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Some properties of the generalized Fibonacci and Lucas sequences related to the extended Hecke groups

Sebahattin İkikardes[1*] and Zehra Sarıgedik[2]

Dedicated to Professor Hari M Srivastava

*Correspondence: skardes@balikesir.edu.tr
[1]Fen-Edebiyat Fakültesi, Matematik Bölümü, Balıkesir Üniversitesi, Balıkesir, 10145, Turkey
Full list of author information is available at the end of the article

Abstract

In this paper, we define a sequence, which is a generalized version of the Lucas sequence, similar to the generalized Fibonacci sequence given in Koruoğlu and Şahin in Turk. J. Math. 2009, doi:10.3906/mat-0902-33. Also, we give some connections between the generalized Fibonacci sequence and the generalized Lucas sequence, and we find polynomial representations of the generalized Fibonacci and the generalized Lucas sequences, related to the extended Hecke groups given in Koruoğlu and Şahin in Turk. J. Math. 2009, doi:10.3906/mat-0902-33.

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1 Introduction

In [1], Hecke introduced groups $H(\lambda)$, generated by two linear fractional transformations

$$T(z) = -\frac{1}{z} \quad \text{and} \quad S(z) = -\frac{1}{z+\lambda},$$

where λ is a fixed positive real number. Hecke showed that $H(\lambda)$ is discrete if and only if $\lambda = \lambda_q = 2\cos\frac{\pi}{q}$, $q \in \mathbb{N}$, $q \geq 3$, or $\lambda \geq 2$. These groups have come to be known as the *Hecke Groups*, and we will denote them $H(\lambda_q)$, $H(\lambda)$ for $q \geq 3$, $\lambda \geq 2$, respectively. The Hecke group $H(\lambda_q)$ is the Fuchsian group of the first kind when $\lambda = \lambda_q$ or $\lambda = 2$, and $H(\lambda)$ is the Fuchsian group of the second kind when $\lambda > 2$. In this study, we focus on the case $\lambda = \lambda_q$, $q \geq 3$. The Hecke group $H(\lambda_q)$ is isomorphic to the free product of two finite cyclic groups of orders 2 and q, and it has a presentation

$$H(\lambda_q) = \langle T, S \mid T^2 = S^q = I \rangle \cong C_2 * C_q, \quad [2]. \tag{1}$$

The first several of these groups are $H(\lambda_3) = \Gamma = \mathrm{PSL}(2,\mathbb{Z})$ (the modular group), $H(\lambda_4) = H(\sqrt{2})$, $H(\lambda_5) = H(\frac{1+\sqrt{5}}{2})$, and $H(\lambda_6) = H(\sqrt{3})$. It is clear that $H(\lambda_q) \subset \mathrm{PSL}(2,\mathbb{Z}[\lambda_q])$, for $q \geq 4$. The groups $H(\sqrt{2})$ and $H(\sqrt{3})$ are of particular interest, since they are the only Hecke groups, aside from the modular group, whose elements are completely known (see, [3]).

The extended Hecke group, denoted by $\overline{H}(\lambda_q)$, has been defined in [4] and [5] by adding the reflection $R(z) = 1/\overline{z}$ to the generators of the Hecke group $H(\lambda_q)$. The extended Hecke group $\overline{H}(\lambda_q)$ has a presentation

$$\langle T, S, R \mid T^2 = S^q = R^2 = I, RT = TR, RS = S^{q-1}R \rangle \cong D_2 *_{\mathbb{Z}_2} D_q. \tag{2}$$

The Hecke group $H(\lambda_q)$ is a subgroup of index 2 in $\overline{H}(\lambda_q)$. It is clear that $\overline{H}(\lambda_q) \subset \mathrm{PGL}(2, \mathbb{Z}[\lambda_q])$ when $q > 3$ and $\overline{H}(\lambda_3) = \mathrm{PGL}(2, \mathbb{Z})$ (the extended modular group $\overline{\Gamma}$).

Throughout this paper, we identify each matrix A in $\mathrm{GL}(2, \mathbb{Z}[\lambda_q])$ with $-A$, so that they each represent the same element of $\overline{H}(\lambda_q)$. Thus, we can represent the generators of the extended Hecke group $\overline{H}(\lambda_q)$ as

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & -1 \\ 1 & \lambda_q \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In [6], Koruoglu and Sahin found that there is a relationship between the generalized Fibonacci numbers and the entries of matrices representations of some elements of the extended Hecke group $\overline{H}(\lambda_q)$. For the elements

$$h = TSR = \begin{pmatrix} \lambda_q & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad f = RTS = \begin{pmatrix} 0 & 1 \\ 1 & \lambda_q \end{pmatrix}$$

in $\overline{H}(\lambda_q)$, then the kth power of h and f are

$$h^k = \begin{pmatrix} a_k & a_{k-1} \\ a_{k-1} & a_{k-2} \end{pmatrix} \quad \text{and} \quad f^k = \begin{pmatrix} a_{k-1} & a_k \\ a_k & a_{k+1} \end{pmatrix},$$

where $a_0 = 0$, $a_1 = 1$, and for $k \geq 2$,

$$a_k = \lambda_q a_{k-1} + a_{k-2}. \tag{3}$$

For all $k \geq 2$,

$$a_k = \frac{1}{\sqrt{\lambda_q^2 + 4}} \left[\left(\frac{\lambda_q + \sqrt{\lambda_q^2 + 4}}{2} \right)^{k+1} - \left(\frac{\lambda_q - \sqrt{\lambda_q^2 + 4}}{2} \right)^{k+1} \right]. \tag{4}$$

Notice that this real numbers sequence is a generalized version of the common Fibonacci sequence. If $\lambda_q = 1$, this sequence coincides with the Fibonacci sequence.

The Fibonacci and the Lucas sequence have been studied extensively and generalized in many ways. For example, you can see in [7–12]. In this paper, firstly, we define a sequence b_k, which is a generalization of the Lucas sequence. Then we give some properties of these sequences and the relationships between them. To do this, we use some results given in [13–15]. In fact, in [14] and [15], Özgür found two sequences, which are the generalization of the Fibonacci sequence and the Lucas sequence, in the Hecke groups $H(\lambda)$, $\lambda \geq 2$ real. But the Hecke groups $H(\lambda)$ are different from the Hecke groups $H(\lambda_q)$, $\lambda_q = 2\cos\frac{\pi}{q}$, $q \in \mathbb{N}$, $q \geq 3$.

2 Some properties of generalized Fibonacci and generalized Lucas sequences

Firstly, we define a sequence b_k by

$$b_k = \lambda_q b_{k-1} + b_{k-2} \tag{5}$$

for $k \geq 2$, where $b_0 = 2$, $b_1 = \lambda_q$.

Proposition 1 *For all* $k \geq 2$,

$$b_k = \left(\frac{\lambda_q + \sqrt{\lambda_q^2 + 4}}{2}\right)^k + \left(\frac{\lambda_q - \sqrt{\lambda_q^2 + 4}}{2}\right)^k. \tag{6}$$

Proof To solve (6), let b_k be a characteristic polynomial r^k. Then we have the equation

$$r^k = \lambda_q r^{k-1} + r^{k-2} \quad \Rightarrow \quad r^2 - \lambda_q r - 1 = 0.$$

The roots of this equation are

$$r_{1,2} = \frac{\lambda_q \pm \sqrt{\lambda_q^2 + 4}}{2}.$$

Using these roots $r_{1,2}$, we can find a general formula of the general term b_k. If we write b_k as combinations of the roots $r_{1,2}$, then we have

$$b_k = A\left(\frac{\lambda_q + \sqrt{\lambda_q^2 + 4}}{2}\right)^k + B\left(\frac{\lambda_q - \sqrt{\lambda_q^2 + 4}}{2}\right)^k.$$

To determine constants A and B, we use two boundary conditions $b_0 = 2$ and $b_1 = \lambda_q$, thus,

$$b_0 = 2 = A + B,$$
$$b_1 = \lambda_q = A\left(\frac{\lambda_q + \sqrt{\lambda_q^2 + 4}}{2}\right) + B\left(\frac{\lambda_q - \sqrt{\lambda_q^2 + 4}}{2}\right).$$

So,

$$2\lambda_q = A\left(\lambda_q + \sqrt{\lambda_q^2 + 4}\right) + (2 - A)\left(\lambda_q - \sqrt{\lambda_q^2 + 4}\right),$$
$$A = 1 \quad \text{and} \quad B = 1.$$

Then we obtain the formula of b_k as

$$b_k = \left(\frac{\lambda_q + \sqrt{\lambda_q^2 + 4}}{2}\right)^k + \left(\frac{\lambda_q - \sqrt{\lambda_q^2 + 4}}{2}\right)^k.$$

This completes the proof. □

Notice that this formula is a generalized Lucas sequence. If $\lambda_q = 1$ (the modular group case $q = 3$), we get the Lucas sequence.

Now, we have two sequences a_k and b_k, which are generalizations of the Fibonacci and the Lucas sequences. Let us write out the first 8 terms of a_k and b_k.

a_k	b_k
$a_0 = 0$	$b_0 = 2$
$a_1 = 1$	$b_1 = \lambda_q$
$a_2 = \lambda_q$	$b_2 = \lambda_q^2 + 2$
$a_3 = \lambda_q^2 + 1$	$b_3 = \lambda_q^3 + 3\lambda_q$
$a_4 = \lambda_q^3 + 2\lambda_q$	$b_4 = \lambda_q^4 + 4\lambda_q^2 + 2$
$a_5 = \lambda_q^4 + 3\lambda_q^2 + 1$	$b_5 = \lambda_q^5 + 5\lambda_q^3 + 5\lambda_q$
$a_6 = \lambda_q^5 + 4\lambda_q^3 + 3\lambda_q$	$b_6 = \lambda_q^6 + 6\lambda_q^4 + 9\lambda_q^2 + 2$
$a_7 = \lambda_q^6 + 5\lambda_q^4 + 6\lambda_q^2 + 1$	$b_7 = \lambda_q^7 + 7\lambda_q^5 + 14\lambda_q^3 + 7\lambda_q$
$a_8 = \lambda_q^7 + 6\lambda_q^5 + 10\lambda_q^3 + 4\lambda_q$	$b_8 = \lambda_q^8 + 8\lambda_q^6 + 20\lambda_q^4 + 16\lambda_q^2 + 2.$

Here, it is possible to extend a_k and b_k backward with the negative subscripts. For example, $a_{-1} = 1$, $a_{-2} = -\lambda_q$, $a_{-3} = \lambda_q^2 + 1$, and so on. Therefore, we can deduce that

$$a_{-k} = (-1)^{k+1} a_k \tag{7}$$

and

$$b_{-k} = (-1)^k b_k. \tag{8}$$

The sequences a_k and b_k have some similar properties of the Fibonacci and the Lucas numbers F_n and L_n. Now, we investigate some properties of these sequences a_k and b_k.

Proposition 2

$$a_k + a_{k+4} = (\lambda_q^2 + 2) a_{k+2} \quad \textit{and} \quad b_k + b_{k+4} = (\lambda_q^2 + 2) b_{k+2}. \tag{9}$$

Proof We will use induction on k. For $k = 0$, we have

$$a_0 + a_4 = 0 + \lambda_q^3 + 2\lambda_q = \lambda_q(\lambda_q^2 + 2) = a_2(\lambda_q^2 + 2).$$

For $k = 1$, we get

$$\begin{aligned} a_1 + a_5 &= 1 + \lambda_q^4 + 3\lambda_q^2 + 1 \\ &= (\lambda_q^2 + 2)(\lambda_q^2 + 1) = (\lambda_q^2 + 2) a_3. \end{aligned}$$

Now let us assume that the proposition holds for $k = 2, \dots, n$. We show that it holds for $k = n + 1$. By assumption, we have

$$a_{n-1} + a_{n+3} = (\lambda_q^2 + 2) a_{n+1} \quad \text{and} \quad a_n + a_{n+4} = (\lambda_q^2 + 2) a_{n+2}.$$

From (3), we obtain

$$\begin{aligned} a_{n+1} + a_{n+5} &= (\lambda_q a_n + a_{n-1}) + (\lambda_q a_{n+4} + a_{n+3}) \\ &= \lambda_q (a_n + a_{n+4}) + a_{n-1} + a_{n+3} \\ &= \lambda_q \left(\lambda_q^2 + 2\right) a_{n+2} + \left(\lambda_q^2 + 2\right) a_{n+1} \\ &= \left(\lambda_q^2 + 2\right)(\lambda_q a_{n+2} + a_{n+1}) \\ &= \left(\lambda_q^2 + 2\right) a_{n+3}. \end{aligned}$$

Then we get

$$a_k + a_{k+4} = \left(\lambda_q^2 + 2\right) a_{k+2}.$$

Similarly, it can be shown that

$$b_k + b_{k+4} = \left(\lambda_q^2 + 2\right) b_{k+2}.$$ □

Proposition 3

$$b_k = a_{k+1} + a_{k-1}. \tag{10}$$

Proof We will use the induction method on k. If $k = 1$, then

$$b_1 = a_2 + a_0.$$

We suppose that the equation holds for $k = 2, 3, \ldots, n-1$, *i.e.*,

$$b_{n-1} = a_{n+1} + a_{n-1}.$$

Now, we show that the equation holds for $k = n$. Then we have

$$\begin{aligned} b_n &= \left(\lambda_q^2 + 2\right) b_{n-2} - b_{n-4} \\ &= \left(\lambda_q^2 + 2\right)(a_{n-1} + a_{n-3}) - (a_{n-5} + a_{n-3}) \\ &= \left(\lambda_q^2 + 2\right) a_{n-1} - a_{n-3} + \left(\lambda_q^2 + 2\right) a_{n-3} - a_{n-5} \\ &= a_{n+1} + a_{n-1}. \end{aligned}$$ □

Proposition 4

$$b_k + b_{k+2} = \left(\lambda_q^2 + 4\right) a_{k+1}. \tag{11}$$

Proof For $k = 0$, we have

$$\begin{aligned} b_0 + b_2 &= 2 + \lambda_q^2 + 2 \\ &= \lambda_q^2 + 4 \\ &= \left(\lambda_q^2 + 4\right) a_1. \end{aligned}$$

For $k = 1$, we have

$$\begin{aligned} b_1 + b_3 &= \lambda_q + \lambda_q^3 + 3\lambda_q \\ &= \lambda_q^3 + 4\lambda_q \\ &= \lambda_q(\lambda_q^2 + 4). \end{aligned}$$

Now, we assume that the proposition holds for $k = 2, \dots, n$. We show that it holds for $k = n + 1$. By assumption, we have

$$b_n + b_{n+2} = (\lambda_q^2 + 4)a_{n+1} \quad \textit{and} \quad b_{n-1} + b_{n+1} = (\lambda_q^2 + 4)a_n.$$

Then we find

$$\begin{aligned} b_{n+1} + b_{n+3} &= (\lambda_q b_n + b_{n-1}) + (\lambda_q b_{n+2} + b_{n+1}) \\ &= \lambda_q(b_n + b_{n+2}) + (b_{n-1} + b_{n+1}) \\ &= \lambda_q(\lambda_q^2 + 4)a_{n+1} + (\lambda_q^2 + 4)a_n \\ &= (\lambda_q^2 + 4)(\lambda_q a_{n+1} + a_n) \\ &= (\lambda_q^2 + 4)a_{n+2}. \end{aligned}$$

□

Proposition 5

$$a_{k-3} + a_{k+3} = (\lambda_q^2 + 1)b_k. \tag{12}$$

Proof We will use induction on k. For $k = 0$, we find

$$\begin{aligned} a_{-3} + a_3 &= (-1)^4 a_3 + a_3 \\ &= 2a_3 \\ &= 2(\lambda_q^2 + 1)b_0. \end{aligned}$$

For $k = 1$, we get

$$\begin{aligned} a_{-2} + a_4 &= (-1)^3 a_2 + a_4 \\ &= -a_2 + a_4 \\ &= -\lambda_q + \lambda_q^3 + 2\lambda_q \\ &= \lambda_q^3 + \lambda_q \\ &= \lambda_q(\lambda_q^2 + 1) \\ &= b_1(\lambda_q^2 + 1). \end{aligned}$$

Now, let us suppose that the proposition holds for $k = 2, \dots, n$. We show that it holds for $k = n + 1$. By assumption, $a_{n-3} + a_{n+3} = (\lambda_q^2 + 1)b_n$ and $a_{n-4} + a_{n+2} = (\lambda_q^2 + 1)b_{n-1}$. Hence we get

$$\begin{aligned} a_{n-2} + a_{n+4} &= \lambda_q a_{n-3} + a_{n-4} + \lambda_q a_{n+3} + a_{n+2} \\ &= \lambda_q (a_{n-3} + a_{n+3}) + a_{n-4} + a_{n+2} \\ &= \lambda_q (\lambda_q^2 + 1) b_n + (\lambda_q^2 + 1) b_{n-1} \\ &= (\lambda_q^2 + 1)(\lambda_q b_n + b_{k-1}) \\ &= (\lambda_q^2 + 1) b_{n+1}. \end{aligned}$$

□

Proposition 6

$$a_{2k} = a_k b_k. \tag{13}$$

Proof We will use the induction method on k. For $k = 0$, we have

$$a_0 b_0 = 0 = a_0.$$

For $k = 1$, we have

$$a_1 b_1 = \lambda_q = a_2.$$

We suppose that the equation holds for $k = 2, \dots, n-1$, *i.e.*,

$$a_{2(n-1)} = a_{n-1} b_{n-1}.$$

Now, we show that the equation holds for $k = n$. By equalities (3), (9) and (10),

$$\begin{aligned} a_n b_n &= a_n (a_{n+1} + a_{n-1}) \\ &= a_n \bigl((\lambda_q^2 + 2) a_{n-1} - a_{n-3} \bigr) + a_{n-1} \bigl((\lambda_q^2 + 2) a_{n-2} - a_{n-4} \bigr) \\ &= (\lambda_q^2 + 2) a_n a_{n-1} + (\lambda_q^2 + 2) a_{n-1} a_{n-2} - a_n a_{n-3} - a_{n-1} a_{n-4} \\ &= (\lambda_q^2 + 2) a_{n-1} (a_n + a_{n-2}) - a_n a_{n-3} - a_{n-1} a_{n-4} \\ &= (\lambda_q^2 + 2) a_{n-1} b_{n-1} - a_n a_{n-3} - a_{n-1} a_{n-4} \\ &= (\lambda_q^2 + 2) a_{n-1} b_{n-1} - a_{n-3} (\lambda_q a_{n-1} + a_{n-2}) - a_{n-1} (a_{n-2} - \lambda_q a_{n-3}) \\ &= (\lambda_q^2 + 2) a_{n-1} b_{n-1} - a_{n-3} a_{n-2} - a_{n-1} a_{n-2} \\ &= (\lambda_q^2 + 2) a_{n-1} b_{n-1} - a_{n-2} (a_{n-3} + a_{n-1}) \\ &= (\lambda_q^2 + 2) a_{n-1} b_{n-1} - a_{n-2} b_{n-2} \\ &= (\lambda_q^2 + 2) a_{2n-2} - a_{2n-4} \quad \text{(by assumption)} \\ &= a_{2n}. \end{aligned}$$

□

Proposition 7

$$b_k^2 - (\lambda_q^2 + 4) a_k^2 = 4(-1)^k. \tag{14}$$

Proof Using (10) and the definitions of a_k and b_k, we have

$$\begin{aligned} b_k^2 - (\lambda_q^2 + 4)a_k^2 &= (a_{k-1} + a_{k+1})^2 - (\lambda_q^2 + 4)a_k^2 \\ &= a_{k-1}^2 + 2a_{k-1}a_{k+1} + a_{k+1}^2 - \lambda_q^2 a_k^2 - 4a_k^2 \\ &= a_{k-1}^2 + 2a_{k-1}(\lambda_q a_k + a_{k-1}) + (\lambda_q a_k + a_{k-1})^2 - \lambda_q^2 a_k^2 - 4a_k^2 \\ &= a_{k-1}^2 + 2\lambda_q a_{k-1}a_k + 2a_{k-1}^2 + \lambda_q^2 a_k^2 + 2\lambda_q a_k a_{k-1} + a_{k-1}^2 - \lambda_q^2 a_k^2 - 4a_k^2 \\ &= 4a_{k-1}^2 + 4\lambda_q a_{k-1}a_k - 4a_k^2 \\ &= 4a_{k-1}(a_{k-1} + \lambda_q a_k) - 4a_k^2 \\ &= 4a_{k-1}a_{k+1} - 4a_k^2 \\ &= 4(a_{k-1}a_{k+1} - a_k^2). \end{aligned}$$

In [10], Yayenie and Edson obtained a generalization of Cassini's identity for the positive real numbers a and b. If we take $a = \lambda_q$ and $b = \lambda_q$ in generalized Cassini's identity, we get

$$a_{k-1}a_{k+1} - a_k^2 = (-1)^n,$$

and so,

$$b_k^2 - (\lambda_q^2 + 4)a_k^2 = 4 \cdot (-1)^n. \qquad \square$$

Proposition 8

$$a_k \cdot a_{k+3} - a_{k+1} \cdot a_{k+2} = (-1)^{k+1}\lambda_q. \tag{15}$$

Proof We will use the induction method on k. For $k = 0$, we have

$$a_0 \cdot a_3 - a_1 \cdot a_2 = -\lambda_q = (-1)^1\lambda_q.$$

For $k = 1$, we have

$$\begin{aligned} a_1 \cdot a_4 - a_2 \cdot a_3 &= \lambda_q^3 + 2\lambda_q - \lambda_q(\lambda_q^2 + 1) \\ &= (-1)^2\lambda_q. \end{aligned}$$

Now, we assume that the proposition holds for $k = 2, \dots, n$. We show that it holds for $k = n + 1$. From assumption $a_n \cdot a_{n+3} - a_{n+1} \cdot a_{n+2} = (-1)^{n+1}\lambda_q$, and, thus,

$$\begin{aligned} a_{n+1} \cdot a_{n+4} - a_{n+2} \cdot a_{n+3} &= a_{n+1}(\lambda_q a_{n+3} + a_{n+2}) - a_{n+3}(\lambda_q a_{n+1} + a_n) \\ &= \lambda_q a_{n+1}a_{n+3} + a_{n+1}a_{n+2} - \lambda_q a_{n+3}a_{n+1} - a_{n+3}a_n \\ &= a_{n+1}a_{n+2} - a_{n+3}a_n \\ &= -(-1)^{n+1}\lambda_q \\ &= (-1)^{n+2}\lambda_q. \end{aligned} \qquad \square$$

Proposition 9

$$a_{2m+2} \cdot a_k - a_{2m} \cdot a_{k-2} = a_{2m+k}\lambda_q. \tag{16}$$

Let m be fixed. We will use the induction method on k. For $k = 0$, we have

$$a_{2m+2} \cdot a_0 - a_{2m} \cdot a_{-2} = \lambda_q a_{2m},$$

since $a_0 = 0$ and $a_{-2} = (-1)^3 a_2 = -\lambda_q$. For $k = 1$, we find

$$\begin{aligned} a_{2m+2} \cdot a_1 - a_{2m} \cdot a_{-1} &= a_{2m+2} - a_{2m} \\ &= \lambda_q a_{2m+1} + a_{2m} - a_{2m} \\ &= \lambda_q a_{2m+1}, \end{aligned}$$

since $a_1 = 1$ and $a_{-1} = 1$. Now, we assume that the proposition holds for $k = 2, \dots, n$. We show that it holds for $k = n + 1$. By assumption,

$$a_{2m+2} \cdot a_n - a_{2m} \cdot a_{n-2} = \lambda_q a_{2m+n}$$

and

$$a_{2m+2} \cdot a_{n-1} - a_{2m} \cdot a_{n-3} = \lambda_q a_{2m+n-1}.$$

Thus, we have

$$\begin{aligned} a_{2m+2} \cdot a_{n+1} - a_{2m} \cdot a_{n-1} &= a_{2m+2}(\lambda_q a_n + a_{n-1}) - a_{2m}(\lambda_q a_{n-2} + a_{n-3}) \\ &= \lambda_q(a_{2m+2}a_n - a_{2m}a_{n-2}) + (a_{2m+2}a_{n-1} - a_{2m}a_{n-3}) \\ &= \lambda_q\lambda_q a_{2m+n} + \lambda_q a_{2m+n-1} \\ &= \lambda_q(\lambda_q a_{2m+n} + a_{2m+n-1}) \\ &= \lambda_q a_{2m+n+1}. \end{aligned}$$

Now, we give a formula for a_k and b_k.

Proposition 10 *For all $k \geq 1$,*

$$a_k = \begin{cases} \frac{1}{2^{k-1}} \sum_{i=0}^{\frac{k-2}{2}} \binom{k}{2i+1} \lambda_q^{k-(2i+1)} (\lambda_q^2 + 4)^i & \textit{if } k \textit{ is even}, \\ \frac{1}{2^{k-1}} \sum_{i=0}^{\frac{k-1}{2}} \binom{k}{2i+1} \lambda_q^{k-(2i+1)} (\lambda_q^2 + 4)^i & \textit{if } k \textit{ is odd} \end{cases} \tag{17}$$

and

$$b_k = \begin{cases} \frac{1}{2^{k-1}} \sum_{i=0}^{\frac{k}{2}} \binom{k}{2i} \lambda_q^{k-2i} (\lambda_q^2 + 4)^{2i} & \textit{if } k \textit{ is even}, \\ \frac{1}{2^{k-1}} \sum_{i=0}^{\frac{k-1}{2}} \binom{k}{2i-1} \lambda_q^{k-(2i-1)} (\lambda_q^2 + 4)^{2i} & \textit{if } k \textit{ is odd}. \end{cases} \tag{18}$$

Proof Let k be even. By (4),

$$\begin{aligned}
a_k &= \frac{1}{\sqrt{\lambda_q^2+4}}\left[\left(\frac{\lambda_q+\sqrt{\lambda_q^2+4}}{2}\right)^k-\left(\frac{\lambda_q-\sqrt{\lambda_q^2+4}}{2}\right)^k\right] \\
&= \frac{1}{2^{k-1}\sqrt{\lambda_q^2+4}}\left[\binom{k}{1}\lambda_q^{k-1}\sqrt{\lambda_q^2+4}+\binom{k}{3}\lambda_q^{k-3}\left(\sqrt{\lambda_q^2+4}\right)^3\right. \\
&\qquad \left.+\cdots+\binom{k}{k-1}\lambda_q\left(\sqrt{\lambda_q^2+4}\right)^{k-1}\right] \\
&= \frac{1}{2^{k-1}}\left[\binom{k}{1}\lambda_q^{k-1}+\binom{k}{3}\lambda_q^{k-3}\left(\lambda_q^2+4\right)^2\right. \\
&\qquad \left.+\cdots+\binom{k}{k-1}\lambda_q\left(\lambda_q^2+4\right)^{k-2}\right] \\
&= \frac{1}{2^{k-1}}\left[\sum_{i=0}^{\frac{k-2}{2}}\binom{k}{2i+1}\lambda_q^{k-(2i+1)}\left(\lambda_q^2+4\right)^{2i}\right].
\end{aligned}$$

Similarly, if k is odd, then we get

$$a_k = \frac{1}{2^{k-1}}\sum_{i=0}^{\frac{k-1}{2}}\binom{k}{2i+1}\lambda_q^{k-(2i+1)}\left(\lambda_q^2+4\right)^{i}.$$

□

Proposition 11

$$ {}^{k+1}_{i=1}a_i = \frac{a_{k+2}+a_{k+1}-1}{\lambda_q} \tag{19}$$

and

$$ {}^{k+1}_{i=1}b_i = \frac{b_{k+2}+b_{k+1}-(\lambda_q+2)}{\lambda_q}. \tag{20}$$

Proof From (3), we have

$$\begin{aligned}
a_{k+2}-a_{k+1} &= \lambda_q a_{k+1}+a_k-a_{k+1} \\
&= (\lambda_q-1)a_{k+1}+a_k,
\end{aligned}$$

and so,

$$\begin{aligned}
n=0 \quad &\Rightarrow \quad a_2-a_1=(\lambda_q-1)a_1+a_0, \\
n=1 \quad &\Rightarrow \quad a_3-a_2=(\lambda_q-1)a_2+a_1, \\
&\vdots \\
n=k-1 \quad &\Rightarrow \quad a_{k+1}-a_k=(\lambda_q-1)a_k+a_{k-1}, \\
n=k \quad &\Rightarrow \quad a_{k+2}-a_{k+1}=(\lambda_q-1)a_{k+1}+a_k.
\end{aligned}$$

If we sum both sides, then we obtain

$$\begin{aligned} a_{k+2} - a_1 &= (\lambda_q - 1)(a_1 + a_2 + \cdots + a_{k+1}) + (a_0 + a_1 + \cdots + a_k) \\ &= \lambda_q(a_1 + a_2 + \cdots + a_{k+1}) + a_0 - a_{k+1}. \end{aligned}$$

Since $a_0 = 0$ and $a_1 = 1$, we have

$$\begin{aligned} &a_{k+2} - 1 = \lambda_q(a_1 + a_2 + \cdots + a_{k+1}) - a_{k+1}, \\ &a_{k+2} + a_{k+1} - 1 = \lambda_q(a_1 + a_2 + \cdots + a_{k+1}), \\ &\frac{a_{k+2} + a_{k+1} - 1}{\lambda_q} = a_1 + a_2 + \cdots + a_{k+1}, \\ &\sum_{i=1}^{k+1} a_i = \frac{a_{k+2} + a_{k+1} - 1}{\lambda_q}. \end{aligned}$$

Similarly, it is easily seen that

$$\sum_{i=1}^{k+1} b_i = \frac{b_{k+2} + b_{k+1} - (\lambda_q + 2)}{\lambda_q}.$$

□

3 Polynomial representations of a_k and b_k

Before we find the polynomial representations of a_k and b_k, note the following identities

$$\binom{k}{p} + 2\binom{k+1}{p-1} - \binom{k}{p-2} = \binom{k+2}{p} \tag{21}$$

and

$$\binom{k}{p} + \binom{k-1}{p-1} = \binom{k-1}{p-1}\frac{p+k}{p} \tag{22}$$

Theorem 1 *Let $\{a_k\}$ denote the generalized Fibonacci sequence. Then, the polynomial representations of a_{2k} and a_{2k+1} are*

$$\begin{aligned} a_{2k} &= (\lambda_q)^{2k-1} + \binom{2k-2}{1}(\lambda_q)^{2k-3} + \binom{2k-3}{2}(\lambda_q)^{2k-5} \\ &\quad + \cdots + \binom{k+2}{k-3}(\lambda_q)^3 + \binom{k+1}{k-2}(\lambda_q) \end{aligned}$$

and

$$\begin{aligned} a_{2k+1} &= (\lambda_q)^{2k} + (2k-1)(\lambda_q)^{2k-2} + \binom{2k-2}{1}\frac{2k-3}{2}(\lambda_q)^{2k-4} \\ &\quad + \binom{2k-3}{2}\frac{2k-5}{3}(\lambda_q)^{2k-6} + \cdots + \binom{k+1}{k-2}\frac{3}{k-1}(\lambda_q)^2 + 1. \end{aligned}$$

Proof We will use the induction method on k. For $k = 1$, we have $a_2 = \lambda_q$, and for $k = 2$, we have $a_4 = (\lambda_q)^3 + 2\lambda_q$. Now, suppose that the equality is true for $k = 1, 2, \ldots, n$. We will show that it holds for $k = n + 1$. By assumption,

$$a_{2n-2} = (\lambda_q)^{2n-3} + \binom{2n-4}{1}(\lambda_q)^{2n-5} + \binom{2n-5}{2}(\lambda_q)^{2n-7} + \cdots + \binom{n+1}{n-4}(\lambda_q)^3 + \binom{n}{n-3}(\lambda_q)$$

and

$$a_{2n} = (\lambda_q)^{2n-1} + \binom{2n-2}{1}(\lambda_q)^{2n-3} + \binom{2n-3}{2}(\lambda_q)^{2n-5} + \cdots + \binom{n+2}{n-3}(\lambda_q)^3 + \binom{n+1}{n-2}(\lambda_q).$$

From (9), we have $a_{2k+2} = (\lambda_q^2 + 2)a_{2k} - a_{2k-2}$, and by definition of a_k, we get

$$\begin{aligned} a_{2n+2} &= (\lambda_q^2 + 2)\left[\begin{matrix} (\lambda_q)^{2n-1} + \binom{2n-2}{1}(\lambda_q)^{2n-3} + \binom{2n-3}{2}(\lambda_q)^{2n-5} \\ + \cdots + \binom{n+2}{n-3}(\lambda_q)^3 + \binom{n+1}{n-2}(\lambda_q) \end{matrix}\right] \\ &\quad - \left[\begin{matrix} (\lambda_q)^{2n-3} + \binom{2n-4}{1}(\lambda_q)^{2n-5} + \binom{2n-5}{2}(\lambda_q)^{2n-7} \\ + \cdots + \binom{n+1}{n-4}(\lambda_q)^3 + \binom{n}{n-3}(\lambda_q) \end{matrix}\right] \\ &= (\lambda_q)^{2n+1} + \left(\binom{2n-2}{1} + 2\right)(\lambda_q)^{2n-1} + \left[\binom{2n-3}{2} + 2\binom{2n-2}{1}\right](\lambda_q)^{2n-3} \\ &\quad + \cdots + \left[\binom{n+1}{n-2} + 2\binom{n+2}{n-3}\right](\lambda_q)^3 + 2\binom{n+1}{n-2}\lambda_q. \end{aligned}$$

From (21), we get

$$a_{2n+2} = (\lambda_q)^{2n+1} + \binom{2n}{1}(\lambda_q)^{2n-1} + \binom{2n-1}{2}(\lambda_q)^{2n-3} + \cdots + \binom{n+3}{n-2}(\lambda_q)^3 + \binom{n+2}{n-1}(\lambda_q).$$

Now, we compute a_{2k+1}. By definition of a_k, we get

$$\begin{aligned} a_{2k+1} &= \frac{1}{\lambda_q}(a_{2k+2} - a_{2k}) \\ &= \frac{1}{\lambda_q}\left[\begin{matrix} ((\lambda_q)^{2k+1} + \binom{2k}{1}(\lambda_q)^{2k-1} + \binom{2k-1}{2}(\lambda_q)^{2k-3} \\ + \cdots + \binom{k+3}{k-2}(\lambda_q)^3 + \binom{k+2}{k-1}(\lambda_q)) \\ -((\lambda_q)^{2k-1} + \binom{2k-2}{1}(\lambda_q)^{2k-3} + \binom{2k-3}{2}(\lambda_q)^{2k-5} \\ + \cdots + \binom{k+2}{k-3}(\lambda_q)^3 + \binom{k+1}{k-2}(\lambda_q)) \end{matrix}\right]. \end{aligned}$$

From (22), we get

$$\begin{aligned} a_{2k+1} &= (\lambda_q)^{2k} + (2k-1)(\lambda_q)^{2k-2} + \binom{2k-2}{1}\frac{2k-3}{2}(\lambda_q)^{2k-4} \\ &\quad + \binom{2k-3}{2}\frac{2k-5}{3}(\lambda_q)^{2k-6} + \cdots + \binom{k+1}{k-2}\frac{3}{k-1}(\lambda_q)^2 + 1. \end{aligned}$$

□

Theorem 2 *Let $\{b_k\}$ denote the generalized Lucas sequence. Then, the polynomial representations of b_{2k} and b_{2k+1} are*

$$b_{2k} = (\lambda_q)^{2k} + (2k)(\lambda_q)^{2k-2} + \binom{2k-3}{1}\frac{2k}{2}(\lambda_q)^{2k-4} + \binom{2k-4}{2}\frac{2k}{3}(\lambda_q)^{2k-6} + \cdots + \binom{k}{k-2}\frac{2k}{k-1}(\lambda_q)^2 + 2$$

and

$$b_{2k+1} = (\lambda_q)^{2k+1} + (2k+1)(\lambda_q)^{2k-1} + \binom{2k-2}{1}\frac{2k+1}{2}(\lambda_q)^{2k-3} + \binom{2k-3}{2}\frac{2k+1}{3}(\lambda_q)^{2k-5} + \cdots + \binom{k+1}{k-2}\frac{2k+1}{k-1}(\lambda_q).$$

Proof From (10), it is easy to find the polynomial representations of b_{2k} and b_{2k+1}. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors completed the paper together. All authors read and approved the final manuscript.

Author details
[1]Fen-Edebiyat Fakültesi, Matematik Bölümü, Balıkesir Üniversitesi, Balıkesir, 10145, Turkey. [2]Köprübasi Meslek Yüksek Okulu, Celal Bayar Üniversitesi, Manisa, 45930, Turkey.

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