More Orthogonal Polynomials as Moments

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To Gian-Carlo Rota with thanks, gratitude and admiration

Abstract. Classical orthogonal polynomials as moments for other classical orthogonal polynomials are obtained via linear functionals. The combinatorics of the Al-Salam-Chihara polynomials is given, and three classification theorems for generalized moments as orthogonal polynomials are proven. Some combinatorial explanations and open problems are discussed.

1. Introduction

The symbolic method consisted of manipulating power series in x, and mapping x^n to α_n, where $\{\alpha_n\}$ is a sequence of combinatorial numbers. This was used by Kaplansky, Mendelsohn and Riordan [K, KR, M] to treat a variety of combinatorial problems. In a beautiful series of papers [RHO, RR, JR], Rota's ideas put the umbral and symbolic calculus on solid foundations and his techniques were applied to study several combinatorial and analytic problems. The purpose of this paper is to use these ideas to consider moments of orthogonal polynomials as other orthogonal polynomials. We thank Gian-Carlo for his insight into these problems and for being the driving force behind the modern theory of the umbral calculus.

In [K2] and [IS2] several families of orthogonal polynomials are shown to be the moment sequences for other orthogonal polynomials. The proofs in [IS2] are by brute force, using the explicit form of the measures. In this paper we motivate and generalize some of these results (Theorems 1, 2, 3 and 4), by evaluating linear functionals on appropriate bases of the vector space of real polynomials. We also combinatorially study the Al-Salam-Chihara polynomials in §5-6. Three characterizations of generalized moment sequences as specialized Al-Salam-Chihara polynomials are given in §7. Some open problems are discussed throughout this work.

The Rotafest, which resulted in these Proceedings, had two components, one on enumeration and a workshop on the umbral calculus. We are pleased that this work overlaps with both components since on one hand our study of functionals is umbral in nature but on the other hand our results on Hermite, Meixner and Al-Salam-Chihara polynomials are combinatorial in nature and use enumerative techniques.

[1]This work was supported by NSF grant DMS-9625459 and a research fellowship from the Leverhulme Foundation.

[2]This work was supported by NSF grant DMS-9400510.

We set some notation. If $\{p_n(x)\}$ is a sequence of monic orthogonal polynomials with real coefficients, it is known [Ch] that they satisfy a recursion relation

$$p_{n+1}(x) = (x - b_n)p_n(x) - \lambda_n p_{n-1}(x), \quad n \geq 0, \tag{1.1}$$

for some real b_n and λ_n, with $p_0(x) := 1$ and $\lambda_0 p_{-1}(x) := 0$. We refer to (1.1) as the three term recurrence relation for $p_n(x)$. We let L denote the linear functional on the vector space of real polynomials for which orthogonality holds,

$$L(p_n p_m) = 0 \quad \text{if } n \neq m. \tag{1.2}$$

The moments μ_n are defined by

$$\mu_n = L(x^n).$$

We note that if $p_n(x)$ satisfies (1.1), and

$$L(p_n) = 0 \text{ for } n > 0, \tag{1.3}$$

then (1.2) holds.

We shall also find the value of L at polynomials of degree n, other than x^n and $p_n(x)$. We shall consider

$$L((x+a)^n) = \sum_{k=0}^{n} \binom{n}{k} \mu_k a^{n-k}, \tag{1.4}$$

$$L((x;q)_n), \tag{1.5}$$

where

$$(A;q)_n := \prod_{i=0}^{n-1} (1 - Aq^i),$$

and

$$L(\phi_n(x;a)), \tag{1.6}$$

where

$$\phi_n(x;a) = (ae^{i\theta};q)_n (ae^{-i\theta};q)_n, \quad x = cos\theta.$$

For many of the cases considered one can find an explicit expansion

$$e_n(x,y) = \sum_{k=0}^{n} c_k p_k(x) s_{n-k}(y), \tag{1.7}$$

where $e_n(x,y)$ is some elementary homogeneous polynomial in x, y of degree n (for instance $(x+y)^n$), the c_k are explicit constants, the p_k form a class of orthogonal polynomials and the s_k form another class of polynomial special functions, often expressible in terms of some class of orthogonal polynomials. The assumption $L(p_0) = 1$ and the expansion (1.7) imply

$$c_n s_n(y) = L(e_n(.,y)). \tag{1.8}$$

This also shows that occurrence of orthogonal polynomials as moments is only a special case of occurrence of orthogonal polynomials as expansion coefficients. Indeed the set up in (1.4), which is used in this paper is just one instance of the more general set up in (1.7) and (1.8).

In some cases, formula (1.7) can be obtained by multiplication of a generating function for $p_k(x)$ with a generating function for $s_l(y)$. It may be possible to obtain (1.7) from (1.8) by substitution of a Rodrigues type formula combined with integration or summation or q-summation by parts. Sometimes one can recognize (1.7) as a degenerate addition formula.

For instance, the Hermite case considered in Section 5 can be obtained by multiplication of the two generating functions

$$e^{2xz-z^2} = \sum_{k=0}^{\infty} \frac{H_k(x)\, z^k}{k!}, \quad e^{2iyz+z^2} = \sum_{l=0}^{\infty} \frac{i^l\, H_l(y)\, z^l}{l!}.$$

The result is

$$(x+iy)^n = \sum_{k+l=n} \binom{n}{l} i^l\, H_k(x)\, H_l(y). \tag{1.9}$$

Motivated by identities such as (1.9), W. Al-Salam and T. Chihara [AC] characterized all triples $\{p_k(x)\}$, $\{s_n(y)\}$, $\{e_n(x,y)\}$ satisfying (1.7) such that $\{p_k(x)\}$ and $\{s_n(y)\}$ are orthogonal polynomials and $\{e_n(x,y)\}$ are orthogonal polynomials in x for infinitely many values of y. In addition to some classical polynomials, Al-Salam and Chihara [AC] identified what has become known as the Al-Salam-Chihara polynomials and their weight function was found recently, see [AI].

Polynomials depending on parameters are orthogonal when the parameters lie in a certain domain. If these polynomials are represented as moments, the integral representation of the functional with respect to a positive measure restricts the parameters to outside this domain. The reason is that an orthogonal polynomial of degree n has n real and simple zeros. One must use other techniques to extend the validity of the results to the domain of orthogonality.

We use the standard notation for hypergeometric and basic hypergeometric series in [GR]. We also use the notion of basic numbers

$$[n]_q = \frac{1-q^n}{1-q}$$

and the q-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}.$$

2. Meixner polynomials as moments

Here we obtain the Meixner polynomials as moments of the translated beta measure. We will see that the moments can be found directly from the orthogonal polynomials via (1.3), without knowledge of a representing measure.

First consider the normalized beta integral on $[0,1]$, and define the associated linear functional L by

$$L(p(x)) = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \int_0^1 p(x) x^\alpha (1-x)^\beta dx. \tag{2.1}$$

The monic orthogonal polynomials for L are constant multiples of the Jacobi polynomials,

$$P_n^{(\alpha,\beta)}(1-2x) = \frac{(\alpha+1)_n}{n!}\ {}_2F_1(-n, n+\alpha+\beta+1; \alpha+1; x).$$

Clearly from (2.1) and the beta function evaluation we have

$$\mu_k = \frac{(\alpha+1)_k}{(\alpha+\beta+2)_k}. \tag{2.2}$$

Thus (1.4) implies

$$L((x+a)^n) = \sum_{k=0}^{n} \binom{n}{k} \frac{(\alpha+1)_k}{(\alpha+\beta+2)_k} a^{n-k}, \tag{2.3}$$

which is a Meixner polynomial under an appropriate choice of α and β. This says that the measure for which the Meixner polynomials are moments is a translate of the orthogonality measure, for Jacobi polynomials, which is stated in [IS2].

Note that (2.2) implies that

$$L(P_n^{(\alpha,\beta)}(1-2x)) = \frac{(\alpha+1)_n}{n!}\ {}_2F_1(-n, n+\alpha+\beta+1; \alpha+\beta+2; 1) = 0 \text{ if } n > 0,$$

from the Chu-Vandermonde evaluation of a terminating ${}_2F_1$ at $x=1$. So we could obtain (2.2) from the explicit formula for $P_n^{(\alpha,\beta)}(1-2x)$ without knowledge of an explicit measure. We shall use this method again in the next section.

3. Three q-versions

In this section we consider three different q-versions of the functional L of §2. These three functionals will be denoted by L_1, L_2 and L_3. They act nicely on x^n, $(x;q)_n$, and $\phi_n(x;a)$, respectively (see Theorems 1, 2, and 3). The corresponding three sets of orthogonal polynomials are the little q-Jacobi, big q-Jacobi, and the Askey-Wilson polynomials. We use the explicit formula for these polynomials to find the value of the linear functional L, in order for (1.3) to hold. Then we change the bases to find orthogonal polynomials as generalized moments.

The little q-Jacobi polynomials are defined by [GR, (7.3.1)]

$$p_n(x;a,b;q) = {}_2\phi_1(q^{-n}, abq^{n+1}; aq; q, xq).$$

For (1.3) to hold, we should try

$$L_1(x^k) = \frac{(aq;q)_k}{(abq^2;q)_k}, \tag{3.1}$$

analogous to §2. In this case the q-analogue of the Chu-Vandermonde evaluation [GR, (II.6)] does imply (1.3). Thus we have found the moments without explicitly knowing any representing measure.

We next obtain the analog of translating the measure by a constant.

Theorem 1. *For the little q-Jacobi functional L_1 we have*

$$L_1((cx;q)_n) = {}_2\phi_1(q^{-n}, aq; abq^2; q, cq^n).$$

Proof. Apply the q-binomial theorem in the form

$$(cx;q)_n = \sum_{k=0}^{n} \frac{(q^{-n};q)_k}{(q;q)_k}(cq^n x)^k$$

to (3.1). □

The big q-Jacobi polynomials of Andrews and Askey are defined by [GR, (7.3.10)]

$$P_n(x;a,b,c;q) = {}_3\phi_2(q^{-n}, abq^{n+1}, x; aq, cq; q, q).$$

As for the little q-Jacobi polynomials again if we put

$$L_2((x;q)_k) = \frac{(aq;q)_k(cq;q)_k}{(abq^2;q)_k},$$

then the q-analogue of the Chu-Vandermonde sum [GR, (II.6)] implies (1.3). To find the moments we expand x^n in terms of $(x;q)_k$, by a limiting case of the above mentioned ${}_2\phi_1$ evaluation

$$x^n = \sum_{k=0}^{n} \frac{(q^{-n};q)_k}{(q;q)_k}(x;q)_k q^k.$$

Theorem 2. *For the big q-Jacobi functional L_2 we have*

$$L_2(x^n) = {}_3\phi_2(q^{-n}, aq, cq; abq^2, 0; q, q).$$

By appropriately choosing the parameters, the moments in Theorem 2 are Al-Salam-Chihara polynomials. Theorem 2 is proven from the explicit big q-Jacobi measure in [IS2, Theorem 3.1].

Finally we consider the Askey-Wilson polynomials, [GR, (7.5.2)]

$$p_n(x;a,b,c,d|q) = {}_4\phi_3(q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta}; ab, ac, ad; q, q).$$

This time

$$L_3(\phi_k(x,a)) = \frac{(ab;q)_k (ac;q)_k (ad;q)_k}{(abcd;q)_k}$$

works. By expanding $\phi_n(x;f)$ in terms of $\phi_n(x;a)$ [I, (2.2)]

$$\phi_n(x;f) = (af, f/a;q)_n \sum_{k=0}^{n} \frac{(q^{-n};q)_k q^k}{(q, af, aq^{1-n}/f;q)_k} \phi_k(x;a)$$

we obtain the following theorem.

Theorem 3. *For the Askey-Wilson functional L_3 we have*

$$L_3(\phi_n(x;f)) = (af, f/a;q)_n \ {}_4\phi_3(q^{-n}, ab, ac, ad; abcd, af, aq^{1-n}/f; q, q).$$

Note that the explicit form of $p_n(x)$ was crucial to determine the appropriate polynomial of degree n, $R_n(x)$, and the value of $L(R_n(x))$ which factored. In §4 we show that this idea can applied even if the explicit form of $p_n(x)$ is not known, but the measure is known.

4. Al-Salam-Chihara polynomials revisited

Theorem 2 gives the Al-Salam-Chihara polynomials as the moments of the measure with respect to which the big q-Jacobi polynomials are orthogonal. In this section we give another measure whose moments are multiples of the Al-Salam-Chihara polynomials. As before we find a polynomial $R_n(x)$ of degree n such that $L(R_n(x))$ factors. However, we do not know an explicit formula for the orthogonal polynomials $\{p_n(x)\}$ with respect to L, nor do we explicitly know the recurrence coefficients given by (1.1).

We consider a measure which is purely discrete with two infinite sequences of jumps,

$$\begin{aligned} L(p(x)) = &\frac{(q/A, q/B)_\infty}{(q, q/D)_\infty} \sum_{n=0}^{\infty} \frac{(A,B;q)_n}{(q,D;q)_n} (Dq/AB)^n p(uq^n) + \\ &\frac{(D/B, D/A)_\infty}{(q, D/q)_\infty} \sum_{n=0}^{\infty} \frac{(Aq/D, Bq/D;q)_n}{(q, q^2/D;q)_n} (Dq/AB)^n p(uq^{n+1}/D). \end{aligned} \tag{4.1}$$

If we let $c = ut$, $e = cq$, in [GR, (III.33)], and consider $L(1/(1-xt))$, we have a sum of two ${}_3\phi_2$'s which is a single infinite product. The result is

$$\frac{(qut/D)_\infty}{(qut/A)_\infty} L(1/(1-xt)) = \frac{(qut/B)_\infty}{(ut)_\infty}. \tag{4.2}$$

Clearly (4.2) is equivalent to a generating function which implies

$$L(R_n(x)) = u^n \frac{(q/B;q)_n}{(q;q)_n}$$

if

$$R_n(x) = \sum_{l=0}^{n} \frac{(A/D;q)_l}{(q;q)_l}(qu/A)^l x^{n-l}. \tag{4.3}$$

We also easily obtain from (4.2) the following theorem, first obtained by Suslov [S].

Theorem 4. *The moments for the linear functional given by (4.1) are*

$$L(x^n) = (qu/D)^n \frac{(D/A)_n}{(q;q)_n} \,{}_2\phi_1(q^{-n}, q/B; Aq^{1-n}/D; q, A).$$

Clearly we could rescale and put $u = 1$.

Note that [GR, (III.6)] implies

$$L(x^n) = (Bu/D)^n \frac{(Dq/AB;q)_n}{(q;q)_n} \,{}_3\phi_2(q^{-n}, q/B, D/B; Dq/AB, 0; q, q),$$

which is multiple of the result in Theorem 2. Thus Theorems 2 and 4 give two possible interpretations for the Al-Salam-Chihara polynomials as moments. There should also be a companion theorem for Theorem 3, but we do not know such a result.

5. Combinatorial applications

In §2-§4 we found that moments of classical orthogonal polynomials may be other classical orthogonal polynomials. There has been much work on combinatorial models for both orthogonal polynomials [FO,FS] and their moments [V]. So if a given orthogonal polynomial is also a moment, these two possibly different combinatorial points of views should be reconciled. In this section we make some remarks in this direction.

The Hermite polynomials, $H_n(x)$, are the simplest limiting case of any classical polynomial. In [IS2], (or from a limiting case of (3.2)) it is shown that a rescaled version, $\tilde{H}_n(a)$ are the moments for a translate of the Hermite measure by a. Thus the Hermite polynomials are the moments for any translate of their own measure.

We give the combinatorial reason for this phenomenon. Consider the set $S = \{1, 2, \cdots, n\}$. A matching m of S is an involution on S. We refer to the 2-cycles of m as edges, and the 1-cycles (fixed points) of m as unmatched vertices.

It is well known [Fo] that, with the proper rescaling, the Hermite polynomials

$$\tilde{H}_n(x) := 2^{-n/2} H_n(x/\sqrt{2}) \tag{5.1}$$

have the representation

$$\tilde{H}_n(x) = \sum_m (-1)^{\#\text{edges in } m} x^{\#\text{fixed points of } m}. \tag{5.2}$$

They are the generating function for all matchings m on a set $\{1, 2, \cdots, n\}$, with edges weighted by -1, and unmatched vertices by x. It is also known that the moments μ_n are the number of complete matchings on $\{1, 2, \cdots, n\}$. Thus

$$L((x+a)^n) = \sum_{k=0}^{n/2} \binom{n}{2k} \mu_{2k} a^{n-2k} \tag{5.3}$$

is the generating function for all matchings of $\{1, 2, \cdots, n\}$, with edges weighted by 1, and unmatched vertices by a. The right-hand side of (5.3) is $i^{-n}\tilde{H}_n(ia)$, hence is just the rescaled Hermite polynomials rescaled again.

Although there is an a priori combinatorial interpretation for Meixner polynomials [V], and another interpretation for moments of general orthogonal polynomials [V], for the Meixner polynomials we do not have a combinatorial reconciliation, as we gave for the Hermite polynomials.

Another example is the Laguerre polynomials, a limiting case of the Meixner, for which there is well-studied combinatorial model [FS]. There are two possible interpretations as moments, corresponding to the limiting cases of Theorems 2 and 4. This would lead to two new models.

6. Combinatorics of Al-Salam-Chihara polynomials

The Al-Salam-Chihara polynomials are a special case of the Askey-Wilson polynomials. Theorems 2 and 4 give linear functionals whose moments are these polynomials. In this section we give the combinatorial interpretations for these polynomials and their moments.

The monic form of the Al-Salam-Chihara polynomials [AI, (3.2)] have the three term recurrence relation

$$p_{n+1}(x) = (x - aq^n)p_n(x) - (c + bq^{n-1})[n]_q p_{n-1}(x). \tag{6.1}$$

An explicit representation for the p_n's as multiples of a ${}_3\phi_2$ function is in Chapter 3 of [AI].

To combinatorially understand these polynomials and their moments, we consider matchings m of $\{1, 2, \cdots, n\}$. A *2-bicoloring* C of a matching m is a 2-coloring of the edges of the matching (say with colors b and c), and an independent 2-coloring of the unmatched vertices (say with colors x and a). We let $b(C)$, $c(C)$, $x(C)$, and $a(C)$ denote the number of these colored edges and unmatched vertices.

If only the edges are 2-colored, and not the unmatched vertices, we call such a coloring D an *edge 2-coloring* of m. We denote by $a(D)$ the number of unmatched vertices, and by $b(D)$ and $c(D)$ the number of edges colored b and c respectively.

Theorem 5. *The Al-Salam-Chihara polynomial $p_n(x)$ is the generating function of all 2-bicolorings C of all matchings m of $\{1, 2, \cdots, n\}$ with weight $w(C)$*

$$p_n(x) = \sum_C w(C),$$

where

$$w(C) = x^{x(C)}(-a)^{a(C)}(-b)^{b(C)}(-c)^{c(C)}q^{s(C)},$$
$$s(C) = s_1(C) + s_2(C) + 2s_3(C),$$

$$s_1(C) = \sum_{a-vertices\, i} |\{z : z < i, m(z) < i\}|,$$
$$s_2(C) = \sum_{all\, edges\, i<j} |\{z : i < z < j, m(z) < j\}|,$$
$$s_3(C) = \sum_{b-edges\, i<j} |\{z : z < i, m(z) < j\}|.$$

Proof. We verify (6.1) by considering $n+1$ in a 2-bicoloring C on $\{1, 2, \cdots, n+1\}$. First ignore the power of q. If $n+1$ is unmatched, then we have an arbitrary 2-bicoloring on $\{1, 2, \cdots, n\}$, with $n+1$ colored either x or a. These are the two terms multiplying $p_n(x)$ in (6.1). If $n+1$ is matched, there are n choices for $m(n+1)$, what remains is an arbitrary 2-bicoloring of $\{1, 2, \cdots, n\} - \{m(n+1)\}$. The colors for the $\{(n+1), m(n+1)\}$ edge are b or c, agreeing with (6.1). So it remains to check the the power of q given by $s(C)$ agrees with (6.1). If $n+1$ is unmatched and colored x, then $n+1$ does not contribute to $s(C)$. If $n+1$ is unmatched and colored a, then (6.1) contributes n to $s(C)$, and n is the number of vertices i to the left of $n+1$ such that $m(i) < n+1$. Any $i < n+1$ with $m(i) > n+1$ is inserted after $n+1$. This gives the term $s_1(C)$. If $n+1$ is matched to $m(n+1) < n+1$, we choose a monomial q^{j-1}, $1 \le j \le n$, from $[n]_q$ to weight the edge. If the edge is colored b we additionally weight the edge by q^{n-1}. We can choose j from left-to-right or right-to-left. For a c-edge $\{n-j+1, n+1\}$, choose q^{j-1}, for the b edge $\{j, n+1\}$ choose $q^{n-1+j-1}$. The term q^{j-1} contributes to $s_2(C)$ for the c-edges, and to $s_3(C)$ for the b-edges. The term q^{n-1} contributes to $s_2(C) + s_3(C)$ for the b-edges. $\square$

It is clear from the proof that several other versions of Theorem 5 could be given, with slight modifications of $s(C)$. For example, if the b-edges are read in the opposite direction, $s_2(C)$ and $2s_3(C)$ would be replaced by

$$\tilde{s}_2(C) = \sum_{all\, edges\, i<j} |\{z : i < z < j, m(z) < j\}|, \tag{6.2}$$
$$\tilde{s}_3(C) = \sum_{b-edges\, i<j} |\{z : z < j, m(z) < j\}|.$$

Note that by taking $a = b = 0$, and $c = 1$, we obtain the continuous q-Hermite polynomials $\tilde{H}_n(x|q)$, which are defined by (1.1) with

$$b_n = 0, \quad \lambda_n = [n]_q.$$

In this case we have only matchings, and Theorem 5 (with (6.2)) becomes Proposition 3.3 in [ISV].

The moments of the continuous q-Hermite polynomials are the generating functions of the crossing numbers of complete matchings [ISV, (3.6)],

$$cross(m) = |\{edges\, i < j, k < l : i < k < j < l\}|.$$

or also the generating functions of the nesting numbers of complete matchings [ISV, (3.9)],

$$nest(m) = |\{edges\, i < j, k < l : i < k < l < j\}|.$$

For the Al-Salam-Chihara polynomials, we need a q-statistic on edge 2-colorings generalizing either of these two statistics.

Theorem 6. *The nth moment for the Al-Salam-Chihara polynomials (6.1) is the generating function for all edge 2-colorings D of matchings m of $\{1, 2, \cdots, n\}$ with weight $w(D)$*

$$\mu_n = \sum_D w(D),$$

where

$$w(D) = a^{a(D)} b^{b(D)} c^{c(D)} q^{t(D)},$$
$$t(D) = c_1(m) + c_2(D) + c_3(m),$$

$$c_1(m) = \sum_{a-vertices} |\{edges\, i < j : i < a < j\}|,$$
$$c_2(D) = \sum_{b-edges\, i<j} |\{edges\, k < l : k < j < l\}|,$$

and $c_3(m)$ is either the crossing number $cross(m)$ or the nesting number $nest(m)$.

Proof. We follow the proof of [ISV, (3.6)]. If $a = b = q = 1$ and $c = 0$, the bijection from Motzkin paths of length n gives matchings on $\{1, 2, \cdots, n\}$. We must weight the unmatched vertices by a, the edges by either b or c, and also an appropriate power of q. This gives Theorem 6, up to the power $t(D)$ of q. An unmatched vertex a has weight aq^n if there are n uncompleted edges preceding a, this contributes the term $c_1(m)$ in Theorem 6. A similar argument applies for the b edges of weight bq^{n-1}, yielding $c_2(D)$. The remaining term $c_3(m)$ appears from the term q^j, $0 \le j \le n-1$, chosen from $[n]_q$ for any edge, b or c. This contributes either $cross(m)$ or $nest(m)$. $\square$

Again by reading the inserted the edges in the opposite order we may find other versions of Theorem 6.

We note that the L^2-norm can be considered as the generating function for the length in Weyl groups of type B_n.

Proposition 1. *Let L be the linear functional for the Al-Salam-Chihara polynomials. Then*

$$L(p_n p_m) = \delta_{n,m} n!_q \prod_{i=0}^{n-1} (c + bq^i).$$

Proof. Since $L(1)=1$, the L^2-norm is always given by $\lambda_n\cdots\lambda_1$, so (6.1) gives the stated constant. Another method is to use the general theory of Viennot [V], giving an involution which proves orthogonality. In this case the fixed points will be all edge 2-colorings of complete matchings of $\{1,2,\cdots,n\}$ to $\{n+1,n+2,\cdots,2n\}$. There are no a-vertices in this case, and Theorem 6 also gives the stated constant. The edge $(m^{-1}(2n-i),2n-i)$ contributes c or bq^i, $0\le i\le n-1$. The crossing number contributes $n!_q$, independent of the coloring. $\square$

It is of interest to consider the q-analog of the Hermite polynomials, which were moments of their own translated measure. If we put $c=0$, $b=-1$ in (6.1) the Al-Salam-Chihara polynomials become Al-Salam-Carlitz (I) polynomials [KS, p. 87]. Then Theorem 6 implies that the moments are the continuous q-Hermite polynomials [ISV, (2.10)].

Corollary 1. *If L is given by $b_n=aq^n$, $\lambda_n=-q^{n-1}[n]_q$, then*

$$L(x^n)=\tilde{H}_n(a|q).$$

Proof. If we apply Theorem 6 with $c=0$, the edges are colored only $b=-1$, while the unmatched vertices are weighted by a. Thus the moments are some q-version of the Hermite polynomials in a. In Theorem 6, $c_2(D)=cross(m)+nest(m)$. If we choose $c_3(m)=nest(m)$, then the q-statistic is $t(m)=c_1(m)+c_2(D)+c_3(m)=c_1(m)+cross(m)+2nest(m)$. If we apply Theorem 5 to $\tilde{H}_n(a|q)$, $(a=0,\ b=0,\ c=1,\ x=a)$, again we have just matchings m, with edges weighted by -1. The power of q is $s(m)=\tilde{s}_2(m)=t(m)$. $\square$

Another q-analog is given by the discrete q-Hermite, [GR, p. 193] $\tilde{H}_n(x;q)$, which have

$$b_n=0,\quad \lambda_n=q^{n-1}[n]_q.$$

The next corollary says that the discrete q-Hermite are the "shifted moments" for the discrete q^{-1}-Hermite.

Corollary 2. *If L is given by $b_n=0$, $\lambda_n=-q^{-n}[n]_{1/q}$, then*

$$L(d^n(-x/d;q)_n)=\tilde{H}_n(d;q).$$

Proof. Clearly

$$L(d^n(-x/d;q)_n)=\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q q^{\binom{k}{2}}d^{n-k}L(x^k). \tag{6.3}$$

We appeal to Theorem 6 to find $L(x^k)$. The choices given for b_n and λ_n correspond to $a=c=0$, $b=-q$, and then q replaced by $1/q$ in Theorem 6. Since $a=0$ the matchings must be complete and k is even. As in the proof of Corollary 1, the q-statistic can be taken to be $t(m)=2cross(m)+nest(m)$. Moreover the generating function for complete matchings is [SS, (5.4)]

$$\sum_m q^{2cross(m)+nest(m)}=[1]_q[3]_q\cdots[k-1]_q,$$

so that [GR, p. 193]

$$L(d^n(-x/d;q)_n) = \sum_{k=0}^{n} \begin{bmatrix} n \\ 2k \end{bmatrix}_q q^{k^2-k}(-1)^k d^{n-2k}[1]_q[3]_q\cdots[2k-1]_q = \tilde{H}_n(d;q).$$

□

Corollaries 1 and 2 are special cases of Corollary 3 ($A=-a$, $B=1$, $q\to 1/q$, and $A=0$, $B=1/q$, respectively). It says that the shifted moment of an Al-Salam-Carlitz (I) polynomial is an Al-Salam-Carlitz (II) polynomial [KS, p. 87].

Corollary 3. *If L is given by $b_n = -A/q^n$, $\lambda_n = -Bq^{1-n}[n]_{1/q}$, then $L(d^n(-x/d;q)_n)$ is an Al-Salam-Chihara polynomial in d of degree n with $a=A$, $c=0$, and $b=Bq$.*

Proof. The choices of b_n and λ_n imply $C=0$ in (6.1), thus force no C-colored edge in Theorem 6. We apply Theorem 6 to (6.3) (with the matching m replacing the edge 2-coloring D) to obtain
(6.4)

$$L(d^n(-x/d;q)_n) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} d^{n-k} \sum_{m \text{ on } \{1,\cdots,k\}} (-A)^{A(m)}(-B)^{B(m)} q^{-t(m)}.$$

The desired conclusion of Corollary 3 also forces $c=0$ in Theorem 5, so we must show

$$L(d^n(-x/d;q)_n) = \sum_{\tilde{m} \text{ on } \{1,\cdots,n\}} (-A)^{A(\tilde{m})}(-Bq)^{B(\tilde{m})} d^{d(\tilde{m})} q^{s(\tilde{m})}. \tag{6.5}$$

Given a subset $S=\{l_1<\cdots<l_k\}$ of $\{1,\cdots,n\}$, and a matching m on $\{1,\cdots,k\}$, define a matching $\tilde{m}$ on $\{1,\cdots,n\}$ by letting the d unmatched vertices be $\{1,\cdots,n\}-S$, and $\tilde{m}(l_i)=l_j$ if $m(i)=j$. If we show that

$$s(\tilde{m}) + t(m) + \#edges\, in\, m = (l_1-1)+\cdots+(l_k-1), \tag{6.6}$$

then Corollary 3 is established, because

$$l_1+\cdots+l_k$$

is a integer partition into k distinct parts, whose largest part is $\le n$. It is well-known that the generating function for these partitions is

$$\begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k+1}{2}}.$$

The following observations verify (6.6).

(1) If $l_i \in S$ is an A-vertex, then each point $z<l_i$ appears exactly once in the l_i contribution to $c_1(m)+s_1(\tilde{m})$.

If $\{l_i < l_j\}$ is an edge, we show that all points $z < l_i$ are counted twice, all points $l_i \le z < l_j$ are counted once, in the $\{l_i < l_j\}$ contribution to the left side of (6.6). This gives a total of $(l_j - 1) + (l_i - 1)$.

(1) If $z < l_i$ and $\tilde{m}(z) < l_i$, then $2s_3(\tilde{m})$ counts z twice.
(2) If $z < l_i$ and $\tilde{m}(z) > l_i$, then z is counted once in $c_2(m)$ and once in $c_3(m) = nest(m)$. (We count the nesting when $\{l_i < l_j\}$ is inside the other edge.)
(3) If $z = l_i$, then the edge $\{l_i < l_j\}$ in (6.6) counts z exactly once.
(4) If $l_i < z < l_j$ and $\tilde{m}(z) < l_i$, then z is counted once in $s_2(\tilde{m})$.
(5) If $l_i < z < l_j$ and $l_i < \tilde{m}(z) < l_j$, then z is counted once in $s_2(\tilde{m})$ and not in $nest(m)$. (We do not count the nesting when $\{l_i < l_j\}$ is outside the other edge.)

□

An interested referee has pointed out that one can prove Corollary 3 from the following pair of generating functions for the Al-Salam-Carlitz polynomials, $\{U_n^{(a)}(x;q)\}$ and $\{V_n^{(a)}(x;q)\}$, [Ch, KS]

$$\frac{(z;q)_\infty\,(az;q)_\infty}{(xz;q)_\infty} = \sum_{k=0}^{\infty} \frac{U_k^{(a)}(x;q)\,z^k}{(q;q)_k} \tag{6.7}$$

and

$$\frac{(yz;q)_\infty}{(z;q)_\infty\,(az;q)_\infty} = \sum_{l=0}^{\infty} \frac{(-1)^l\,q^{l(l-1)/2}\,V_l^{(a)}(y;q)\,z^l}{(q;q)_l}. \tag{6.8}$$

Clearly multiplying (6.7) and (6.8) and using the q-binomial theorem implies

$$(y/x;q)_n\,x^n = \sum_{k+l=n} \begin{bmatrix} n \\ l \end{bmatrix}_q (-1)^l\,q^{l(l-1)/2}\,U_k^{(a)}(x;q)\,V_l^{(a)}(y;q). \tag{6.9}$$

Then (1.8), (6.9) and rescaling imply Corollary 3.

It is perhaps worth noting that $|q| < 1$ is necessary for the generating function expansions (6.7) and (6.8). This restriction is removed when applied to Corollary 3. Combinatorially the two Al-Salam-Carlitz polynomials $\{U_n^{(a)}(x;q)\}$ and $\{V_n^{(a)}(x;q)\}$ are identical, because

$$U_n^{(a)}(x;q^{-1}) = V_n^{(a)}(x;q),$$

The combinatorial proof of Corollary 3 requires no assumption on q.

7. Remarks

In [IS2] several applications of Theorem 2 are given to generating functions. All of the techniques given there apply, in particular new generating functions for

Al-Salam-Chihara polynomials may be given via Theorem 4. A more elementary example is given by applying the linear functional given by Corollary 1 (the Al-Salam-Carlitz measure [Ch, p. 197]), to the generating function for the continuous q-Hermite polynomials. The result is the q-analog of Mehler's formula, [IS1, (2.2)].

One may ask if it is possible to characterize which orthogonal polynomials are moments. Since any sequence is a moment sequence [Ch, p. 74] (possibly not of a positive definite measure), we must put a restriction on the types of functionals which are available. We give two such results, below, motivated by Corollaries 1 and 3.

Proposition 2. *If $b_n = aq^n$, and λ_n is independent of a, then $L(x^n)$ is an orthogonal polynomial in a of degree n only when $\lambda_n = q^{n-1}[n]_q\lambda_1$.*

Proof. From Corollary 1 the stated choice of λ_n works. It is easy to see from [V] that $L(x^n)$ is an even function of a for n even, and an odd function of a for n odd. The remainder, upon division of $L(x^{2n}) - aL(x^{2n-1})$ by $L(x^{2n-2})$, is a linear polynomial in λ_n, so λ_n is uniquely determined for $n > 1$. $\square$

This raises the question of characterizing orthogonal polynomials of the form $L(d^n(-x/d;q)_n)$.

Proposition 3. *If λ_n and b_n are independent of d and $|q| \neq 0, 1$, then $L(d^n(-x/d;q)_n)$ is an orthogonal polynomial in d of degree n only when*

$$\lambda_n = q^{2-2n}[n]_q\lambda_1, \quad b_n = q^{-n}b_0.$$

Proof. The proof is similar to the proof of Proposition 2. From Corollary 3 the stated choices of b_n and λ_n work. The two leading terms of $L(d^n(-x/d;q)_n)$ are

$$L(d^n(-x/d;q)_n) = d^n + \begin{bmatrix} n \\ 1 \end{bmatrix}_q b_0 d^{n-1} + \cdots . \tag{7.1}$$

The possible three term recurrence relation for $p_n(d) = L(d^n(-x/d;q)_n)$ is

$$p_{n+1}(d) = (d + \tilde{b}_n)p_n(d) + \tilde{\lambda}_n p_{n-1}(d). \tag{7.2}$$

Clearly (7.1) implies that $\tilde{b}_n = b_0 q^n$. The remainder when $p_{n-1}(d)$ divides $p_{n+1}(d) - (d + q^n b_0)p_n(d)$ as a polynomial in d must be 0. Note that [V] implies that $p_{2m}(d)$ has a unique monomial containing λ_m,

$$\lambda_m\lambda_{m-1}\cdots\lambda_1 q^{\binom{2m}{2}},$$

while $p_{2m-1}(d)$ and $p_{2m-2}(d)$ do not contain λ_m. So λ_m will appear in the remainder for $n = 2m-1$ if $\lambda_{m-1}\cdots\lambda_1 \neq 0$, which is the case since $|q| \neq 0, 1$. This uniquely determines λ_m from $\{\lambda_{m-1}, \cdots, \lambda_1, b_{m-1}, \cdots, b_0\}$. An analogous argument on b_m and $p_{2m+1}(d)$, with monomial

$$b_m\lambda_m\lambda_{m-1}\cdots\lambda_1 q^{\binom{2m+1}{2}},$$

shows that b_m is uniquely determined. For $n = 0, 1$ there is no remainder, so b_0 and λ_1 are arbitrary. $\square$

Theorem 7. *Suppose λ_n and b_n are independent of d, $b_1 \neq 0$, $a_n \neq 0$, $a_0 + \cdots + a_n \neq 0$ for all n. $L(\prod_{i=0}^{n-1}(d + a_i x))$ is an orthogonal polynomial in d of degree n only when*

$$a_i = a_0 q^i, \quad b_i = b_0/q^i, \quad \lambda_i = q^{2-2n}[n]_q \lambda_1,$$

where $q = b_0/b_1$.

Proof. Let $p_n(d) = L(\prod_{i=0}^{n-1}(d + a_i x))$, so that

$$p_n(d) = \sum_{i=0}^{n} e_i(a_0, \cdots, a_{n-1}) d^{n-i} \mu_i, \tag{7.3}$$

where e_i is the elementary symmetric function of degree i. By equating the coefficients of d^{n+1-i} in (7.2) we have

$$\begin{aligned} \mu_i e_i(a_0, \cdots, a_n) =& \mu_i e_i(a_0, \cdots, a_{n-1}) + \tilde{b}_n e_{i-1}(a_0, \cdots, a_{n-1}) \\ &+ \tilde{\lambda}_n e_{i-2}(a_0, \cdots, a_{n-2}). \end{aligned} \tag{7.4}$$

If $i = 1$ in (7.4) we have

$$\tilde{b}_n = \mu_1 a_n = b_0 a_n.$$

If $i = 2$ in (7.4) we have

$$\begin{aligned} \tilde{\lambda}_n =& \mu_2 (e_2(a_0, \cdots, a_n) - e_2(a_0, \cdots, a_{n-1}) - \mu_1 \tilde{b}_n e_1(a_0, \cdots, a_{n-1}) \\ =& a_n \lambda_1 (a_0 + \cdots + a_{n-1}). \end{aligned}$$

If $i = 3$ in (7.4) we have

$$a_n(\mu_3 - b_0\mu_2) e_2(a_0 + \cdots + a_{n-1}) = b_0 a_n \lambda_1 (a_0 + \cdots + a_{n-1})(a_0 + \cdots + a_{n-2}). \tag{7.5}$$

Since $\mu_3 = b_0^3 + 2b_0\lambda_1 + b_1\lambda_1$, $\mu_2 = b_0^2 + \lambda_1$, $a_n \neq 0$, $\lambda_1 \neq 0$, (7.5) implies

$$(b_0 + b_1) e_2(a_0 + \cdots + a_{n-1}) - b_0(a_0 + \cdots + a_{n-1})(a_0 + \cdots + a_{n-2}) = 0. \tag{7.6}$$

The coefficient of a_{n-1} in (7.6) is $b_1(a_0 + \cdots + a_{n-2}) \neq 0$, so a_{n-1} is uniquely determined from $b_0, b_1, a_0, \cdots, a_{n-2}$. The solution is

$$a_i = (\frac{b_0}{b_1})^i a_0.$$

The values of b_n and λ_n are determined either by applying Proposition 3, or by considering (7.4) for $i = 4$. $\square$

If the combinatorics of $p_n(x)$ and μ_n are known, then the combinatorics of the associated orthogonal polynomials is often easy to find. For the associated Hermite polynomials, $b_n = 0$, $\lambda_n = n-1+c$. To combinatorially interpret these polynomials and their moments, we weight one the n choices for $m(n+1)$ in the matching m by c instead of 1. An analogous technique can be applied to the associated q-Hermite polynomials (see [Ke]) and gives associated versions of Theorems 5 and 6.

Acknowledgments. We gratefully acknowledge many helpful comments by the referees, including the discussion of (1.7). This work was done while the first author was visiting Imperial College and he gratefully acknowledges the research support of both Imperial College and the University of South Florida.

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