One Parameter Generalizations of the Fibonacci and Lucas Numbers

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Abstract

We give one parameter generalizations of the Fibonacci and Lucas numbers denoted by $\{F_n(\theta)\}$ and $\{L_n(\theta)\}$, respectively. We evaluate the Hankel determinants with entries $\{1/F_{j+k+1}(\theta) : 0 \le i,j \le n\}$ and $\{1/L_{j+k+1}(\theta) : 0 \le i,j \le n\}$. We also find the entries in the inverse of $\{1/F_{j+k+1}(\theta) : 0 \le i,j \le n\}$ and show that all its entries are integers. Some of the identities satisfied by the Fibonacci and Lucas numbers are extended to more general numbers. All integer solutions to three diophantine equtions related to the Pell equation are also found.

Running Title. Generalized Fibonacci and Lucas Numbers

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1 Introduction

The Hilbert matrix H_n has entries $1/(i + j + 1) : 0 \le i,j \le n$. It is well-known that, for all n, H_n is non singular and the elements of its inverse matrix are all integers, see for example [6]. The determinant of H_n has a closed form expression which shows that the determinant is very small for large n. This is important in numerical analysis because the smaller the determinant, the larger the condition number becomes and computing the inverse numerically becomes unstable. Many other applications of the Hilbert matrix are in [6].

The Fibonacci numbers have many interesting properties and appear in many areas of mathematics. [12], [17]. One unexpected result is due to Richardson who showed in [15] that the "Filbert matrix" is also non singular and its inverse has only integer entries. The i, j entry of the Filbert matrix is $1/F_{i+j+1}$ where $0 \le i, j \le n$ and $\{F_n : n \ge 1\}$ are the Fibonacci numbers.

One way to compute the determinant of H_n is to note that it is the Hankel determinant associated with a constant weight function supported on $[0, 1]$. Berg [5] observed that the reciprocals of the Fibonacci numbers form a moment sequence of a special little q-Jacobi weight, [9, §18.4]. He used Lemma 1.1, to be stated below, to prove Richardson's result.

Recall that the Chebyshev polynomials of the first and second kinds are

$$T_n(\cos\theta) = \cos(n\theta), \quad U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}, \tag{1.1}$$

respectively. Askey [3], [4] observed that the Fibinacci numbers $\{F_n\}$ and the Lucas numbers $\{L_n\}$ are related to the Chebyshev polynomials via

$$F_{n+1} = (-i)^n U_n(i \sinh(\theta_0)), \quad L_{n+1} = 2(-i)^n T_n(i \sinh\theta_0)) \tag{1.2}$$

where $\theta_0 > 0$ and $\sinh\theta_0 = 1/2$.

The purpose of this paper is to give one parameter generalizations of the Fibonacci and Lucas numbers. Our generalization comes from the Chebyshev polynomials of the first and second kinds. Our generalizations of the Fibonacci and Lucas numbers satisfy the recurrence relation

$$y_{n+1}(\theta) = 2\sinh\theta y_n(\theta) + y_{n-1}(\theta) \tag{1.3}$$

and the initial conditions (2.12) and (3.2). We generalize Richardson's result by replacing the Fibonacci numbers by our generalized Fibonacci numbers. This will be done in §2. In §3 we introduce the generalized Lucas numbers and study some of their properties. We also give a closed form evaluation of a Hankel determinant whose elements are reciprocals of Lucas numbers.

The book [12] contains many results on Fibonacci and Lucas numbers with detailed proofs. On the other hand we found Vajda's book [17] to be very comprehensive but concise. In §4 we extend some of the properties of the Fibonacci and Lucas numbers to our numbers. We have only included a sample of the identities involving $\{F_n(\theta)\}$ and $\{L_n(\theta)\}$. There are many other relationships involving the Fibonacci and Lucas numbers which extend to our more general sequences $\{F_n(\theta)\}$ and $\{L_n(\theta)\}$ but we made no attempt to include them. In §5 we describe all integer solutions to

$$y^2 - kxy - x^2 = \pm 1,$$

for a given integer $k > 1$. We also characterize all integers n for which $n^2(1 + k^2) \pm 4$ is a perfect square when k is odd. When $k = 1$ these results reduce to known facts involving the Fibonacci numbers.

The connection between Fibonacci numbers, hyperbolic functions, and Chebyshev polynomials was observed but some how never fully exploited, see for example [17, Chapter 11], and [13]. Another recently development is due to Kalman and Mena [10] who treated sequences which satisfy the three term recurrence relation

$$y_{n+1} = ay_n + by_{n-1},$$

under general initial conditions. They derived many of the properties that their generalized sequence share with the Fibonacci or Lucas numbers. Our numbers being less general than the Kalman-Mena numbers have additional properties. For example the inverse matrix to $1/L_{1+i+j}$ does not have integer coefficients, while the inverse matrix to $1/F_{i+j+1}$ as well as $1/F_{i+j+1}(\theta)$ have integer entries, $\{F_n(\theta)$ being our generalization of the Fibonacci number. Some of the other refined properties involving congruences and integer points on algebraic curves or surfaces do not extend to the very general setting of Kalman and Mena.

We now come to Lemma 1.1.

Let μ be a measure whose moments, $s_0 \neq 0$, $s_n := \int_{\mathbb{R}} x^n d\mu(x)$ exist for all $n = 0, 1, \ldots$, and let $\{p_n(x)\}$ be the sequence of polynomials orthogonal with respect to μ, that is

$$\int_{\mathbb{R}} p_m(x)p_n(x)d\mu(x) = \zeta_n\delta_{m,n}, \quad \zeta_n \neq 0 \tag{1.4}$$

for $n = 0, 1, 2, \ldots$. We shall always normalize μ by $\zeta_0 = 1$, so that μ has a unit total mass. The corresponding Hankel matrix and Hankel determinant are

$$H_n = \begin{pmatrix} s_0 & s_1 & \ldots & s_n \\ s_1 & s_2 & \ldots & s_{n+1} \\ \vdots & \vdots & \ldots & \vdots \\ s_n & s_{n+1} & \ldots & s_{2n} \end{pmatrix}, \; D_n = \begin{vmatrix} s_0 & s_1 & \ldots & s_n \\ s_1 & s_2 & \ldots & s_{n+1} \\ \vdots & \vdots & \ldots & \vdots \\ s_n & s_{n+1} & \ldots & s_{2n} \end{vmatrix}, \tag{1.5}$$

respectively, and n = 0, 1, ... The kernel polynomials are

$$K_n(x, y) = \sum_{k=0}^{n} p_k(x)p_k(y)/\zeta_n. \tag{1.6}$$

Lemma 1.1. *Let*

$$K_n(x, y) = \sum_{j,k=0}^{n} a_{j,k}(n)\, x^j\, y^k. \tag{1.7}$$

Then $a_{j,k}(n) = a_{k,j}(n)$ and the matrix A_n whose entries are $\{a_{j,k}(n)\}$ is the inverse of H_n.

Lemma 1.1 is in the paper [16] by Tracy and Widom and in Berg's paper [5].

2 Generalized Fibonacci Numbers

Consider the Chebyshev polynomials of the second kind

$$
\begin{aligned}
U_n(i\sinh\theta) = U_n(\cos(\pi/2 - i\theta) &= \frac{e^{i(\pi/2-i\theta)(n+1)} - e^{-i(\pi/2-i\theta)(n+1)}}{e^{i(\pi/2-i\theta)} - e^{-i(\pi/2-i\theta)}} \\
&= i^n\, \frac{e^{(n+1)\theta} + (-1)^n e^{-(n+1)\theta}}{e^{\theta}+e^{-\theta}}.
\end{aligned} \tag{2.1}
$$

Set

$$F_{n+1}(\theta) = (-i)^n\, U_n(\sinh(i\theta)) = \frac{e^{(n+1)\theta} + (-1)^n e^{-(n+1)\theta}}{e^{\theta}+e^{-\theta}}. \tag{2.2}$$

The explicit representation of the Chebyshev polynomials of the second kind, see for example [9, §4.5] leads to

$$F_n(\theta) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n+1}{2k+1} \sinh^{n-2k}(\theta)\cosh^{2k}(\theta). \tag{2.3}$$

Choose $\theta_0 > 0$ so that $\cosh\theta_0 = \sqrt{5}/2$. Thus $\sinh\theta_0 = 1/2$ and $e^{\theta_0} = \phi$ in Berg's notation in [5]. Clearly $e^{-\theta_0} = (\sqrt{5}-1)/2 - \hat{\phi}$ in Berg's notation. Thus $F_n(\theta_0) = F_n$, $n = 1, 2, \ldots$, the Fibonacci sequence. Moreover

$$F_n(\theta) = e^{n\theta}\, \frac{1-(-1)^n e^{-2n\theta}}{e^{\theta}+e^{-\theta}}. \tag{2.4}$$

For positive integer α we have

$$\frac{F_\alpha(\theta)}{F_{n+\alpha}(\theta)} = e^{-n\theta}\, \frac{1-(-e^{-2\theta})^{\alpha}}{1-(-e^{-2\theta})^{n+\alpha}}. \tag{2.5}$$

With

$$q = -e^{-2\theta} \tag{2.6}$$

we arrive at

$$F_n(\theta) = e^{(n-1)\theta}\, \frac{1-q^n}{1-q}. \tag{2.7}$$

Formula (2.7) enables us to extend the definition of $F_n(\theta)$ to nonpositive values of n. This agrees with defining $F_n(\theta)$ for $n \le 0$ from (1.3) and the initial conditions (2.12) below. Indeed it is easy to see that

$$F_{-n}(\theta) = (-1)^{n-1} F_n(\theta). \tag{2.8}$$

From (2.7) it follows that

$$\frac{F_\alpha(\theta)}{F_{n+\alpha}(\theta)} = (1-q^{\alpha}) \sum_{k=0}^{\infty} (q^k/e^{\theta})^n\, q^{\alpha k}. \tag{2.9}$$

Now use $\binom{n}{k}_{\mathbb{F}}$ to denote the binomial coefficient relative to $\{F_n(\theta)\}$, that is

$$\binom{n}{0}_{\mathbb{F}} := 1, \quad \binom{n}{k}_{\mathbb{F}} = \frac{F_n(\theta)F_{n-1}(\theta)\dots F_{n-k+1}(\theta)}{F_1(\theta)\; F_2(\theta)\; \dots\; F_k(\theta)}. \tag{2.10}$$

Theorem 2.1. *We have*

$$\binom{n}{k}_{\mathbb{F}} = F_{k-1}(\theta)\binom{n-1}{k}_{\mathbb{F}} + F_{n-k+1}(\theta)\binom{n-1}{k-1}_{\mathbb{F}}. \tag{2.11}$$

Proof. It is easy to write the right-hand side of (2.11) in the form

$$\begin{aligned} &\frac{1}{F_n(\theta)}\binom{n}{k}_{\mathbb{F}}[F_{k-1}(\theta)F_{n-k}(\theta) + F_k(\theta)F_{n-k+1}(\theta)] \\ &= \frac{(1-q)^{-2}}{F_n(\theta)}\binom{n}{k}_{\mathbb{F}} e^{(n-1)\theta}[(1-q^k)(1-q^{n-k+1}) - q(1-q^{k-1})(1-q^{n-k})]. \end{aligned}$$

The quantity in the square bracket simplifies to $(1-q)(1-q^n)$ and the result follows. □

It is clear from (2.2) that

$$F_1(\theta) = 1, \quad F_2(\theta) = 2\sinh\theta. \tag{2.12}$$

We now choose θ such that

$$\sinh\theta = \text{a positive integer}. \tag{2.13}$$

It then follows from the three term recurrence relation for Chebyshev polynomials that $\{F_n(\theta)\}$ solves (1.3) under the initial conditions (2.12). This and (2.13) show that $F_n(\theta)$ is a positive integer for all $n, n > 0$. Theorem 1.1 implies that $\binom{n}{k}_{\mathbb{F}}$ is always a positive integer when $n > 0$.

We can express (2.9) as the nth moment of the measure

$$\nu(x) = (1-q^\alpha)\sum_{k=0}^{\infty} q^{\alpha k}\delta(x - q^k e^{-\theta}), \tag{2.14}$$

where $\delta(x-c)$ is a unit atomic measure located at $x = c$. When α is even this is a positive measure with total mass $= 1$, otherwise ν is a unit signed measure. In view of (18.4.11) and (18.4.13) in [9] we see that the corresponding orthogonal polynomials are little q-Jacobi polynomials $\{p_n(xe^\theta; q^{\alpha-1}, 1)\}$, where

$$\begin{aligned} p_n(x; a, b) &= {}_2\phi_1(q^{-n}, abq^{n+1}; aq; q, qx) \\ &= \sum_{j=0}^{n} \frac{(q;q)_n(abq^{n+1};q)_j}{(q;q)_j(q;q)_{n-j}} q^{\binom{j+1}{2}} \frac{(-x)^j}{(aq;q)_j}, \end{aligned} \tag{2.15}$$

and the q-shifted factorials are

$$(\lambda; q)_s = (1-\lambda)(1-\lambda q)\dots(1-\lambda q^{s-1}).$$

The above q is a base for the q-shifted factorials and is not the same as in (2.6)

In terms of the generalized Fibonacci coefficients the polynomials are expressed as

$$\begin{aligned} p_n^{(\alpha)}(x) &:= \binom{n+\alpha-1}{n}_{\mathbb{F}} p_n(xe^{\theta}; q^{\alpha-1}, 1) \\ &= \sum_{k=0}^{n} \binom{n}{k}_{\mathbb{F}} \binom{\alpha+n+k-1}{n}_{\mathbb{F}} (-1)^{nk+\binom{k}{2}} x^k. \end{aligned} \tag{2.16}$$

The orthogonality relation is [8]

$$\int_{\mathbb{R}} p_m^{(\alpha)}(x) p_n^{(\alpha)}(x)\, d\nu(x) = (-1)^{\alpha n} \frac{F_\alpha(\theta)}{F_{\alpha+n}(\theta)} \delta_{m,n}. \tag{2.17}$$

Recall that if the orthonormal polynomial of degree n is

$$\gamma_n x^n + \text{lower order terms}$$

then the Hankel determinant D_n is given by

$$D_n = \prod_{j=1}^{n} \gamma_j^{-2}. \tag{2.18}$$

Consequently

$$\begin{aligned} &\det\{1/F_{\alpha+i+j}(\theta) : 0 \le i, j \le n\} \\ &= (-1)^{\alpha\binom{n+1}{2}} F_\alpha^{-n}(\theta) \left[\prod_{k=1}^{n} F_{\alpha+2k} \binom{\alpha+2k-1}{k}_{\mathbb{F}}\right]^{-1}. \end{aligned} \tag{2.19}$$

Theorem 2.2. *Let A be the the matrix $\{1/F_{\alpha+j+k} : 0 \le j, k \le n\}$. Then A^{-1} has the matrix elements*

$$\begin{aligned} &(-1)^{(\alpha+j+k)n-\binom{j}{2}-\binom{k}{2}} F_{\alpha+j+k}(\theta) \binom{\alpha+n+j}{n-k}_{\mathbb{F}} \\ &\times \binom{\alpha+n+k}{n-j}_{\mathbb{F}} \binom{\alpha+j+k-1}{j}_{\mathbb{F}} \binom{\alpha+j+k-1}{k}_{\mathbb{F}}. \end{aligned}$$

Proof. Use Lemma 1.1, (2.16), and (2.17). □

3 Generalized Lucas Numbers

We now consider the Chebyshev polynomials of the first kind

$$\begin{aligned} T_n(i\sinh\theta) = T_n(\cos(\pi/2 - i\theta) &= [e^{i(\pi/2-i\theta)n} + e^{-i(\pi/2-i\theta)n}] \\ &= \frac{i^n}{2}\left[e^{n\theta} + (-1)^n e^{-n\theta}\right] \end{aligned}$$

Define the generalized Lucas numbers by

$$L_n(\theta) = 2(-i)^n\, T_n(\cos(i\theta - \pi/2) = \left[e^{n\theta} + (-1)^n e^{-n\theta}\right], \tag{3.1}$$

for $n = 0, 1, \dots$. Thus

$$L_0(\theta) = 2, \quad L_1(\theta) = 2\sinh\theta. \tag{3.2}$$

Hence $L_2(\theta) = \cosh(2\theta)$. It readily follows that $\{L_n(\theta)\}$ solves (1.3) under the initial conditions (3.2). Assume

$$2\sinh\theta = \text{a positive integer}. \tag{3.3}$$

Consequently $L_n(\theta)$ is a positive integer for all n, $n = 1, 2, \dots$. Clearly there are infinitely many such θs. Moreover $L_n(\theta_0) = L_n$.

In view of (2.6) we see that

$$L_n(\theta) = e^{n\theta}\,[1 + q^n]. \tag{3.4}$$

We extend the definition of $L_n(\theta)$ to $n \le 0$ by (3.4). The explicit representation of $T_n(x)$, [9, §4.5] establishes the representation

$$L_n(\theta) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \sinh^{n-2k}(\theta)\cosh^{2k}(\theta). \tag{3.5}$$

It readily follows from (3.4) that

$$\frac{L_\alpha(\theta)}{L_{n+\alpha}(\theta)} = (1 + q^\alpha)\sum_{k=0}^{\infty}(-q^\alpha)^k (q^k e^{-\theta})^n.$$

Define a measure ψ by

$$\psi = (1 + q^\alpha)\sum_{k=0}^{\infty}(-q^\alpha)^k \delta(x - q^k e^{-\theta}), \tag{3.6}$$

where, as before, $\delta(x-c)$ is a unit atomic measure located at $x = c$. Analogous to the definition (2.10) the binomial coefficient relative to the generalized Lucas numbers $\{L_n(\theta)\}$ is

$$\binom{n}{0}_{\mathbb{L}} := 1, \quad \binom{n}{k}_{\mathbb{L}} = \frac{L_n(\theta)L_{n-1}(\theta)\dots L_{n-k+1}(\theta)}{L_1(\theta)\; L_2(\theta)\;\dots\; L_k(\theta)}. \tag{3.7}$$

It is unlikely that the binomial coefficients relative to the generalized Lucas numbers are integers, but they may be integers if we only use the generalized Lucas numbers of odd indices.

The polynomials

$$\begin{aligned} q_n^{(\alpha)}(x) &:= \binom{n+\alpha-1}{n}_{\mathbb{L}} p_n(xe^\theta; -q^{\alpha-1}, 1) \\ &= \sum_{k=0}^{n} \binom{n}{k}_{\mathbb{L}} \binom{\alpha+n+k-1}{n}_{\mathbb{L}} (-1)^{nk+\binom{k}{2}}\, x^k \end{aligned} \tag{3.8}$$

are special little q-Jacobi polynomials and satisfy the orthogonality relation

$$\int_{\mathbb{R}} p_m^{(\alpha)}(x)p_n^{(\alpha)}(x)\,d\nu(x) = (-1)^{\alpha n}\,\frac{L_\alpha(\theta)}{L_{\alpha+n}(\theta)}\,\delta_{m,n}. \tag{3.9}$$

The proof of (2.19) can be modified to establish

$$\begin{aligned} &\det\{1/L_{\alpha+i+j}(\theta) : 0 \le i, j \le n\} \\ &= (-1)^{\alpha\binom{n+1}{2}} L_\alpha^{-n}(\theta)\left[\prod_{k=1}^{n} L_{\alpha+2k}\binom{\alpha+2k-1}{k}_{\mathbb{L}}\right]^{-1}. \end{aligned} \tag{3.10}$$

4 Relations

Let y_n be a solution to (1.3) with integer initial conditions and y_1 and y_2. If y_0 and y_1 are relatively prime then y_n and y_{n+1} are relatively prime. This follows by induction form (1.3). Consequently $F_n(\theta)$ and $F_{n+1}(\theta)$ are relatively prime, and so are $L_n(\theta)$ and $L_{n+1}(\theta)$.

The following result follows from (2.6)–(2.7) and (3.4).

Theorem 4.1. *For all integers α, n, i, j, the following identities hold*

$$\begin{aligned} &F_{\alpha+n+i}(\theta)F_{\alpha+n+j}(\theta) - (-1)^{\alpha+i+j}F_{n-i}(\theta)F_{n-j}(\theta) \\ &\qquad = F_{\alpha+2n}(\theta)F_{\alpha+i+j}(\theta), \end{aligned} \tag{4.1}$$

$$F_{n+i}(\theta)F_{n+j}(\theta) - F_n(\theta)F_{n+i+j}(\theta) = (-1)^n F_i(\theta)F_j(\theta), \tag{4.2}$$

together with their companion formulas

$$\begin{aligned} &L_{\alpha+n+i}(\theta)L_{\alpha+n+j}(\theta) - (-1)^{\alpha+i+j}(4+k^2)L_{n-i}(\theta)L_{n-j}(\theta) \\ &\qquad = L_{\alpha+2n}(\theta)L_{\alpha+i+j}(\theta), \end{aligned} \tag{4.3}$$

$$L_{n+i}(\theta)L_{n+j}(\theta) - L_n(\theta)L_{n+i+j}(\theta) = (-1)^{n+1}(k^2+4)F_i(\theta)F_j(\theta), \tag{4.4}$$

where $k = 2\sinh\theta$.

One interesting application of (4.2) is to take $i = -j = 2$, replace n by $2n \pm 1$ and conclude that

$$F_{2n+1}^2(\theta) \equiv -k^2 (\mathrm{mod}\, F_{2n-1}(\theta)) \quad\text{and}\quad F_{2n-1}^2(\theta) \equiv -k^2 (\mathrm{mod}\, F_{2n+1}(\theta)). \tag{4.5}$$

Thus given an integer k a solution to the system of congruences

$$a^2 \equiv -k^2 (\mathrm{mod}\, b), \quad\text{and}\quad a^2 \equiv -k^2 (\mathrm{mod}\, b),$$

is $(a, b) = (F_{2n-1}(\theta), F_{2n+1}(\theta))$. The converse to this may be true, at least for certain values of k and it is interesting to characterize such values. In the case $k = 1$ the converse is due to Owings [14].

One of the topics in §32.3–32.4 in [12] is the question of evaluating the sums

$$\sum_{i,j,k>0, i+j+k=n} F_i F_j F_k.$$

We consider the more general question of evaluating S_n,

$$S_m(n) := \sum_{j_1, j_2, \ldots, j_m : j_1+j_2+\cdots+j_m=n} F_{j_1} F_{j_2} \ldots F_{j_m}. \tag{4.6}$$

Since $S_m(n)$ is an m-fold Cauchy convolution we find

$$\sum_{n=0}^{\infty} S_m(n) t^n = t^m (1-t-t^2)^{-m}.$$

The ultraspherical polynomials polynomials $\{C_n^{\nu}(x)\}$ have the generating function

$$\sum_{n=0}^{\infty} C_n^{\nu}(x) t^n = (1-2xt+t^2)^{-\nu},$$

[9, §4.5]. They have the explicit formula

$$C_n^{\nu}(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(2\nu)_n \, x^{n-2k} \, (x^2-1)^k}{4^k \, k! \, (\nu+1/2)_k \, (n-2k)!}.$$

It is clear that $S_m(n) = 0$ if $n < m$. Therefore

$$\begin{aligned} S_m(n+m) &= (-i)^n C_n^m(i/2) \\ &= \frac{(2m)_n}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(5/4)^k}{k! \, (m+1/2)_k \, (n-2k)!}. \end{aligned} \tag{4.7}$$

Formula (4.7) generalizes many of the formulas in [12]. The only drawback of (4.7) is that it does not show that $S_m(n)$ is an integer. Indeed the individual terms in the sum (after multiplication by $2^{-n}(2m)_n$ are not integers but their sum is an integer.

Theorem 5.9 in [12] asserts that

$$F_{n+k} F_{n-k} - F_n^2 = (-1)^{n+k+1} F_k^2,$$

and is attributed to Catalan. Equations (2.6) and (2.7) yield the identical result

$$F_{n+k}(\theta) \, F_{n-k}(\theta) - F_n^2(\theta) = (-1)^{n+k+1} F_k^2(\theta). \tag{4.8}$$

A consequence of (4.8) is that if $p \mid F_n(\theta)$ and $p \mid F_{n \pm k}(\theta)$ then $p \mid F_k(\theta)$. The case $k = 1$ of (4.8) is

$$F_{n+1}(\theta) \, F_{n-1}(\theta) - F_n^2(\theta) = (-1)^n. \tag{4.9}$$

and generalizes the Cassini formula, [12, Theorem 5.3]. Moreover

$$\begin{pmatrix} F_2(\theta) & F_1(\theta) \\ F_1(\theta) & F_0\theta) \end{pmatrix}^n = \begin{pmatrix} F_{n+1}(\theta) & F_n(\theta) \\ F_n(\theta) & F_{n-1}(\theta) \end{pmatrix}, \tag{4.10}$$

follows from equations (2.6) and (2.7), and the fact that

$$\begin{pmatrix} F_2(\theta) & F_1(\theta) \\ F_1(\theta) & F_0\theta) \end{pmatrix} = \frac{1}{1+e^{-2\theta}} \begin{pmatrix} 1 & e^{-\theta} \\ e^{-\theta} & -1 \end{pmatrix} \begin{pmatrix} e^{\theta} & 0 \\ 0 & -e^{-\theta} \end{pmatrix} \begin{pmatrix} 1 & e^{-\theta} \\ e^{-\theta} & -1 \end{pmatrix}$$

One can prove a formula similar to (4.10) and involving the generalized Lucas numbers. The relationship (4.9) follows from (4.10) by evaluating the determinants of both sides.

Theorem 4.2. *The generalized Fibonacci numbers have the property*

$$\arctan(k/F_{2m+1}(\theta)) + \arctan(1/F_{2m+2}(\theta)) = \arctan(1/F_{2m}(\theta)), \tag{4.11}$$

where $k = 2\sinh\theta$. *Moreover*

$$\sum_{n=0}^{\infty} \arctan(k/F_{2n+3}(\theta)) = \arctan(1/k). \tag{4.12}$$

Proof. Clearly (4.11) is equivalent to

$$\left[\frac{k}{F_{2m+1}(\theta)} + \frac{1}{F_{2m+2}(\theta)}\right] \Big/ \left[1 - \frac{k}{F_{2m+1}(\theta)F_{2m+2}(\theta)}\right] = \frac{1}{F_{2m}(\theta)}.$$

In other words we need to show that

$$[kF_{2m+2}(\theta) + F_{2m+1}(\theta)]F_{2m}(\theta) = F_{2m+1}(\theta)F_{2m+2}(\theta) - k.$$

The above can be rewritten as

$$k[1 + F_{2m+2}(\theta)F_{2m}(\theta)] = F_{2m+1}(\theta)[F_{2m+2}(\theta) - F_{2m}(\theta)] = kF_{2m+1}^2(\theta),$$

which follows from (4.9). Finally (4.12) follows by telescopy from (4.11). ☐

It is easy to prove the following result

$$\sum_{j=1}^{n} F_j^2(\theta) = 2\sinh\theta F_n(\theta)F_{n+1}(\theta). \tag{4.13}$$

which reduces to a theorem of Lucas when $\theta = \theta_0$, see Theorem 5.5 in [12]. One can also prove

$$\begin{aligned} F_{n+1}^2(\theta) + F_n^2(\theta) &= F_{2n+1}(\theta), \\ F_{n+1}^2(\theta) - F_n^2(\theta) &= 2\sinh\theta\, F_{2n+1}(\theta). \end{aligned} \tag{4.14}$$

When $\theta = \theta_0$ the above identities reduce to results of Lucas, [12, Corollary 5.4].

The Lucas numbers are related to the Fibonacci numbers via

$$L_m(\theta) = F_{m+1}(\theta) + F_{m-1}(\theta) \tag{4.15}$$

which follows from a calculation using (3.4) and (2.7).

The identities (5) and (7a) in [17] extend to

$$L_{n+1}(\theta) + L_{n-1}(\theta) = 4\cosh^2\theta F_n(\theta) \tag{4.16}$$

and

$$F_{n+2}(\theta) - L_{n-2}(\theta) = 2\sinh\theta L_n(\theta), \tag{4.17}$$

respectively.

Another identity which follows from (3.4) and (2.7) is

$$L_m(\theta)F_n(\theta) - F_{m+n}(\theta) = (-1)^m L_{n-m}(\theta) \tag{4.18}$$

With $n = tm$ we iterate (4.18) and derive the finite continued fraction expansion

$$\frac{F_{m(t+1)}(\theta)}{F_{mt}(\theta)} = L_m(\theta) - \frac{(-1)^m}{L_m(\theta)-}\,\frac{(-1)^m}{L_m(\theta)-}\cdots\frac{(-1)^m}{L_m(\theta)} \tag{4.19}$$

In the above equation $L_m(\theta)$ appears m times.

Theorem 4.3. *The following identities hold*

$$F_{2n}(\theta) = F_n(\theta)L_n(\theta), \tag{4.20}$$

$$F_{n+m}(\theta) + (-1)^m F_{n-m}(\theta) = F_n(\theta)L_m(\theta) \tag{4.21}$$

$$\sum_{j=0}^{n} \frac{1}{F_j(\theta)} = \frac{1+\sinh\theta}{\sinh\theta} - \frac{F_{2^n-1}(\theta)}{F_{2^n}(\theta)}. \tag{4.22}$$

Proof. Formula (4.20) is the special case $m = n$ of (4.21). The proof of (4.22) is by induction. It clearly holds when $n = 1$. The induction step uses

$$-\frac{F_{2^n-1}(\theta)}{F_{2^n}(\theta)} + \frac{1}{F_{2^{n+1}}(\theta)} = -\frac{F_{2^n}(\theta)}{F_{2^{n+1}}(\theta)}$$

which follows from (4.20) and (4.21). □

By letting $n \to \infty$ in Theorem 4.3 we find

$$\sum_{n=0}^{\infty} \frac{1}{F_n(\theta)} = 1 + e^{-\theta}\coth\theta.$$

In the case of the Fibonacci numbers $\theta = \theta_0$ and the above sum reduces to (77) page 60 in [17]

5 Integer Points on Algebraic Curves and Surfaces

In this section we prove two theorems describing all the integral points on the curves $y^2 - kxy - x^2 = \pm 1$ for a positive integer k.

Theorem 5.1. *Let $\theta > 0$ be given and assume that $k := 2sinh\theta > 1$ is an odd integer. A positive integer n is a generalized Fibonacci number if and only if $4n^2\cosh^2\theta + 4$ or $4n^2\cosh^2\theta - 4$ is a perfect square.*

Proof. We will only consider positive solutions to

$$x^2(k^2+4) - y^2 = \pm 4. \tag{5.1}$$

It is clear that

$$(k^2+4)F_m^2(\theta) - L_m^2(\theta) = 4\cosh^2\theta F_m^2(\theta) - L_m^2(\theta)$$
$$= (e^\theta + e^{-\theta})^2 e^{2(m-1)\theta}\left(\frac{1-q^m}{1-q}\right)^2 - (1+q^m)^2 e^{2m\theta}$$

which simplifies to $4(-1)^{m+1}$, so it is equal to ± 1. Hence $n = F_m(\cosh\theta)$ makes $4n^2\cosh^2\theta \pm 4$ a perfect square. To prove the converse assume that $4x_1^2\cosh^2\theta \pm 4$ is a perfect square $= y_1^2$ say. We assume $x_1 > 1$ and the case $x_1 = 1$ we considered at the end. Thus

$$y_1^2 = x_1^2(k^2+4) \pm 4 \quad \text{with} \quad x_1 > 1.$$

It can be easily seen that $x_1 > 1$ implies $y_1 > kx_1$. Let

$$x_2 = (y_1 - kx_1)/2, \quad y_2 = |(ky_1 - (k^2+4)x_1)/2|.$$

Both x_2 and y_2 are positive integers since x_1 and y_1 have the same parity. A calculation shows that $x = x_2, y = y_2$ solve (5.1). Moreover $x_2 < x_1$ if and only if $y_1 < (k+2)x_1$, that is if and only if $x_1^2(k^2+4) \pm 4 < (k+2)^2x_1^2$, since the left-hand side is y_1^2. Clearly the latter inequality holds, hence $0 < x_2 < x_1$. We continue in this manner until we reach $x_n = 1$. Thus $y_n^2 = k^2 + 4 \pm 4$.The case $-$ leads to $y_n = k$ but the case $+$ makes $(y_n - k)(y_n + k) = 8$, hence $y_n = k + 2^j$ and $y_n = -k + 2^{3-j}$, for some $j = 0, 1, 2, 3$. This forces $y_n = (2^j + 2^{3-j})/2$ so that j must equal 1 or 2, that is $y_n = 3$ which contradicts $k > 1$. Thus the only solution is $y_n = k = L_1(\theta)$ and $x_1 = F_1(\theta)$. By reversing the above steps, and using (4.15) and (1.3) we see that $x_1 = F_n(\theta)$ and $y_1 = L_n(\theta)$. $\square$

Note that in the process of proving Theorem 4.3 we also proved the following.

Corollary 5.2. *We have*

$$4\cosh^2 F_n^2(\theta) - L_n^2(\theta) = 4(-1)^{n+1} \tag{5.2}$$

In particular $F_n(\theta)\}$ and $\{L_n(\theta)\}$ can not have any common divisor larger than 2. Moreover $F_n(\theta)$ and $L_n(\theta)$ have the same parity.

Note that the Diophantine equation (5.1) is a special case of the Pell equation.

Let

$$k := 2 \sinh \theta \tag{5.3}$$

Observe that

$$F_{n+1}^2(\theta) - kF_n(\theta)F_{n+1}(\theta) - F_n^2(\theta) = (-1)^n \tag{5.4}$$

follows from replacing the F's in the left-hand side by the corresponding expressions from (2.7). We now prove a converse to (5.4). Consider the diophantine equations

$$y^2 - kxy - x^2 = 1, \tag{5.5}$$

$$y^2 - kxy - x^2 = -1. \tag{5.6}$$

The integer solutions to (5.5) or (5.6) will be denoted by (x, y). It is clear that if (x, y) is such a pair then $(-x, -y)$ will satisfy the same equation. Moreover if (x, y) satisfy (5.5) (or (5.6)) then $(y, -x)$ will solve (5.6) (respectively (5.5)). Hence there is no loss of generality in assuming $x \geq 1$ and $y \geq 1$.

Theorem 5.3. *Let k be an integer, $k > 1$, and related to θ through* (5.3)*. Assume that (x, y) solve* (5.5)*. Then there exists a positive integer n such that $(x, y) = (F_{2n}(\theta), F_{2n+1}(\theta))$. On the other hand if (x, y) solve* (5.6) *then there exists a positive integer n such that $(x, y) = (F_{2n-1}(\theta), F_{2n}(\theta))$.*

Proof. The proof consists of three step.
Step 1. We show that the smallest positive x satisfying (5.5) is $x = k$. To see this write (5.5) in the form $y(y - kx) = x^2 + 1$, hence $y = kx + z$ and $z > 1$. Thus (5.5) becomes $x(x - kz) = z^2 - 1$, which shows that $x \geq k = F_2(\theta)$. The only possible answer for y is $y = F_3(\theta)$. Indeed the point $(F_2(\theta), F_3(\theta))$ lies on the curve (5.5).
Step2 We use induction. Assume that all solutions to (5.5) are of the form $(F_{2j}(\theta), F_{2j+1}(\theta))$, for $1 \leq j \leq m$. Let $x > F_{2m}(\theta)$ and assume that x is the smallest integer such that (x, y) solves (5.5). Rewrite (5.5) as

$$(y - kx)^2 - 1 = (k^2 + 1)x^2 - kxy = x[(k^2 + 1)x - ky].$$

Thus $(k^2 + 1)x - ky > 0$. We have already shown that $y > kx$. Define (x_0, y_0) by

$$x_0 = (k^2 + 1)x - ky, \qquad y_0 = y - kx. \tag{5.7}$$

Both x_0 and y_0 are positive integers. Moreover $x_0 - x = k(x - ky) < 0$, that is $x_0 < x$, hence $x_0 \leq F_{2m}(\theta)$. By direct computation we see that (x_0, y_0) solves (5.5), hence there is a positive integer r such that $x_0 = F_{2r}(\theta)$ and $y_0 = F_{2r+1}(\theta)$. From (5.7) it follows that

$$x = x_0 + ky_0, \quad \text{and} \quad y = kx_0 + (1 + k^2)y_0.$$

Hence $x = F_{2r+2}(\theta)$ and $y = F_{2r+3}(\theta)$.
Step 3 Assume that (x, y) solve (5.6) and set $(x_0, y_0) = (y, x+ky)$. A calculation shows that (x_0, y_0) satisfies (5.5), hence $(y, x + ky) = (F_{2j}(\theta), F_{2j-1}(\theta))$, for some positive integer j, which implies $(x, y) = (F_{2j}(\theta), F_{2j-1}(\theta))$, and the proof is complete. □

We next extend the following identities of Carlitz [12, Ex 91-91]:

$$Z^3_{n+1} - Z^3_n - Z^3_{n-1} = 3Z_{n+1}Z_nZ_{n-1}, \quad Z_j = F_j \text{ or } L_j.$$

Theorem 5.4. *With $k = \sinh\theta$ the identity*

$$\begin{aligned} Z^3_{n+1}(\theta) - k^3Z^3_n(\theta) - Z^3_{n-1}(\theta) \\ = 3kZ_{n+1}(\theta)Z_n(\theta)Z_{n-1}(\theta), \end{aligned} \tag{5.8}$$

holds for $Z_n(\theta) = F_n(\theta)$ or $Z_n(\theta) = L_n(\theta)$.

Proof. After using (1.3) we see that the left-hand side of the above equation in the Fibonacci case is

$$\begin{aligned} &2\sinh\theta F_n(\theta)[F^2_{n+1}(\theta) + F^2_{n-1}(\theta) + F_{n+1}(\theta)F_{n-1}(\theta)] - (2\sinh\theta)^3F^3_n(\theta) \\ &= 2\sinh\theta F_n(\theta)F_{n-1}(\theta)[F_{n+1}(\theta) + 2\sinh\theta F_n(\theta) + F_{n-1}(\theta) + F_{n+1}(\theta)] \end{aligned}$$

which simplifies to the right-hand side of (5.8). We only used the recurrence relation (1.3) to establish (5.8). Thus (5.8) also holds for $\{L_n(\theta)\}$ since it also satisfies (1.3). □

It is interesting to determine all the positive integer points on the surface $z^3 - y^3 - z^3 = 3xyz$. We suspect that the only solutions are $(x, y, z) = (F_{n-1}, F_n, F_{n+1})$ or (L_{n-1}, L_n, L_{n+1}). This would give a converse to Carlitz's identities (5.8). Similarly it is of interest to determine all the the positive integer points (x, y, z) which lie on the surface $z^3 - k^3y^3 - z^3 = 3kxyz$ for a given positive integer k.

Fairgrieve and Gould [7] studied formulas involving differences of products of Fibonacci numbers. They claim that computer searches yielded only the list of formulas stated below. They pointed out that some of these formulas were already known and references are given in [7].

$$F_{n+1}F_{n+2}F_{n+6} - F^2_{n+3} = (-1)^nF_n, \tag{5.9}$$
$$F_nF_{n+4}F_{n+5} - F^3_{n+1} = (-1)^{n+1}F_{n+6}, \tag{5.10}$$
$$F_{n-2}F^2_{n+1} - F^3_n = (-1)^{n-1}F_{n-1}, \tag{5.11}$$
$$F_{n+2}F^2_{n-1} - F^3_n = (-1)^nF_{n+1}, \tag{5.12}$$
$$F_{n-3}F^3_{n+1} - F^4_n = (-1)^n\left[F_{n-1}F_{n+3} + 2F^2_n\right] \tag{5.13}$$
$$F_{n+3}F^3_{n-1} - F^4_n = (-1)^n\left[F^2_n + F_nF_{n-1} + 2F^2_{n-1}\right]. \tag{5.14}$$

It is clear that we can rewrite the last equation above as

$$F_{n+3}F^3_{n-1} - F^4_n = (-1)^n\left[F_nF_{n+1} + 2F^2_{n-1}\right] \tag{5.15}$$

These can be extended to the numbers $F_n(\theta)$. The extensions are given below.

$$\text{(5.16)}\quad F_{n+1}(\theta)F_{n+2}(\theta)F_{n+6}(\theta) - F_{n+3}^2(\theta) = (-1)^n\left[k^2F_n(\theta) + (k^3-1)F_{n+1}(\theta)\right],$$

$$\text{(5.17)}\quad F_n(\theta)F_{n+4}(\theta)F_{n+5}(\theta) - F_{n+1}^3(\theta) = (-1)^{n+1}\left[F_{n+6}(\theta) + k(k-1)F_{n+4}(\theta)\right],$$

$$\text{(5.18)}\quad F_{n-2}(\theta)F_{n+1}^2(\theta) - F_n^3(\theta) = (-1)^{n-1}\left[kF_{n-1}(\theta) + (k^2-1)F_n(\theta)\right],$$

$$\text{(5.19)}\quad F_{n+2}(\theta)F_{n-1}^2(\theta) - F_n^3(\theta) = (-1)^n[F_n(\theta) + kF_{n-1}(\theta)],$$

$$\text{(5.20)}\quad F_{n-3}(\theta)F_{n+1}^3(\theta) - F_n^4(\theta) = (-1)^n\left[F_{n-1}(\theta)F_{n+3}(\theta) + 2F_n^2(\theta) + (k^2-1)F_n(\theta)F_{n+2}(\theta)\right]$$

$$\text{(5.21)}\quad F_{n+3}(\theta)F_{n-1}^3(\theta) - F_n^4(\theta) = (-1)^n\left[F_n^2(\theta) + F_n(\theta)F_{n-1}(\theta) + 2F_{n-1}^2(\theta)\right].$$

The proofs use (4.8)–(4.9) and (1.3).

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