

$$\theta_n(L) := \sum_0^n \begin{bmatrix} n \\ j \end{bmatrix}_q (-1)^{n-j} q^{(n-j)(n-j-1)/2} L^j, \tag{5.5}$$

$$L^0 = \varepsilon_{1,1}, \qquad L^1 = L, \qquad L^j = L^{j-1} L \qquad \text{for} \quad j > 1. \tag{5.6}$$

THEOREM 5.2. *Under the above functional multiplication we have*

$$\langle \theta_n(\varepsilon_{q,1}) \mid \theta_m(x, y)\rangle = (-1)^n (q; q)_n\, q^{n(n-1)/2} \delta_{m,n}. \tag{5.7}$$

Proof. Using (5.4) we obtain

$$\langle \varepsilon_{a,b} \varepsilon_{c,d} \mid \theta_m(x, y)\rangle = \sum_0^m \begin{bmatrix} m \\ j \end{bmatrix}_q \theta_j(qa; 1)\, \theta_{m-j}(1, d) = \theta_m(qa, d),$$

by (4.15). Thus

$$\langle \varepsilon_{a,b}\varepsilon_{c,d} \mid \theta_m(x, y)\rangle = \langle \varepsilon_{aq,d} \mid \theta_m(x, y)\rangle$$

holds. A calculation gives

$$\langle \varepsilon_{q,1}^j \mid \theta_m(x, y)\rangle = \theta_m(q^j, 1),$$

that is,

$$\langle \varepsilon_{q,1}^j \mid \theta_m(x, y)\rangle = \theta_m(q^j) = \langle \varepsilon_{q^j} \mid \theta_m(x)\rangle.$$

Recall that in the single variable case with the comultiplication $\Delta x = x \otimes x$ the functional ε_{q^j} is nothing but ε_q^j, see the proof of Theorem 3.10. Therefore

$$\langle \varepsilon_{q,1}^j \mid \theta_m(x, y)\rangle = \langle \varepsilon_q^j \mid \theta_m(x)\rangle. \tag{5.8}$$

Now (5.8) implies

$$\langle \theta_n(\varepsilon_{q,1}) \mid \theta_m(x, y)\rangle = \langle \theta_n(\varepsilon_q) \mid \theta_m(x)\rangle$$

and the result (5.7) follows from (4.22). This completes the proof.

DEFINITION 5.3. (Andrews [1]). We say that $\{p_n(x, y)\}$ is a homogeneous Eulerian family of polynomials if each $p_n(x, y)$ is a homogeneous polynomial of degree n in x and y, $p_0(x, y) = 1$, $p_n(x, 0) \neq 0$ and

$$p_n(x, y) = \sum_0^n \begin{bmatrix} n \\ k \end{bmatrix}_q p_k(x, z)\, p_{n-k}(z, y). \tag{5.9}$$

Observe that homogeneous Eulerian families of polynomials satisfy

$$p_n(x, x) = 0, \tag{5.10}$$

as can be seen from (5.9). The polynomials $\theta_n(x, y)$ form a homogeneous Eulerian families of polynomials.

LEMMA 5.4. *Let $\{p_n(x, y)\}$ be a homogeneous Eulerian family of polynomials with $p_1(x, y) = x - y$. Then there exists an isomorphism S from $(K[x], \cdot)$ to $(K[x], *)$, $K := \mathbb{R}[y]$ such that*

$$\theta_n^*(x, y) := S(\theta_n(x, y)) = p_n(x, y). \tag{5.11}$$

Furthermore we have

$$\theta_n^*(a, y) = \theta_n(a, y), \qquad \theta_n^*(x, a) = p_n(x, a), \qquad \theta_n^*(ax, y) = p_n(ax, y),$$
$$\textit{for all} \quad a \in K = \mathbb{R}[y]. \tag{5.12}$$

Proof. Let

$$\theta_n(x, y) = \sum_{k,j} c_{kj}^n x^k y^j,$$

and define S and $\theta_n^*(x, y)$ as in (5.11). Therefore

$$\theta_n^*(x, y) = \sum_{k,j} c_{kj}^n x^{k*} y^{j*} = \sum_{k,j} c_{kj}^n x^{k*} y^j,$$

because

$$x^{k*} y^{j*} = S(x^k, y^j) = y^j S(x^k) = y^j x^{k*}.$$

This proves the first two equalities in (5.12). The third equality follows from the homogeneity of both $\theta_n(x, y)$ and $p_n(x, y)$ and

$$\theta_n^*(ax, y) = a^n \theta_n^*(x, y/a) = a^n p_n(x, y/a) = p_n(ax, y).$$

This completes the proof.

We now extend the functional expansion theorem (Theorem 4.7) to homogeneous Eulerian families of polynomials.

THEOREM 5.5. *If* $\{p_n(x, y)\}$ *is a homogeneous Eulerian family of polynomials then there exists an associated functional* M *such that*

$$\langle \theta_n(M) \mid p_m(x, y) \rangle = (-1)^n (q; q)_n q^{n(n-1)/2} \delta_{mn}, \tag{5.13}$$

where the functional multiplication is as in (5.3) *and* (5.6).

Proof. Apply the map S of Lemma 5.4 to (5.7) to find

$$\langle \theta_n^*(M) \mid p_m(x, y) \rangle = (-1)^n (q; q)_n q^{n(n-1)/2} \delta_{mn}, \tag{5.14}$$

where $M = \varepsilon_{q,1}^*$. We claim that

$$\langle \theta_n^*(M) \mid p_m(x, y) > = \langle \theta_n(M) \mid p_m(x, y) \rangle. \tag{5.15}$$

Once we establish (5.15) the assertion (5.13) will follow from (5.14) and (5.15). First we show that

$$\langle L_1 L_2 \mid p_n \rangle = \langle L_1 * L_2 \mid p_n \rangle, \tag{5.16}$$

where $L_i = \tilde{L}_i \varepsilon_{x,1}$ and $\tilde{L}_i$ are functionals defined on $\mathbb{R}[x]$. We have

$$\langle L_1 L_2 \mid p_n \rangle = \sum_0^n \begin{bmatrix} n \\ j \end{bmatrix}_q \langle L_1 \mid p_j(qx, 1) \rangle \langle L_2 \mid p_{n-j}(1, y) \rangle \tag{5.17}$$

and

$$\begin{aligned} \langle L_1 * L_2 \mid p_n \rangle &= \langle L_1 * L_2 \mid \theta_n^* \rangle \\ &= \sum_0^n \begin{bmatrix} n \\ j \end{bmatrix}_q \langle L_1 \mid \theta_j^*(qx, 1) \rangle \langle L_2 \mid \theta_{n-j}^*(1, y) \rangle \\ &= \sum_0^n \begin{bmatrix} n \\ j \end{bmatrix}_q \langle L_1 \mid p_j(qx, 1) \rangle \langle L_2 \mid \theta_{n-j}(1, y) \rangle \end{aligned}$$

using 5.12. Now $\langle L_2 \mid \theta_{n-j}(1, y) \rangle = \langle \tilde{L}_2 \mid \theta_{n-j}(1, 1) \rangle = \delta_{n-j,0} \langle \tilde{L}_2 \mid 1 \rangle$. Thus $\langle L_1 * L_2 \mid p_n \rangle = \langle \tilde{L}_2 \mid 1 \rangle \langle L_1 \varepsilon_{qx,1} \mid p_n \rangle$. Since $\langle L_2 \mid p_{n-j}(1, y) \rangle = \delta_{n-j,0} \langle \tilde{L}_2 \mid 1 \rangle$ we also find $\langle L_1 L_2 \mid p_n \rangle = \langle \tilde{L}_2 \mid 1 \rangle \langle L_1 \varepsilon_{qx,1} \mid p_n \rangle$ using (5.17). This shows (5.16). We next observe that $L_1 L_2 = (\tilde{L}_1 \varepsilon_{qx,1}) \varepsilon_{x,1}$ so $L_1 L_2$ is again of the form $\tilde{L}_3 \varepsilon_{x,1}$, where $\tilde{L}_3$ is a functional on $\mathbb{R}[x]$. Now (5.15) follows easily from the following induction statement: For each n if $\deg p_n(x) = n$ then $\langle p_n(M) \mid \theta_m(x) \rangle = \langle p_n^*(M) \mid \theta_m(x) \rangle$ and $p_n(M) = \tilde{M} \varepsilon_{x,1}$ for some functional $\tilde{M}$ on $\mathbb{R}[x]$. For $n = 0$ or $n = 1$ this statement is clear since $M = \varepsilon_{q,1}^* = \varepsilon_{q,1}^* \varepsilon_{x,1}$. For $n \geqslant 2$ we may write $p_n(x)$ as $a_n x^n + p(x)$, where $\deg(p(x)) \leqslant n-1$. By induction M^{n-1} and $p(M)$ are each of the desired form with $\langle M^{n-1} \mid \theta_m(x) \rangle = \langle M^{*(n-1)} \mid \theta_m(x) \rangle$ and

$$\langle p(M) \mid \theta_m(x) \rangle = \langle p^*(M) \mid \theta_m(x) \rangle. \tag{5.18}$$

Thus by the above $a_n M^n = a_n M(M^{n-j})$ will be of the correct form and will satisfy $\langle a_n M^n \mid \theta_n(x) \rangle = \langle a_n M^{*n} \mid \theta_n(x) \rangle$. Add this to (518) to give the result.

We now use this result to give an expansion theorem in one variable.

THEOREM 5.6. *Define $\Delta(x) = qx \otimes 1$ and $\Delta p(x) = p(\Delta(x))$. If $p_n(x)$ is an Eulerian family of polynomials there is a functional L such that*

$$p(x) = \sum \frac{\langle \theta_n(L) \mid p(x) \rangle}{(-1)^n (q; q)_n q^{n(n-1)/2}} p_n(x).$$

Proof. Let $p(x) = \sum_m b_m p_m(x)$. We show $(-1)^n (q; q)_n q^{n(n-1)/2} b_n = \langle \theta_n(L) \mid p(x) \rangle$. Define $p_n(x, y) = y^n p_n(x/y)$. $P_n(x, y)$ forms a homogeneous Eulerian family ([1]) and $p_n(x, 1) = p_n(x)$. One can easily show by induction that $L^n = L\varepsilon_{q^n x}$, where L is any functional on $\mathbb{R}[x]$. Let $L = \varepsilon_q^*$. Then the M in the previous theorem is $L\varepsilon_{x,1}$. We have

$$
\begin{aligned}
(-1)^n (q;q)_n q^{(n-1)/2}\delta_{nm} &= \langle \theta_n(M) \mid p_m(x,y)\rangle \\
&= \left\langle \sum_j a_{nj} M^j + a_{n0}\varepsilon_{1,1} \mid p_m(x,y)\right\rangle \\
&= \left\langle \sum_j a_{nj} L\varepsilon_{qj_x} + a_{n0}\varepsilon_1 \mid p_m(x,1)\right\rangle \\
&= \langle \theta_n(L) \mid p_m(x)\rangle.
\end{aligned}
$$

Writing out $\langle \theta_n(L) \mid p(x)\rangle = \sum_m b_m \langle \theta_n(L) \mid p_n(x)\rangle$ gives the desired equation for b_m.

References

1. G. E. Andrews, On the foundations of combinatorial theory. V. Eulerian differential operators, *Stud. Appl. Math.* **50** (1971), 345–375.
2. G. E. Andrews, The Theory of Partitions, *in* "Encyclopedia of Mathematics and its Applications," Vol. 2, Addison–Wesley, Reading, Mass., 1976.
3. G. E. Andrews and R. A. Askey, Another q-extension of the beta function, *Proc. Amer. Math. Soc.* **81** (1981), 97–100.
4. J. M. Freeman, A. Weyl algebra approach to connection constants, talk delivered at the conference on Umbral Calculus and Hopf Algebras, May 1978, at Norman, Okla., unpublished.
5. J. M. Freeman, New solutions to the Rota/Mullin problem of connection constants, *in*, "Proceedings, 9th S. E. Conference on Combinatorics, Graph Theory and Computing, 1978," p. 301–305.
6. J. M. Freeman, Pincherle's formula, *in* "Proceddings, 10th Southeastern Conference on Combinatorics, Graph Theory and Computing, 1979, p. 447–464.
7. J. Goldman and G. C. Rota, On the foundations of combinatorical theory. IV. Finite vector spaces and Eulerian generating functions, *Stud. Appl. Math.* **49** (1970), 239–258.
8. A. P. Guinand, The umbral method: A survey of elementary mnemonic and manipulative uses, *Amer. Math. Monthly* **86** (1979), 187–195.
9. G. H. Hardy and E. M. Wright, "An Introduction to the Theory of Numbers," 4th ed., Oxford Univ. Press, Oxford, 1960.
10. E. C. Ihrig and M. E. H. Ismail, On an umbral calculus, *in* "Proceedings, 10th Southeastern Conference on Combinatorics, Graph Theory and Computing, 1979," p. 523–528.
11. M. E. H. Ismail, A simple proof of Ramanujan's ${}_1\Psi_1$ sum, *Proc. Amer. Math. Soc.* **63** (1977), 185–186.
12. S. A. Joni and G. C. Rota, Coalgebras and bialgebras in combinatorics, *Stud. Appl. Math.* **61** (1978), 93–139.
13. R. Mullin and G. C. Rota, On the foundations of combinatorial theory. III. Theory of binomial enumeration, *in* "Graph Theory and Its Applications" (B. Harris, Ed.), p. 167–213, Academic Press, New York, 1970.
14. S. M. Roman and G. C. Rota, The umbral calculus, *Adv. in Math.* **27** (1978), 95–188.
15. G. C. Rota, On the foundations of combinatorial theory. I. Theory of Möbius functions. *Zeit. Wahrsch. Verw. Gebiete* **2** (1964), 340–368.

16. G. C. ROTA, D. KAHANER, AND A. ODLYZKO, On the foundations of combinatorial Theory. VIII. Finite operator calculus, *J. Math. Anal. Appl.* **42** (1973), 685–760.
17. I. M. SHEFFER, Some properties of polynomials of type zero, *Duke Math. J.* **5** (1939), 590–622.
18. L. J. SLATER, "Generalized Hypergeometric Functions," Cambridge Univ. Press, Cambridge, 1966.