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# Polynomial Expansions and Generating Functions

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We study expansions in polynomials $\{P_n(x)\}_0^\infty$ generated by $\sum_{n=0}^\infty P_n(x)t^n = A(t)\,\phi(xt^k\theta(t))$, $\theta(0) \neq 0$, and $\sum_{n=0}^\infty P_n(x)t^n = \sum_{j=1}^k A_j(t)\,\phi(xt\epsilon_j)$, $\epsilon_1, ..., \epsilon_k$ being the $k$ roots of unity. The case $k = 1$ is contained in a recent work by Fields and Ismail. We also prove a new generalization of Vandermond's inverse relations.

*Notation.* We use the contracted notation

$$ {}_pF_q\left(\begin{matrix} a_P \\ b_Q \end{matrix}\,\middle|\, z\right) = \sum_{n=0}^{\infty} \frac{(a_P)_k}{(b_Q)_k} \frac{z^k}{k!} $$

for the hypergeometric function

$$ {}_pF_q\left(\begin{matrix} a_1, ..., a_p \\ b_1, ..., b_q \end{matrix}\,\middle|\, z\right), $$

where $(a_P)_k = \prod_{j=1}^{P} (a_j)_k$, with $(\alpha)_k = \Gamma(\alpha + k)/\Gamma(\alpha)$.

## 1. INTRODUCTION

Polynomial expansions of analytic functions and obtaining generating functions are old subjects that have attracted several mathematicians lately. Fields and Wimp [6] expanded hypergeometric functions in terms of Jacobi and Laguerre type polynomials. They essentially proved

$$ \begin{aligned} &{}_{p+r}F_{q+s}\left(\begin{matrix} a_P, c_R \\ b_Q, d_S \end{matrix}\,\middle|\, zw\right) \\ &= \sum_{n=0}^{\infty} \frac{(a_P)_n}{(b_Q)_n (n+\gamma)_n} \frac{(-z)^n}{n!}\, {}_pF_{q+1}\left(\begin{matrix} n + a_P \\ 1 + 2n + \gamma, n + b_Q \end{matrix}\,\middle|\, z\right) \qquad (1.1) \\ &\quad \cdot {}_{r+2}F_s\left(\begin{matrix} -n, \gamma + n, c_R \\ d_S \end{matrix}\,\middle|\, w\right) \end{aligned} $$





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via Laplace transform and induction, and derived

$$ {}_{p+r}F_{q+s}\left(\begin{matrix} a_P\,,\, c_R \\ b_Q\,,\, d_S \end{matrix} \middle| zw\right) = \sum_{n=0}^{\infty} \frac{(a_P)_n}{(b_Q)_m} \frac{(-z)^n}{n!} \, {}_pF_q\left(\begin{matrix} n + a_P \\ n + b_Q \end{matrix} \middle| z\right) {}_{r+1}F_s\left(\begin{matrix} -n,\, c_R \\ d_S \end{matrix} \middle| w\right) \tag{1.2} $$

from (1.1) by confluence. Verma [14] generalized (1.1) to

$$ \sum_{n=0}^{\infty} a_n b_n (zw)^n = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!(\gamma + n)_n} \left\{ \sum_{r=0}^{\infty} \frac{b_{n+r}}{r!(\gamma + 2n + 1)_r} z^r \right\} \left\{ \sum_{s=0}^{n} \frac{(-n)_s (n + \gamma)_s}{s!} a_s w^s \right\}, \tag{1.3} $$

and obtained a result corresponding to (1.2), also by confluence. He also established a basic analog as well as a two-dimensional analog of the above formula. However, these analogs and extensions did not shed any light on the structure behind these expansions. Later, Verma [15] generalized a result of Niblett to

$$ \sum_{n=0}^{\infty} a_n b_n \frac{(zw)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!} \left\{ \sum_{k=0}^{n} \frac{(-n)_k}{k!} [h + k(1 - \alpha)] \, a_k w^k \right\} \cdot \left\{ \sum_{s=0}^{\infty} (k + c_U)_{s+n-k} (h + k + 1 - n\alpha)_{s+n-k} \, b_{s+n} \frac{x^s}{s!} \right\}, \tag{1.4} $$

which, at first sight, seems to be unrelated to (1.3). Later, Fields and Ismail [7] observed that (1.3) and (1.4) are formal expansions of the form

$$ \sum_{n=0}^{\infty} a_n b_n (zw)^n = \sum_{n=0}^{\infty} z^n P_n(w) \, R_n(z), \tag{1.5} $$

where the $R_n(z)$'s are power series and the $P_n(w)$'s are generated by

$$ \sum_{n=0}^{\infty} P_n(w) \, t^n = A(t) \, \phi(wH(t)), \tag{1.6} $$

where $A(t)$, $H(t)$, and $\phi(t)$ are formal power series with $A(0) \, H'(0) \neq 0$. They also used a different approach based on orthogonality relations or inverse relations. For other related results see [1, 3–5, 12, 17]. For finding generating functions of the form (1.6) for orthogonal polynomials see [8].

The present work is a continuation of Fields and Ismail's. We start, in Section 2, by establishing a generalization of Vandermonde's inversion relations and use it to obtain a new generalization of (1.2). In Section 3 we study expansions in polynomials generated by

$$\sum_{n=0}^{\infty} P_n(x)\, t^n = \sum_{j=1}^{k} A_j(t)\, \phi(xt\epsilon_j), \tag{1.7}$$

where $\epsilon_1, \epsilon_2, \ldots, \epsilon_k$ are the $k$-roots of unity. This includes, in particular, expansions in the Euler polynomials $\{S_n(z)\}$ and $\{C_n(z)\}$ generated by

$$\frac{\sinh tz - \cosh t(1-z)}{\cosh t} = \sum_{n=0}^{\infty} S_n(z)\, t^n \tag{1.8}$$

and

$$\frac{\cosh tz - \sinh t(1-z)}{\cosh t} = \sum_{n=0}^{\infty} C_n(z)\, t^n. \tag{1.9}$$

These polynomials were studied by Bernstein [2]. Expansions in polynomials generated by (1.7) when $\phi(x)$ is the exponential function have also been investigated; for example, see [9, 10]. The last section, Section 4, is devoted to studying expansions in polynomials $\{P_n(x)\}_0^{\infty}$ generated by

$$\sum_{n=0}^{\infty} P_n(x)\, t^n = A(t)\, \phi(xt^k\theta(t)), \tag{1.10}$$

where $A(t)$, $\phi(t)$, and $\theta(t)$ are power series with $A(0)\,\theta(0) \neq 0$ and $k$ is a positive integer. It is obvious that (1.6) is the special case $k = 1$ of (1.10). A generating function of type (1.10) appears in [13].

## 2. A Generalization of Vandermonde Inversion Relations

A sequence $\{c_n\}_0^{\infty}$ is called [16] a fundamental sequence if $c_0 = 0$ and $c_n \neq 0$ for $n > 0$. We follow Ward's notation and write $[n]$ for $c_n$ and define factorials $[n]!$ and binomial coefficients $\left[{n \atop j}\right]$ by

$$[n]! = 0 \quad \text{if} \quad n = 0, \qquad\qquad \left[{n \atop j}\right] = 0 \quad \text{if} \quad j > n,$$
$$\text{and}$$
$$= [1] \cdots [n] \quad \text{if} \quad n > 0, \qquad\qquad = \frac{[n]!}{[j]\,[n-j]!} \quad \text{if} \quad j < n,$$

respectively. A fundamental sequence is called normal if $\sum_{j=0}^{n} \left[{n \atop j}\right](-1)^j$ is $\delta_{n,0}$. Clearly $\sum_{j=0}^{n} \left[{n \atop j}\right](-1)^j$ vanishes for $n$ odd since $\left[{n \atop j}\right] = \left[{n \atop n-j}\right]$. Therefore the restriction of being normal is only a restriction on $c_2, c_4, \ldots,$ while $c_1, c_3, \ldots$ remain arbitrary.

We now state our generalization of Vandermonde's inversion relations.

THEOREM. *If* $\{[n]\}_0^\infty$ *is a normal sequence then*

$$f_n = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} g_j \qquad \text{if and only if} \qquad g_n = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} (-1)^{n+j} f_j .$$

The proof follows by substitution from one relation into the other and using the normality requirement.

Let

$$P_n(w) = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} a_j w^j. \tag{2.1}$$

Then, by the above theorem we get

$$a_n w^n = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} (-1)^{n+j} P_j(w). \tag{2.2}$$

Multiplying (2.2) by $b_n z^n$ and summing over $n = 0, 1, \ldots$, we obtain

$$\sum_{n=0}^{\infty} a_n b_n (zw)^n = \sum_{n=0}^{\infty} (-z)^n \left\{ \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix} a_l w^l \right\} \left\{ \sum_{j=0}^{\infty} \begin{bmatrix} n+j \\ j \end{bmatrix} b_{n+j} z^j \right\} . \tag{2.3}$$

Formula (2.3) generalizes Fields and Wimp's formula (1.2), and is self-dual; see [7]. Note that the polynomials $P_n(w)$ of (2.1) do not, in general, have a generating function of type (1.6).

## 3. EXPANSIONS IN POLYNOMIALS GENERATED BY (1.7)

Ozegov [11] proved that a polynomial set $\{P_n(x)\}_0^\infty$ satisfies $D^k P_n(x) = P_{n-k}(x)$, $D = d/dx$, $P_j(x) = 0$ if $j < 0$ if and only if (1.7) is satisfied with $\phi(x) = e^x$ and

$$\sum_{j=1}^{k} A_j(0)\, \epsilon_j^{\,n} \neq 0, \qquad n = 0, 1, 2, \ldots, k-1. \tag{3.1}$$

Given any fundamental sequence $\{[n]\}_0^\infty$, we associate a linear operator $\mathscr{D}$ defined on formal power series by $\mathscr{D} x^n = [n]\, x^{n-1}$, $n = 0, 1, \ldots$ . It is easy to see that $\mathscr{D}^k P_n(x) = P_{n-k}(x)$ if and only if (1.7) and (3.1) are satisfied with $\phi(x) = \sum_{n=0}^{\infty} x^n/[n]!$.

Consider polynomials $\{P_n(x)\}_0^\infty$ satisfying (1.7) and (3.1). Set $\phi(x) = \sum_{n=0}^{\infty} \phi_n x^n$, $A_j(t) = \sum_{n=0}^{\infty} a_{j,n} t^n$. It is clear that

$$P_n(x) = \sum_{r=0}^{n} \phi_r x^r \mu_{r,n-r} , \tag{3.2}$$

where

$$\mu_{r,n-r} = \sum_{j=1}^{k} a_{j,n-r}\epsilon_j^{\,r}. \tag{3.3}$$

Relation (1.7) implies, upon choosing $\epsilon_j = \exp(2\pi i/j)$,

$$\sum_{n=0}^{\infty} P_n(x)\,(t/\epsilon_l)^n = \sum_{j=1}^{k} A_j(t/\epsilon_l)\,\phi(xt\epsilon_{j-l}), \qquad l = 0, 1,..., k-1. \tag{3.4}$$

If we denote $\sum_{n=0}^{\infty} P_n(x)\,t^n$ by $G(x, t)$, then (3.4) can be written in the matrix form

$$\begin{bmatrix} G(x,t) \\ G(x, t/\epsilon_1) \\ \cdots \\ G(x, t/\epsilon_{k-1}) \end{bmatrix} = \begin{bmatrix} A_1(t) & A_2(t) & \cdots & A_k(t) \\ A_2(t/\epsilon_1) & A_3(t/\epsilon_1) & \cdots & A_1(t/\epsilon_1) \\ \cdots & \cdots & \cdots & \cdots \\ A_k(t/\epsilon_{k-1}) & A_1(t/\epsilon_{k-1}) & \cdots & A_{k-1}(t/\epsilon_{k-1}) \end{bmatrix} \begin{bmatrix} \phi(xt) \\ \phi(xt\epsilon_1) \\ \cdots \\ \phi(xt\epsilon_{k-1}) \end{bmatrix}. \tag{3.5}$$

We make the additional assumption that the above $k \times k$ matrix formed from the $A$'s is nonsingular at $t = 0$. Thus we can express $\phi(xt)$ in terms of $P_n(x)$'s. The resulting relation is an inverse to (3.2) and of the form

$$\phi_n x^n = \sum_{r=0}^{n} \lambda_{r,n-r} P_r(x). \tag{3.6}$$

It is clear that $\lambda_{n,0} \neq 0$ since $\mu_{n,0} \neq 0$, by (3.3). Now multiply (3.6) by $\psi_n y^n$ and add for $n = 0, 1,...,$ to get

$$\sum_{n=0}^{\infty} \psi_n \phi_n x^n y^n = \sum_{n=0}^{\infty} y^n \left\{ \sum_{r=0}^{n} \mu_{r,n-r}\phi_r x^r \right\} \left\{ \sum_{s=0}^{\infty} \lambda_{n,s}\psi_{n+s} y^s \right\}. \tag{3.7}$$

Relationship (3.7) is the sought expansion. Note that because the inverse relations (3.2) and (3.6) are essentially inversions of infinite triangular matrices we can interchange the $\lambda$'s and $\mu$'s. This leads to the dual expansion

$$\sum \psi_n \phi_n (xy)^n = \sum_{n=0}^{\infty} y^n \left\{ \sum_{r=0}^{n} \lambda_{r,n-r}\phi_r x^r \right\} \left\{ \sum_{s=0}^{\infty} \mu_{n,s}\psi_{n+s} y^s \right\}. \tag{3.8}$$

EXAMPLE. Consider the case $k = 2$, $A_1(t) = (1 + e^{-t})/(e^t + e^{-t})$, $A_2(t) = (1 - e^t)/(e^t + e^{-t})$. This contains the Euler's polynomials $\{C_n(x)\}_0^\infty$ generated by (1.8) as the case $\phi_n = 1/n!$. In this case the matrix

$$\begin{bmatrix} A_1(0) & A_2(0) \\ A_2(0) & A_1(0) \end{bmatrix}$$

is not singular and we have

$$\phi(xt) = 2\left(\frac{1-e^t}{e^t+e^{-t}}\right)\sum_{n=0}^{\infty} P_n(x)\, t^n - 2\left(\frac{1-e^t}{e^t+e^{-t}}\right)\sum_{n=0}^{\infty} P_n(x)\,(-t)^n.$$

In this case (3.2) and (3.6) will be

$$P_n(x) = \frac{1}{2}\sum_{r=0}^{n} \frac{\phi_r x^r}{(n-r)!}\left[E_{n-r}(1+(-1)^r) + (-1)^r \sum_{l=0}^{n-r}\binom{n-r}{l} E_l((-1)^{n-l}-1)\right] \tag{3.9}$$

and

$$\phi_n x^n = \sum_{r=0}^{n}\left\{2E_{n-r} + (1+(-1)^r)\sum_{l=0}^{n-r-1} E_l\binom{n-r}{l}\right\}\frac{P_r(x)}{(n-r)!}, \tag{3.10}$$

respectively, where $E_0$, $E_1$,... are the Eulerian numbers generated by

$$2/(e^t+e^{-t}) = \sum_{n=0}^{\infty} E_n t^n/(n!).$$

From (3.10) one may derive (3.7) in the present special case as well as its dual (3.8).

We would like to emphasize that the key in the above expansions is the inverse relations (3.2) and (3.6). However, generating functions provide a rich source of these relationships.

## 4. Polynomials Generated by (1.10)

Let $\{P_n(x)\}_0^\infty$ be a sequence of polynomials generated by (1.10). Set

$$\phi(x) = \sum_0^{\infty} \phi_k x^k \tag{4.1}$$

and

$$A(t)\,\{t^k\theta(t)\}^n = \sum_{j=0}^{\infty} \mu_{n,j} t^{j+kn}, \qquad \mu_{n,0} \neq 0. \tag{4.2}$$

From (1.10), (4.1), and (4.2) we get

$$P_n(x) = \sum_{j=0}^{[n/k]} \phi_j \mu_{j,n-kj} x^j. \tag{4.3}$$

On the other hand, performing the change of variable $t^k\theta(t) = u$ so that $t = u^{1/k}\psi(u)$, $\psi(0) \neq 0$, we obtain

$$[A(u^{1/k}\psi(u))]^{-1} \sum_{n=0}^{\infty} P_n(x)\, u^{n/k}\psi^n(u) = \phi(xu). \tag{4.4}$$

Let

$$\{u^{1/k}\psi(u)\}^n/A(u^{1/k}\psi(u)) = \sum_{j=0}^{\infty} \lambda_{n,j} u^{(n+j)/k}. \tag{4.5}$$

From (4.4) and (4.5) we get

$$\phi_n w^n = \sum_{j=0}^{kn} \lambda_{j,kn-j} P_j(w). \tag{4.6}$$

Multiplying (4.6) by $b_n z^{kn}$ and adding for $n = 0, 1, \ldots$, we obtain

$$\sum_{n=0}^{\infty} \phi_n \psi_n (z^k w)^n = \sum_{n=0}^{\infty} P_n(w)\, z^{kn} R_n(z), \tag{4.7}$$

where

$$R_n(z) = \sum_{m=0}^{\infty} z^m \lambda_{n,km} \psi_{m+1+[(n-1)/k]}\,. \tag{4.8}$$

An example of this type is the main result of [13; see formula (11)]. This is indeed the case

$$A(t) = \frac{(1-t)^{a+1}}{(1+bt)}, \qquad \mu_{j,n-kj} = (-1)^n \binom{a+bn+n}{n} (-n)_{kj}/(1+a+b^n)_{kj}\,.$$

It is easy to perform the aforementioned operations to get

$$\lambda_{n,j} = (a+b+b_j)(a+1)_{j-1}/j!.$$

Therefore (4.7) reduces to

$$\begin{aligned}\sum_{n=0}^{\infty} \phi_n \psi_n (z^k w)^n = \sum_{n=0}^{\infty} \frac{z^{kn}}{a} \binom{a+bn+n}{n} &\left\{ \sum_{j=0}^{[n/k]} \frac{(-n)_{kj}}{(1+a+bn)_{kj}} c_j w^j \right\} \\ &\cdot \left\{ \sum_{m=0}^{\infty} \frac{z^m (a)_{mk}\,(a+km+bkm)}{(km)!} \psi_{m+1+[(n-1)/k]} \right\},\end{aligned} \tag{4.9}$$

which obviously contains Eq. (11) of [13] and seems to be new. Furthermore its dual expansion also seems to be new.

For known special cases of (4.9) and applications of these generating functions or expansions we refer the reader to [7, 13].

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