The Laplace and Mellin transforms of powers of the Riemann zeta-function

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Abstract

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This paper gives a survey of known results concerning the Laplace transform

$$L_k(s) := \int_0^\infty |\zeta(\tfrac{1}{2}+ix)|^{2k}\mathrm{e}^{-sx}\,\mathrm{d}x \qquad (k \in \mathbb{N},\ \Re\mathrm{e}\, s > 0),$$

and the (modified) Mellin transform

$$\mathcal{Z}_k(s) := \int_1^\infty |\zeta(\tfrac{1}{2}+ix)|^{2k}x^{-s}\,\mathrm{d}x \qquad (k \in \mathbb{N}),$$

where the integral is absolutely convergent for $\Re\mathrm{e}\, s \ge c(k) > 1$. Also some new results on these integral transforms of $|\zeta(\frac{1}{2}+ix)|^{2k}$ are given, which have important connections with power moments of the Riemann zeta-function $\zeta(s)$.

1 Introduction

A central place in Analytic Number Theory is occupied by the Riemann zeta-function $\zeta(s)$, defined by the representations

$$\zeta(s) = \sum_{n=1}^\infty n^{-s} = \prod_{p\,:\,\mathrm{prime}} (1-p^{-s})^{-1} \qquad (s = \sigma + it,\ \sigma > 1), \tag{1.1}$$

and otherwise by analytic continuation. It admits meromorphic continuation to the whole complex plane, its only singularity being the simple pole $s = 1$ with residue 1. For general information on $\zeta(s)$ the reader is referred to the monographs [5] and [30]. From the functional equation

$$\zeta(s) = \chi(s)\zeta(1-s), \quad \chi(s) = 2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s), \tag{1.2}$$

which is valid for any complex s, it follows that $\zeta(s)$ has zeros at $s = -2, -4, \dots$. These zeros are traditionally called the "trivial" zeros of $\zeta(s)$, to distinguish them from the complex zeros of $\zeta(s)$, of which the smallest ones (in absolute value) are $\frac{1}{2} \pm 14.134725\dots i$. It is well-known that all complex zeros of $\zeta(s)$ lie in the so-called "critical strip" $0 < \sigma = \Re\mathrm{e}\, s < 1$, and if $N(T)$ denotes the number of zeros $\rho = \beta + i\gamma$ (β, γ real) of $\zeta(s)$ for which $0 < \gamma \le T$, then

$$N(T) = \frac{T}{2\pi}\log(\frac{T}{2\pi}) - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O(\frac{1}{T}) \tag{1.3}$$

with

$$S(T) = \frac{1}{\pi}\arg\zeta(\tfrac{1}{2} + iT) = O(\log T). \tag{1.4}$$

Here $S(T)$ is obtained by continuous variation along the straight lines joining $2, 2 + iT$, $\frac{1}{2} + iT$, starting with the value 0; if T is the ordinate of a zero, let $S(T) = S(T + 0)$. This is the so-called Riemann–von Mangoldt formula. *The Riemann hypothesis* (RH) is the conjecture, stated by B. Riemann in his epoch-making memoir [29], that "*very likely all complex zeros of* $\zeta(s)$ *have real parts equal to* $\frac{1}{2}$". For this reason the line $\sigma = \frac{1}{2}$ is called the "critical line" in the theory of $\zeta(s)$. The RH is undoubtedly one of the most celebrated and difficult open problems in whole Mathematics. Its proof (or disproof) would have very important consequences in multiplicative number theory, especially in problems involving the distribution of primes. It would also very likely lead to generalizations to many other zeta-functions (Dirichlet series) sharing similar properties with $\zeta(s)$.

The aim of this paper is to present known results on the Laplace transform

$$L_k(s) := \int_0^\infty |\zeta(\tfrac{1}{2} + ix)|^{2k} \mathrm{e}^{-sx}\,\mathrm{d}x \qquad (k \in \mathbb{N},\ \sigma = \Re\mathrm{e}\, s > 0), \tag{1.5}$$

and the (modified) Mellin transform

$$\mathcal{Z}_k(s) := \int_1^\infty |\zeta(\tfrac{1}{2} + ix)|^{2k} x^{-s}\,\mathrm{d}x \qquad (k \in \mathbb{N}), \tag{1.6}$$

where the integral in (1.6) is absolutely convergent for $\Re\mathrm{e}\, s \ge c(k) > 1$, and also to present some new results. The term "modified" refers to the fact that the lower bound of integration in (1.6) is 1, and not 0 which is usual for Mellin transforms, and also we have x^{-s} instead of x^{s-1}. These modifications are technical, as the lower bound 1 dispenses with convergence problems which may occur when $|s|$ is large.

Apart from the distribution of complex zeros of $\zeta(s)$, there are several central topics in zeta-function theory. One of them is doubtlessly the evaluation (or estimation) of power moments of $|\zeta(\sigma + it)|$, that is, integrals of the form $\int_0^T |\zeta(\sigma + it)|^{2k}\,\mathrm{d}t$ $(0 < \sigma < 1)$, where σ is assumed to be fixed. The most important case, in view of the functional equation (1.2), is when $\sigma = 1/2$ and also when $k \in \mathbb{N}$, although the case when k is not an integer is also of interest (see e.g., [6, Chapter 6], and in general for power moments of the zeta-function see [5], [6] and [30]). Thus the most important object of study involving power moments of $|\zeta(\frac{1}{2}+it)|$ is

$$I_k(T) \;:=\; \int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k}\,\mathrm{d}t \qquad (k\in\mathbb{N}). \tag{1.7}$$

One trivially has

$$I_k(T) \le \mathrm{e}\int_0^\infty |\zeta(\tfrac{1}{2}+it)|^{2k}\mathrm{e}^{-t/T}\,\mathrm{d}t = \mathrm{e}L_k\left(\frac{1}{T}\right). \tag{1.8}$$

Therefore any nontrivial bound of the form

$$L_k(\sigma) \;\ll_\varepsilon\; \left(\frac{1}{\sigma}\right)^{c_k+\varepsilon} \qquad (\sigma\to 0+,\; c_k\ge 1) \tag{1.9}$$

gives, in view of (1.8) ($\sigma = 1/T$), the bound

$$I_k(T) \;\ll_\varepsilon\; T^{c_k+\varepsilon}. \tag{1.10}$$

Conversely, if (1.10) holds, then we obtain (1.9) from the identity

$$L_k\left(\frac{1}{T}\right) \;=\; \frac{1}{T}\int_0^\infty I_k(t)\mathrm{e}^{-t/T}\,\mathrm{d}t,$$

which is easily established by integration by parts. We note that here and later ε denotes arbitrarily small constants, not necessarily the same ones at each occurrence. The symbol $f(x) \ll g(x)$ (same as $f(x) = O(g(x))$) means that $|f(x)| \le Cg(x)$ for some $C > 0$ and $x \ge x_0$, while $f(x) \ll_{a,b,\ldots} g(x)$ means that the $\ll$–constant depends on $a, b, \ldots$.

One of the possible applications of $\mathcal{Z}_k(s)$ consists of the following. If

$$F(s) \;=\; \int_0^\infty f(x)x^{s-1}\,\mathrm{d}x$$

is the (classical) Mellin transform of $f(x)$, then by (4.2) obtains, for suitable $c > 1$,

$$\int_1^\infty f\left(\frac{x}{T}\right)|\zeta(\tfrac{1}{2}+ix)|^{2k}\,\mathrm{d}x$$

$$= \int_1^\infty \frac{1}{2\pi i}\int\limits_{(c)} F(s)\left(\frac{T}{x}\right)^s \mathrm{d}s|\zeta(\tfrac{1}{2}+ix)|^{2k}\,\mathrm{d}x = \frac{1}{2\pi i}\int_{(c)} F(s)T^s\mathcal{Z}_k(s)\,\mathrm{d}s, \tag{1.11}$$

where as usual $\int_{(c)}$ denotes integration over the line $\Re\mathrm{e}\, s = c$. If $f(x) \in C^\infty$ is a nonnegative function of compact support such that $f(x) = 1$ for $1 \le x \le 2$, then $F(s)$ is entire of fast decay, and (1.11) (with $c = 1+\varepsilon$) yields a weak form of the $2k$–th moment for $|\zeta(\frac{1}{2}+it)|$, namely

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k}\,\mathrm{d}t \;\ll_{k,\varepsilon}\; T^{1+\varepsilon}, \tag{1.12}$$

provided that $\mathcal{Z}_k(s)$ has analytic continuation to the half-plane $\sigma > 1$, where it is regular and of polynomial growth in $|t|$. Conversely, if (1.12) holds, then integrating by parts it is seen that $\mathcal{Z}_k(s)$ is regular for $\sigma > 1$ (see Theorem 2 below for the precise statement).

Note that, by a change of variables $x = \mathrm{e}^t$, $z = s - 1$, (1.5) becomes

$$\int_0^\infty |\zeta(\tfrac{1}{2} + i\mathrm{e}^t)|^{2k}\mathrm{e}^{-zt}\,\mathrm{d}t \qquad (\Re\mathrm{e}\, z > 0),$$

which is the Laplace transform of $|\zeta(\frac{1}{2} + i\mathrm{e}^t)|^{2k}$. Indeed, it is well-known that the Laplace and Mellin transforms are closely connected, as (by a change of variable) both of them can be regarded as special cases of Fourier transforms, and their theory built from the theory of Fourier transforms.

2 Laplace transforms of powers of the zeta-function

Laplace transforms play an important rôle in analytic number theory. Of special interest in the theory of the Riemann zeta-function $\zeta(s)$ are the Laplace transforms (1.5), namely the functions

$$L_k(s) \;:=\; \int_0^\infty |\zeta(\tfrac{1}{2} + ix)|^{2k}\mathrm{e}^{-sx}\,\mathrm{d}x \qquad (k \in \mathbb{N},\ \Re\mathrm{e}\, s > 0).$$

E.C. Titchmarsh's well-known monograph [30, Chapter 7] gives a discussion of $L_k(s)$ when $s = \sigma$ is real and $\sigma \to 0+$, especially detailed in the cases $k = 1$ and $k = 2$. Indeed, a classical result of H. Kober [24] says that, as $\sigma \to 0+$,

$$L_1(2\sigma) = \frac{\gamma - \log(4\pi\sigma)}{2\sin\sigma} + \sum_{n=0}^{N} c_n\sigma^n + O_N(\sigma^{N+1}) \tag{2.1}$$

for any given integer $N \geq 1$, where the c_n's are effectively computable constants and $\gamma = -\Gamma'(1) = 0.577\ldots$ is Euler's constant. For complex values of s the function $L_1(s)$ was studied by F.V. Atkinson [1], and more recently by M. Jutila [22], who noted that Atkinson's argument gives

$$L_1(s) = -i\mathrm{e}^{\frac{1}{2}is}(\log(2\pi) - \gamma + (\frac{\pi}{2} - s)i) + 2\pi\mathrm{e}^{-\frac{1}{2}is}\sum_{n=1}^{\infty} d(n)\exp(-2\pi i n\mathrm{e}^{-is}) + \lambda_1(s) \tag{2.2}$$

in the strip $0 < \Re\mathrm{e}\, s < \pi$, where the function $\lambda_1(s)$ is holomorphic in the strip $|\Re\mathrm{e}\, s| < \pi$. Moreover, in any strip $|\Re\mathrm{e}\, s| \leq \theta$ with $0 < \theta < \pi$, we have

$$\lambda_1(s) \;\ll_\theta\; (|s| + 1)^{-1}.$$

In [21] M. Jutila gave a discussion on the application of Laplace transforms to the evaluation of sums of coefficients of certain Dirichlet series.

For $L_2(\sigma)$ F.V. Atkinson [2] obtained the asymptotic formula

$$L_2(\sigma) = \frac{1}{\sigma}\Big(A\log^4\frac{1}{\sigma} + B\log^3\frac{1}{\sigma} + C\log^2\frac{1}{\sigma} + D\log\frac{1}{\sigma} + E\Big) + \lambda_2(\sigma), \tag{2.3}$$

where $\sigma \to 0+$,

$$A = \frac{1}{2\pi^2},\ B = \pi^{-2}(2\log(2\pi) - 6\gamma + 24\zeta'(2)\pi^{-2})$$

and

$$\lambda_2(\sigma) \ll_\varepsilon \left(\frac{1}{\sigma}\right)^{\frac{13}{14}+\varepsilon}. \tag{2.4}$$

He also indicated how, by the use of estimates for Kloosterman sums, one can improve the exponent $\frac{13}{14}$ in (2.4) to $\frac{8}{9}$. This is of historical interest, since it is one of the first instances of an application of Kloosterman sums to analytic number theory. Atkinson in fact showed that ($\sigma = \Re\mathrm{e}\, s > 0$)

$$L_2(s) \;=\; 4\pi \mathrm{e}^{-\frac{1}{2}s}\sum_{n=1}^\infty d_4(n)K_0(4\pi i\sqrt{n}\mathrm{e}^{-\frac{1}{2}s}) + \phi(s), \tag{2.5}$$

where $d_4(n)$ is the divisor function generated by $\zeta^4(s)$, K_0 is the Bessel function, and the series in (2.5) as well as the function $\phi(s)$ are both analytic in the region $|s| < \pi$. When $s = \sigma \to 0+$ one can use the asymptotic formula

$$K_0(z) \;=\; \tfrac{1}{2}\sqrt{\pi}z^{-1/2}\mathrm{e}^{-z}\left(1 - 8z^{-1} + O(|z|^{-2})\right) \quad \left(|\arg z| < \theta < \frac{3\pi}{2},\ |z| \ge 1\right)$$

and then, by delicate analysis, one can deduce (2.3)–(2.4) from (2.5).

The author [7] gave explicit, albeit complicated expressions for the remaining coefficients C, D and E in (2.3). More importantly, he applied a result on the fourth moment of $|\zeta(\frac{1}{2}+it)|$, obtained jointly with Y. Motohashi [19], [20] (see also [28]), to establish that

$$\lambda_2(\sigma) \;\ll\; \sigma^{-1/2} \qquad (\sigma \to 0+), \tag{2.6}$$

which in view of Theorem 1 below is best possible.

For $k \ge 3$ not much is known about $L_k(s)$, even when $s = \sigma \to 0+$. This is not surprising, since not much is known about upper bounds for the integral $I_k(T)$ (see (1.7)). For a discussion on $I_k(T)$ the reader is referred to the author's monographs [5] and [6].

One can consider $L_2(s)$, where s is a complex variable, and prove a result analogous to (2.2), valid in a certain region in $\mathbb{C}$. This was achieved by the author in [10]. The main tools that were used are the results and methods from spectral theory, by which recently much advance has been made in connection with $I_2(T)$ (cf. (1.7); see [6]–[20], and [26]–[28] for some of the relevant papers on this subject). For a competent and extensive account of spectral theory the reader is referred to Y. Motohashi's monograph [28].

We shall state here very briefly the necessary notation involving the spectral theory of the non-Euclidean Laplacian. As usual $\{\lambda_j = \kappa_j^2 + \frac{1}{4}\} \cup \{0\}$ will denote the discrete spectrum of the non-Euclidean Laplacian acting on $SL(2,\mathbb{Z})$–automorphic forms, and $\alpha_j = |\rho_j(1)|^2(\cosh \pi\kappa_j)^{-1}$, where $\rho_j(1)$ is the first Fourier coefficient of the Maass wave form corresponding to the eigenvalue λ_j to which the Hecke series $H_j(s)$ is attached. We note that

$$\sum_{\kappa_j \le K} \alpha_j H_j^3(\tfrac12) \ll K^2 \log^C K \qquad (C > 0). \tag{2.7}$$

Our result on $L_2(s)$, proved in [10], is the following

THEOREM 1. *Let $0 \le \phi < \frac{\pi}{2}$ be given. Then for $0 < |s| \le 1$ and $|\arg s| \le \phi$ we have*

$$L_2(s) = \frac{1}{s}\left(A\log^4\frac{1}{s} + B\log^3\frac{1}{s} + C\log^2\frac{1}{s} + D\log\frac{1}{s} + E\right)$$

$$+ s^{-\frac12}\left\{\sum_{j=1}^{\infty}\alpha_j H_j^3(\tfrac12)\Big(s^{-i\kappa_j}R(\kappa_j)\Gamma(\tfrac12 + i\kappa_j) + s^{i\kappa_j}R(-\kappa_j)\Gamma(\tfrac12 - i\kappa_j)\Big)\right\} + G_2(s), \tag{2.8}$$

where

$$R(y) \ := \ \sqrt{\frac{\pi}{2}}\Big(2^{-iy}\frac{\Gamma(\frac14 - \frac{i}{2}y)}{\Gamma(\frac14 + \frac{i}{2}y)}\Big)^3\Gamma(2iy)\cosh(\pi y) \tag{2.9}$$

and in the above region $G_2(s)$ is a regular function satisfying ($C > 0$ is a suitable constant)

$$G_2(s) \ll |s|^{-1/2}\exp\left\{-\frac{C\log(|s|^{-1}+20)}{(\log\log(|s|^{-1}+20))^{2/3}(\log\log\log(|s|^{-1}+20))^{1/3}}\right\}. \tag{2.10}$$

Remark 1. The constants A, B, C, D, E in (2.8) are the same ones as in (2.3).

Remark 2. From Stirling's formula for the gamma-function it follows that $R(\kappa_j) \ll \kappa_j^{-1/2}$. In view of (2.7) this means that the series in (2.8) is absolutely convergent and uniformly bounded in s when $s = \sigma$ is real. Therefore, when $s = \sigma \to 0+$, (2.8) gives a refinement of (2.6).

Remark 3. From (2.3) and (2.6) it transpires that $\lambda(\sigma)$ is an error term when $0 < \sigma < 1$. For this reason we considered the values $0 < |s| \le 1$ in (2.8), although one could treat the case $|s| > 1$ as well.

Remark 4. From (2.8) and elementary properties of the Laplace transform one can easily obtain the Laplace transform of

$$E_2(T) := \int_0^T |\zeta(\tfrac12 + it)|^4\,\mathrm{d}t - TP_4(\log T), \quad P_4(x) = \sum_{j=0}^{4} a_j x^j, \tag{2.11}$$

where $a_4 = 1/(2\pi^2)$ (for the evaluation of the remaining coefficients a_j, see [7]).

3 Analytic continuation of the Mellin transform

Remarks on the general problem of analytic continuation of the modified Mellin transform $\mathcal{Z}_k(s)$ were given in [11] and [17]. We start here by proving a general result, which links the problem to the moments of $|\zeta(\frac{1}{2}+it)|$. This is a new result, which we state as

THEOREM 2. *Let $k \in \mathbb{N}$ be fixed. The bound*

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k}\,\mathrm{d}t \ll_\varepsilon T^{c+\varepsilon} \tag{3.1}$$

holds for some constant c, if and only if $\mathcal{Z}_k(s)$ is regular for $\Re\mathrm{e}\, s > c$, and for any given $\varepsilon > 0$

$$\mathcal{Z}_k(c+\varepsilon+it) \ll_\varepsilon 1. \tag{3.2}$$

Proof of Theorem 2. The constant c must satisfy $c \ge 1$ in view of the known lower bounds for moments of $|\zeta(\frac{1}{2}+it)|$ (see e.g., [5, Chapter 9]). Suppose that (3.1) holds. Then we have

$$\int_X^{2X} |\zeta(\tfrac{1}{2}+ix)|^{2k} x^{-s}\,\mathrm{d}x \ll X^{-\sigma}\int_0^{2X} |\zeta(\tfrac{1}{2}+ix)|^{2k}\,\mathrm{d}x \ll_\varepsilon X^{c-\sigma+\varepsilon/2},$$

where $\sigma = \Re\mathrm{e}\, s$. Therefore

$$\int_1^\infty |\zeta(\tfrac{1}{2}+ix)|^{2k} x^{-s}\,\mathrm{d}x = \sum_{j=0}^\infty \int_{2^j}^{2^{j+1}} |\zeta(\tfrac{1}{2}+ix)|^{2k} x^{-s}\,\mathrm{d}x \ll \sum_{j=0}^\infty 2^{j(c-\sigma+\varepsilon/2)} \ll_\varepsilon 1$$

if $\sigma = c+\varepsilon$, since $\sum_j 2^{-\varepsilon j/2}$ converges. This shows that $\mathcal{Z}_k(s)$ is regular for $\sigma > c$ and that (3.2) holds.

Conversely, suppose that $\mathcal{Z}_k(s)$ is regular for $\sigma > c$ and that (3.2) holds. Using the classical integral

$$\mathrm{e}^{-x} = \frac{1}{2\pi i}\int_{(d)} x^{-s}\Gamma(s)\,\mathrm{d}s \qquad (\Re\mathrm{e}\, x > 0,\ d > 0),$$

we have

$$\begin{aligned}\int_1^\infty \mathrm{e}^{-x/T}|\zeta(\tfrac{1}{2}+ix)|^{2k}\,\mathrm{d}x &= \int_1^\infty \frac{1}{2\pi i}\int_{(c+\varepsilon)} \Gamma(s)(\frac{x}{T})^{-s}\,\mathrm{d}s|\zeta(\tfrac{1}{2}+ix)|^{2k}\,\mathrm{d}x \\ &= \frac{1}{2\pi i}\int_{(c+\varepsilon)} \Gamma(s)T^s\mathcal{Z}_k(s)\,\mathrm{d}s \ll_\varepsilon T^{c+\varepsilon},\end{aligned}$$

by absolute convergence and the fast decay of the gamma-function. This yields

$$\begin{aligned}\int_0^T |\zeta(\tfrac{1}{2}+ix)|^{2k}\,\mathrm{d}x &\le O(1) + \mathrm{e}\int_1^T \mathrm{e}^{-x/T}|\zeta(\tfrac{1}{2}+ix)|^{2k}\,\mathrm{d}x \\ &\ll 1 + \int_1^\infty \mathrm{e}^{-x/T}|\zeta(\tfrac{1}{2}+ix)|^{2k}\,\mathrm{d}x \ll_\varepsilon T^{c+\varepsilon},\end{aligned}$$

which proves (3.1).

Corollary 1. The Lindelöf hypothesis ($|\zeta(\frac{1}{2}+it)| \ll_\varepsilon |t|^\varepsilon$) is equivalent to the statement that, for every $k \in \mathbb{N}$, $\mathcal{Z}_k(s)$ is regular for $\sigma > 1$ and satisfies $\mathcal{Z}_k(1+\varepsilon+it) \ll_{k,\varepsilon} 1$.

Indeed, the Lindelöf hypothesis is equivalent (see e.g., [6, Section 1.9]) to (3.1) with $c = 1$ for every $k \in \mathbb{N}$. Therefore the assertion follows from Theorem 1.

Corollary 2. If we define

$$\sigma_k := \inf \left\{ d_k \ : \ \int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k}\,\mathrm{d}t \ll_k T^{d_k} \right\}, \tag{3.3}$$

$$\rho_k := \inf \{ r_k \ : \ \mathcal{Z}_k(s) \text{ is regular for } \Re\mathrm{e}\, s > r_k \}, \tag{3.4}$$

then

$$\rho_k \ = \ \sigma_k, \quad \sigma_k \geq \ 1. \tag{3.5}$$

Note that from the bounds on $I_k(T)$ in [5, Chapter 8] we obtain

$$\sigma_1 = \sigma_2 = 1, \quad \sigma_k \leq \frac{k+2}{4} \quad (3 \leq k \leq 6), \tag{3.6}$$

and upper bounds for σ_k when $k > 7$ may be obtained by using results on the corresponding higher power moments of $|\zeta(\frac{1}{2}+it)|$ (op. cit.). The Lindelöf hypothesis may be reformulated as $\sigma_k = 1$ ($\forall k \geq 1$).

Thus at present we have two situations regarding analytic continuation of $\mathcal{Z}_k(s)$:

a) For $k = 1, 2$, one can obtain analytic continuation of $\mathcal{Z}_k(s)$ to the left of $\Re\mathrm{e}\, s = 1$ (in fact to $\mathbb{C}$). This will be discussed in Section 5 and Section 6, respectively.

b) For $k > 2$ only upper bounds for σ_k (cf. (3.6)) are known. A challenging problem is to improve these bounds, which would entail progress on bounds of power moments of $|\zeta(\frac{1}{2}+it)|$, one of the central topics in the theory of $\zeta(s)$.

In what concerns power moments of $|\zeta(\frac{1}{2}+it)|$ one expects to have a formula analogous to (2.11). Namely for any fixed $k \in \mathbb{N}$, we expect

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k}\,\mathrm{d}t = TP_{k^2}(\log T) + E_k(T) \tag{3.7}$$

to hold, where it is generally assumed that

$$P_{k^2}(y) = \sum_{j=0}^{k^2} a_{j,k} y^j \tag{3.8}$$

is a polynomial in y of degree k^2 (the integral in (3.7) is $\gg_k T \log^{k^2} T$; see e.g., [5, Chapter 9]). The function $E_k(T)$ is to be considered as the error term in (2.7), namely one supposes that

$$E_k(T) \ = \ o(T) \qquad (T \to \infty). \tag{3.9}$$

So far (3.7)–(3.9) are known to hold only for $k = 1$ and $k = 2$ (see [6] and [28] for a comprehensive account). Therefore in view of the existing knowledge on the higher moments of $|\zeta(\frac{1}{2}+it)|$, embodied in (3.6), at present the really important cases of (3.7) are $k = 1$ and $k = 2$. Plausible heuristic arguments for the values of the coefficients $a_{j,k}$ were recently given by Conrey et al. [3], by using methods from Random Matrix Theory (see also Keating–Snaith [23]).

In case (3.7)–(3.9) hold, this may be used to obtain the analytic continuation of $\mathcal{Z}_k(s)$ to the region $\sigma \ge 1$ (at least). Indeed, by using (3.7)-(3.9) we have

$$\mathcal{Z}_k(s) = \int_1^\infty |\zeta(\tfrac{1}{2}+ix)|^{2k} x^{-s}\,\mathrm{d}x = \int_1^\infty x^{-s}\,\mathrm{d}\left(xP_{k^2}(\log x) + E_k(x)\right)$$

$$= \int_1^\infty (P_{k^2}(\log x) + P'_{k^2}(\log x))x^{-s}\,\mathrm{d}x - E_k(1) + s\int_1^\infty E_k(x)x^{-s-1}\,\mathrm{d}x. \tag{3.10}$$

But for $\Re\mathrm{e}\, s > 1$ change of variable $\log x = t$ gives

$$\int_1^\infty (P_{k^2}(\log x) + P'_{k^2}(\log x))x^{-s}\,\mathrm{d}x = \int_1^\infty \left\{\sum_{j=0}^{k^2} a_{j,k}\log^j x + \sum_{j=0}^{k^2-1}(j+1)a_{j+1,k}\log^j x\right\} x^{-s}\,\mathrm{d}x$$

$$= \int_0^\infty \left\{\sum_{j=0}^{k^2} a_{j,k}t^j + \sum_{j=0}^{k^2-1}(j+1)a_{j+1,k}t^j\right\}\mathrm{e}^{-(s-1)t}\,\mathrm{d}t = \frac{a_{k^2,k}(k^2)!}{(s-1)^{k^2+1}} + \sum_{j=0}^{k^2-1}\frac{a_{j,k}j! + a_{j+1,k}(j+1)!}{(s-1)^{j+1}}. \tag{3.11}$$

Hence inserting (3.11) in (3.10) and using (3.9) we obtain the analytic continuation of $\mathcal{Z}_k(s)$ to the region $\sigma \ge 1$. As we know (e.g., see [6] and [28]) that

$$\int_1^T E_1^2(t)\,\mathrm{d}t \ll T^{3/2}, \qquad \int_1^T E_2^2(t)\,\mathrm{d}t \ll T^2\log^{22} T, \tag{3.12}$$

it follows on applying the Cauchy–Schwarz inequality to the last integral in (3.10) that (3.9)-(3.11) actually provides the analytic continuation of $\mathcal{Z}_1(s)$ to the region $\Re\mathrm{e}\, s > 1/4$, and of $\mathcal{Z}_2(s)$ to $\Re\mathrm{e}\, s > 1/2$.

4 Recurrence formulas and identities

There is a possibility to obtain analytic continuation of $\mathcal{Z}_k(s)$ by using a recurrent relation involving $\mathcal{Z}_r(s)$ with $r < k$, which was mentioned in [11] and [17]. This result of ours is

THEOREM 3. *For $k \ge 2$, $r = 1, \dots, k-1$, $\Re\mathrm{e}\, s$ and $c = c(k,r)$ sufficiently large, we have*

$$\mathcal{Z}_k(s) = \frac{1}{2\pi i}\int_{(c)} \mathcal{Z}_{k-r}(w)\mathcal{Z}_r(1+s-w)\,\mathrm{d}w. \tag{4.1}$$

Proof of Theorem 3. For $\Re\mathrm{e}\,(1-s)$ sufficiently large we have

$$\mathcal{Z}_k(1-s) = \int_0^\infty \zeta^*(x)x^{s-1}\,\mathrm{d}x,$$

where $\zeta^*(x) = |\zeta(\frac12 + ix)|^{2k}$ if $x \ge 1$ and zero otherwise. If $F(s)$ is the Mellin transform of $f(x)$ (see e.g., the Appendix of [5] for conditions under which this holds) then one has the Mellin inversion formula

$$\tfrac12(f(x+0)+f(x-0)) = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\sigma-iT}^{\sigma+iT} F(s)x^{-s}\,\mathrm{d}s. \tag{4.2}$$

The use of this relation gives

$$|\zeta(\tfrac12+ix)|^{2k} = \frac{1}{2\pi i}\int_{(c)} \mathcal{Z}_k(1-s)x^{-s}\,\mathrm{d}s, \qquad (c \le c_0(k) < 0,\ x \ge 1). \tag{4.3}$$

Therefore, for $k, r \in \mathbb{N}$, $k \ge 2$, $1 \le r < k$, by using (4.3) we obtain

$$\begin{aligned}\int_1^\infty |\zeta(\tfrac12+ix)|^{2k}x^{-s}\,\mathrm{d}x &= \int_1^\infty |\zeta(\tfrac12+ix)|^{2r}|\zeta(\tfrac12+ix)|^{2(k-r)}x^{-s}\,\mathrm{d}x \\ &= \int_1^\infty |\zeta(\tfrac12+ix)|^{2r}\left(\frac{1}{2\pi i}\int_{(c)}\mathcal{Z}_{k-r}(1-w)x^{-w}\,\mathrm{d}w\right)x^{-s}\,\mathrm{d}x \\ &= \frac{1}{2\pi i}\int_{(c)}\mathcal{Z}_r(w+s)\mathcal{Z}_{k-r}(1-w)\,\mathrm{d}w \quad (\sigma \ge \sigma_0(k) > 1-c).\end{aligned}$$

Changing $1-w$ to w we obtain (4.1).

In particular, by using (3.6), we obtain the identities

$$\begin{aligned}\mathcal{Z}_3(s) &= \frac{1}{2\pi i}\int_{(1+\varepsilon)}\mathcal{Z}_1(w)\mathcal{Z}_2(1-w+s)\,\mathrm{d}w \qquad (\sigma > \tfrac54),\\ \mathcal{Z}_4(s) &= \frac{1}{2\pi i}\int_{(\frac54+\varepsilon)}\mathcal{Z}_2(w)\mathcal{Z}_2(1-w+s)\,\mathrm{d}w \qquad (\sigma > \tfrac32).\end{aligned}$$

The following result provides an integral representation for $\mathcal{Z}_k^2(s)$. This is

THEOREM 4. *In the region of absolute convergence we have*

$$\mathcal{Z}_k^2(s) = 2\int_1^\infty x^{-s}\left(\int_{\sqrt{x}}^x |\zeta(\tfrac12+iu)|^{2k}\Big|\zeta\Big(\tfrac12+i\,\frac{x}{u}\Big)\Big|^{2k}\frac{\mathrm{d}u}{u}\right)\mathrm{d}x. \tag{4.4}$$

Proof of Theorem 4. Clearly the region of validity of (4.4) depends on k, and by using e.g. (3.6) one can provide explicit $\bar\sigma_k$ such that (4.4) holds for $\sigma > \bar\sigma_k$. To prove the assertion, we set $f(x) = |\zeta(\frac{1}{2}+ix)|^{2k}$ and make the change of variables $xy = X$, $x/y = Y$, so that the absolute value of the Jacobian of the transformation is equal to $1/(2Y)$. Therefore

$$\mathcal{Z}_k^2(s) = \int_1^\infty \int_1^\infty (xy)^{-s} f(x)f(y)\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{2}\int_1^\infty X^{-s}\int_{1/X}^{X} \frac{1}{Y} f(\sqrt{XY}\,)f(\sqrt{X/Y}\,)\,\mathrm{d}Y\,\mathrm{d}X.$$

But as we have $(y = 1/u)$

$$\int_{1/x}^{x} f(\sqrt{xy}\,)f(\sqrt{x/y}\,)\frac{\mathrm{d}y}{y} = \int_1^x f(\sqrt{x/u})f(\sqrt{xu}\,)\frac{\mathrm{d}u}{u},$$

we obtain that, in the region of absolute convergence, the identity

$$\mathcal{Z}_k^2(s) = \tfrac{1}{2}\int_1^\infty x^{-s}\left(\int_1^x f(\sqrt{xy}\,)f(\sqrt{x/y}\,)\frac{\mathrm{d}y}{y}\right)\mathrm{d}x$$

is valid. The inner integral here becomes, after the change of variable $\sqrt{xy} = u$,

$$2\int_{\sqrt{x}}^{x} f(u)f(\frac{x}{u})\frac{\mathrm{d}u}{u},$$

and (4.4) follows. The argument also shows that, for $0 < a < b$ and any integrable function f on $[a,\,b]$,

$$\left(\int_a^b f(x)x^{-s}\,\mathrm{d}x\right)^2 = 2\int_{a^2}^{b^2} x^{-s}\left\{\int_{\sqrt{x}}^{\min(x/a,b)} f(u)f(\frac{x}{u})\frac{\mathrm{d}u}{u}\right\}\mathrm{d}x.$$

5 The modified Mellin transform, $k = 1$

We begin our discussion of the function $\mathcal{Z}_1(s)$ by obtaining its analytic continuation over $\mathbb{C}$. The relevant result is contained in

THEOREM 4. *The function $\mathcal{Z}_1(s)$ continues meromorphically to $\mathbb{C}$, having only a double pole at $s = 1$, and at most simple poles at $s = -1, -3, \ldots$. The principal part of its Laurent expansion at $s = 1$ is given by*

$$\frac{1}{(s-1)^2} + \frac{2\gamma - \log(2\pi)}{s-1}, \tag{5.1}$$

where γ is Euler's constant.

Proof of Theorem 4. It was shown in [17] that $\mathcal{Z}_1(s)$ continues analytically to a function that is regular for $\sigma > -3/4$. In [22] M. Jutila proved that $\mathcal{Z}_1(s)$ continues meromorphically to $\mathbb{C}$, having only a double pole at $s = 1$ and at most double poles for $s = -1, -2, \ldots$. The present form of Theorem 4 was obtained by the author in [13], and a different proof is to be found in the dissertation of M. Lukkarinen [25]. As our proof of Theorem 4 is fairly simple, it will be given now. Let

$$\bar{L}_1(s) := \int_1^\infty |\zeta(\tfrac12 + iy)|^2 \mathrm{e}^{-ys}\,\mathrm{d}y \quad (\Re\mathrm{e}\, s > 0). \tag{5.2}$$

Then we have by absolute convergence, taking $\sigma = \Re\mathrm{e}\, s$ sufficiently large and making the change of variable $xy = t$,

$$\begin{aligned}\int_0^\infty \bar{L}_1(x)x^{s-1}\,\mathrm{d}x &= \int_0^\infty \Big(\int_1^\infty |\zeta(\tfrac12 + iy)|^2 \mathrm{e}^{-yx}\,\mathrm{d}y\Big)\, x^{s-1}\,\mathrm{d}x\\ &= \int_1^\infty |\zeta(\tfrac12 + iy)|^2 \Big(\int_0^\infty x^{s-1}\mathrm{e}^{-xy}\,\mathrm{d}x\Big)\,\mathrm{d}y\end{aligned}$$

$$= \int_1^\infty |\zeta(\tfrac12 + iy)|^2 y^{-s}\,\mathrm{d}y \int_0^\infty \mathrm{e}^{-t}t^{s-1}\,\mathrm{d}t = \mathcal{Z}_1(s)\Gamma(s). \tag{5.3}$$

Further we have

$$\begin{aligned}\int_0^\infty \bar{L}_1(x)x^{s-1}\,\mathrm{d}x &= \int_0^1 \bar{L}_1(x)x^{s-1}\,\mathrm{d}x + \int_1^\infty \bar{L}_1(x)x^{s-1}\,\mathrm{d}x\\ &= \int_1^\infty \bar{L}_1(1/x)x^{-1-s}\,\mathrm{d}x + A(s) \quad (\sigma > 1),\end{aligned}$$

say, where $A(s)$ is an entire function. Since (see (1.1))

$$\bar{L}_1(1/x) = L_1(1/x) - \int_0^1 |\zeta(\tfrac12 + iy)|^2 \mathrm{e}^{-y/x}\,\mathrm{d}y \qquad (x \ge 1),$$

it follows from (5.3) by analytic continuation that, for $\sigma > 1$,

$$\mathcal{Z}_1(s)\Gamma(s) = \int_1^\infty L_1(1/x)x^{-1-s}\,\mathrm{d}x - \int_1^\infty \Big(\int_0^1 |\zeta(\tfrac12 + iy)|^2 \mathrm{e}^{-y/x}\,\mathrm{d}y\Big) x^{-1-s}\,\mathrm{d}x + A(s)$$

$$= I_1(s) - I_2(s) + A(s), \tag{5.4}$$

say. Clearly, for any integer $M \ge 1$, we have

$$I_2(s) = \int_1^\infty \int_0^1 |\zeta(\tfrac12 + iy)|^2 \left(\sum_{m=0}^{M} \frac{(-1)^m}{m!}\left(\frac{y}{x}\right)^m + O_M(x^{-M-1})\right) \mathrm{d}y\, x^{-1-s}\,\mathrm{d}x$$

$$= \sum_{m=0}^{M} \frac{(-1)^m}{m!} h_m \cdot \frac{1}{m+s} + H_M(s), \tag{5.5}$$

say, where $H_M(s)$ is a regular function of s for $\sigma > -M - 1$, and h_m is a constant. Note that, for $\sigma = 1/T$ ($T \to \infty$) and any $N \ge 0$, (2.1) gives

$$L_1\left(\frac{1}{T}\right) = \left(\log\left(\frac{T}{2\pi}\right) + \gamma\right) \sum_{n=0}^{N} a_n T^{1-2n} + \sum_{n=0}^{N} b_n T^{-2n} + O_N(T^{-1-2N}\log T)$$

with suitable a_n, b_n $(a_0 = 1)$. Inserting this formula in $I_1(s)$ in (5.4) we have

$$I_1(s) = \int_1^\infty (\log \frac{x}{2\pi} + \gamma) \sum_{n=0}^{N} a_n x^{-2n-s}\,\mathrm{d}x + \int_1^\infty \sum_{n=0}^{N} b_n x^{-1-2n-s}\,\mathrm{d}x + K_N(s)$$
$$= \sum_{n=0}^{N} a_n \left(\frac{1}{(2n+s-1)^2} + \frac{\gamma - \log 2\pi}{2n+s-1} \right) + K_N(s) \quad (\sigma > 1), \tag{5.6}$$

say, where $K_N(s)$ is regular for $\sigma > -2N$. Taking $M = 2N$ it follows from (5.4)–(5.6) that

$$\mathcal{Z}_1(s)\Gamma(s) = \sum_{n=0}^{N} a_n \left(\frac{1}{(2n+s-1)^2} + \frac{\gamma - \log 2\pi}{2n+s-1} \right) + \sum_{m=0}^{2N} \frac{(-1)^m}{m!} h_m \cdot \frac{1}{m+s} + R_N(s), \tag{5.7}$$

say, where $R_N(s)$ is a regular function of s for $\sigma > -2N$. This holds initially for $\sigma > 1$, but by analytic continuation it holds for $\sigma > -2N$. Since N is arbitrary and $\Gamma(s)$ has no zeros, it follows that (5.7) provides meromorphic continuation of $\mathcal{Z}_1(s)$ to $\mathbb{C}$. Taking into account that $\Gamma(s)$ has simple poles at $s = -m$ $(m = 0, 1, 2, \dots)$ we obtain then the analytic continuation of $\mathcal{Z}_1(s)$ to $\mathbb{C}$, showing that besides $s = 1$ the only poles of $\mathcal{Z}_1(s)$ can be simple poles at $s = 1 - 2n$ for $n \in \mathbb{N}$, as asserted by Theorem 4. With more care the residues at these poles could be explicitly evaluated. Finally using (5.7) and

$$\frac{1}{\Gamma(s)} = 1 + \gamma(s-1) + \sum_{n=2}^{\infty} d_n (s-1)^n$$

we obtain that the principal part of the Laurent expansion at $s = 1$ is given by (5.1).

Another major problem is to determine the order of growth of $\mathcal{Z}_1(\sigma + it)$. Concerning pointwise bounds of $\mathcal{Z}_1(s)$, we have (see M. Jutila [22])

$$\mathcal{Z}_1(\sigma + it) \ll_\varepsilon t^{\frac{5}{6}-\sigma+\varepsilon} \qquad (\tfrac{1}{2} \le \sigma \le 1,\ t \ge t_0). \tag{5.9}$$

We also have the mean square bounds (see [17] for proof)

$$\int_1^T |\mathcal{Z}_1(\sigma + it)|^2\,\mathrm{d}t \ll_\varepsilon T^{3-4\sigma+\varepsilon} \qquad (0 \le \sigma \le \tfrac{1}{2}), \tag{5.10}$$

and

$$\int_1^T |\mathcal{Z}_1(\sigma + it)|^2\,\mathrm{d}t \ll_\varepsilon T^{2-2\sigma+\varepsilon} \qquad (\tfrac{1}{2} \le \sigma \le 1). \tag{5.11}$$

The bound in (5.11) is essentially best possible since, for any given $\varepsilon > 0$,

$$\int_1^T |\mathcal{Z}_k(\sigma + it)|^2\,\mathrm{d}t \gg_\varepsilon T^{2-2\sigma-\varepsilon} \qquad (k = 1, 2;\ \tfrac{1}{2} < \sigma < 1). \tag{5.12}$$

This assertion follows from

$$\int_T^{2T} |\zeta(\tfrac{1}{2} + it)|^{2k}\,\mathrm{d}t \ll_\varepsilon T^{2\sigma-1} \int_0^{T^{1+\varepsilon}} |\mathcal{Z}_k(\sigma + it)|^2\,\mathrm{d}t \quad (k = 1, 2;\ \tfrac{1}{2} < \sigma < 1) \tag{5.13}$$

and lower bounds for the integral on the left-hand side (see [5, Chapter 9]). The proof of (5.13) when $k = 2$ appeared in [11], and the proof of the bound when $k = 1$ is on similar lines. It is plausible to conjecture that (5.9) holds with the exponent $1/2$ instead of $5/6$ on the right-hand side.

6 The modified Mellin transform, $k = 2$

The function $\mathcal{Z}_2(s)$ has quite a different analytic behaviour from the function $\mathcal{Z}_1(s)$. It was introduced and studied by Y. Motohashi [26] (see also his monograph [28]). He has shown that $\mathcal{Z}_2(s)$ has meromorphic continuation over $\mathbb{C}$. In the half-plane $\Re\mathrm{e}\, s > 0$ it has the following singularities: the pole $s = 1$ of order five, simple poles at $s = \frac{1}{2} \pm i\kappa_j$ ($\kappa_j = \sqrt{\lambda_j - \frac{1}{4}}$) and poles at $s = \frac{1}{2}\rho$, where ρ denotes complex zeros of $\zeta(s)$. The residue of $\mathcal{Z}_2(s)$ at $s = \frac{1}{2} + i\kappa_h$ equals (see Section 2 for definitions concerning spectral theory)

$$R(\kappa_h) := \sqrt{\frac{\pi}{2}}\Big(2^{-i\kappa_h}\frac{\Gamma(\frac{1}{4} - \frac{1}{2}i\kappa_h)}{\Gamma(\frac{1}{4} + \frac{1}{2}i\kappa_h)}\Big)^3 \Gamma(2i\kappa_h)\cosh(\pi\kappa_h) \sum_{\kappa_j = \kappa_h} \alpha_j H_j^3(\tfrac{1}{2}),$$

and the residue at $s = \frac{1}{2} - i\kappa_h$ equals $\overline{R(\kappa_h)}$. The principal part of $\mathcal{Z}_2(s)$ has the form (this may be compared with (5.1))

$$\sum_{j=1}^{5} \frac{A_j}{(s-1)^j}, \tag{6.1}$$

where $A_5 = 12/\pi^2$, and the remaining A_j's can be evaluated explicitly by following the analysis in [26]. The function $\mathcal{Z}_2(s)$ was used to furnish several strong results on $E_2(T)$ (see (2.7)), the error term in the asymptotic formula for the fourth moment of $|\zeta(\frac{1}{2} + it)|$. Y. Motohashi [26] used it to show that $E_2(T) = \Omega_\pm(T^{1/2})$, which sharpens the earlier result of Ivić-Motohashi (see [18]) that $E_2(T) = \Omega(T^{1/2})$. The same authors (see e.g., [19], [20] and [28]) have proved that

$$E_2(T) \ll T^{2/3}\log^{C_1} T\ (C_1 > 0), \quad \int_0^T E_2(t)\,\mathrm{d}t \ll T^{3/2}, \tag{6.2}$$

as well as the second bound in (2.12). In [8] and [9] the author has applied the theory of $\mathcal{Z}_2(s)$ to obtain the following quantitative omega-results: There exist constants $A, B > 0$ such that for $T \ge T_0 > 0$ every interval $[T,\ AT]$ contains points t_1, t_2, t_3, t_4 such that

$$E_2(t_1) > Bt_1^{1/2},\ E_2(t_2) < -Bt_2^{1/2},\ \int_0^{t_3} E_2(t)\,\mathrm{d}t > Bt_3^{3/2},\ \int_0^{t_4} E_2(t)\,\mathrm{d}t < -Bt_4^{3/2}.$$

Moreover, we have (see [9])

$$\int_0^T E_2^2(t)\,\mathrm{d}t \gg T^2, \tag{6.3}$$

which complements the upper bound in (6.2).

As for the estimation of $\mathcal{Z}_2(s)$, we have (see the author's work [12])

$$\mathcal{Z}_2(\sigma + it) \ll_\varepsilon t^{\frac{4}{3}(1-\sigma)+\varepsilon} \qquad (\tfrac{1}{2} < \sigma \le 1;\ t \ge t_0 > 0), \tag{6.4}$$

and (similarly to (5.9)) I have conjectured in [12] that the exponent on the right-hand side of (6.4) can be replaced by $1/2 - \sigma$. This is very strong, as it implies (6.12) with $\theta = 1$ and $E_2(T) \ll_\varepsilon T^{1/2+\varepsilon}$, and both bounds (up to "ε") are best possible. It was proved in [17] that

$$\int_0^T |\mathcal{Z}_2(\sigma + it)|^2 \,\mathrm{d}t \ll_\varepsilon T^\varepsilon \left(T + T^{\frac{2-2\sigma}{1-c}}\right) \qquad (\tfrac12 < \sigma \le 1), \tag{6.5}$$

and we also have unconditionally

$$\int_0^T |\mathcal{Z}_2(\sigma + it)|^2 \,\mathrm{d}t \ \ll \ T^{\frac{10-8\sigma}{3}} \log^C T \qquad (\tfrac12 < \sigma \le 1,\ C > 0). \tag{6.6}$$

The constant c appearing in (6.5) is defined by $E_2(T) \ll_\varepsilon T^{c+\varepsilon}$, so that by (6.2) and (6.3) we have $\frac12 \le c \le \frac23$. In (6.4)–(6.6) σ is assumed to be fixed, as $s = \sigma + it$ has to stay away from the line $\Re\mathrm{e}\, s = \frac12$ where $\mathcal{Z}_2(s)$ has poles. Lastly, the author [14] proved that, for $\frac56 \le \sigma \le \frac54$ we have,

$$\int_1^T |\mathcal{Z}_2(\sigma + it)|^2 \,\mathrm{d}t \ll_\varepsilon T^{\frac{15-12\sigma}{5}+\varepsilon}. \tag{6.7}$$

The lower limit of integration in (6.7) is unity, because of the pole $s = 1$ of $\mathcal{Z}_2(s)$. By taking $c = 2/3$ in (6.5) and using the convexity of mean values (see [5, Lemma 8.3]) it follows that

$$\int_1^T |\mathcal{Z}_2(\sigma + it)|^2 \,\mathrm{d}t \ \ll_\varepsilon \ T^{\frac{7-6\sigma}{2}+\varepsilon} \qquad (\tfrac12 < \sigma \le \tfrac56). \tag{6.8}$$

Note that (6.7) and (6.8) combined provide the sharpest known bounds in the whole range $\frac12 < \sigma \le \frac56$.

Both pointwise and mean square estimates for $\mathcal{Z}_2(s)$ may be used to estimate $E_2(T)$ and the eighth moment of $|\zeta(\frac12 + it)|$. This connection is furnished by the following result, proved by the author in [12].

THEOREM 6. *Suppose that, for some $\rho \ge 0$ and $r \ge 0$,*

$$\mathcal{Z}_2(\sigma + it) \ \ll_\varepsilon \ |t|^{\rho+\varepsilon}, \quad \int_1^T |\mathcal{Z}_2(\sigma + it)|^2 \,\mathrm{d}t \ \ll_\varepsilon \ T^{1+2r+\varepsilon} \quad (\tfrac12 < \sigma \le 1), \tag{6.9}$$

where σ is fixed and $|t| \ge t_0 > 0$. Then we have

$$E_2(T) \ \ll_\varepsilon \ T^{\frac{2\rho+1}{2\rho+2}+\varepsilon}, \quad E_2(T) \ \ll_\varepsilon \ T^{\frac{2r+1}{2r+2}+\varepsilon} \tag{6.10}$$

and

$$\int_0^T |\zeta(\tfrac12 + it)|^8 \,\mathrm{d}t \ \ll_\varepsilon \ T^{\frac{4r+1}{2r+1}+\varepsilon}. \tag{6.11}$$

Note that the conditions $\rho \ge 0$, $r \ge 0$ must hold in view of (5.12). Also note that from (6.6) with $\sigma = \frac12 + \varepsilon$ one can take in (6.9) $r = \frac12$, hence (6.10) gives $E_2(T) \ \ll_\varepsilon \ T^{\frac23+\varepsilon}$, which is essentially the strongest known bound (see (6.2)). Thus any improvement of the existing mean square bound for $\mathcal{Z}_2(s)$ at $\sigma = \frac{1}{2} + \varepsilon$ would result in the bound for $E_2(T)$ with the exponent strictly less than 2/3, which would be important. Of course, if the first bound in (6.9) holds with some ρ, then trivially the second bound will hold with $r = \rho$, i.e. $r \le \rho$ has to hold. Observe that the known value $r = \frac{1}{2}$ and (6.11) yield

$$\int_0^T |\zeta(\tfrac{1}{2} + it)|^8 \,\mathrm{d}t \ \ll_\varepsilon\ T^{\theta+\varepsilon} \tag{6.12}$$

with $\theta = 3/2$, which is, up to"ε", currently the best known upper bound (see [5, Chapter 8]) for the eighth moment, and any value $r < \frac{1}{2}$ in (6.9) would reduce the exponent $\theta = 3/2$ in (6.12). The connections between upper bounds for the integral in (6.12) and mean square estimates involving $\mathcal{Z}_2(s)$ and related functions are also given as

THEOREM 7. *The eighth moment bound, namely* (6.12) *with* $\theta = 1$, *is equivalent to the mean square bound*

$$\int_1^T |\mathcal{Z}_2(1 + it)|^2 \,\mathrm{d}t \ \ll_\varepsilon\ T^\varepsilon, \tag{6.13}$$

and to

$$\int_T^{2T} I^2(t, G) \,\mathrm{d}t \ \ll_\varepsilon\ T^{1+\varepsilon} \qquad (T^\varepsilon \le G = G(T) \le T), \tag{6.14}$$

where

$$I(T, G) := \frac{1}{\sqrt{\pi}G} \int_{-\infty}^{\infty} |\zeta(\tfrac{1}{2} + iT + iu)|^4 \mathrm{e}^{-(u/G)^2} \,\mathrm{d}u. \tag{6.15}$$

Proof of Theorem 7. We suppose first that (6.13) holds. Then (6.12) with $\theta = 1$ follows from (5.13) with $k = 2$. Conversely, if (6.12) holds with $\theta = 1$, note that we have, by [11, Lemma 4],

$$\int_T^{2T} \left| \int_X^{2X} |\zeta(\tfrac{1}{2} + ix)|^4 x^{-s} \,\mathrm{d}x \right|^2 \,\mathrm{d}t \ll \int_X^{2X} |\zeta(\tfrac{1}{2} + ix)|^8 x^{1-2\sigma} \,\mathrm{d}x \ll_\varepsilon X^{2-2\sigma+\varepsilon} \tag{6.16}$$

for $s = \sigma + it$, $\frac{1}{2} < \sigma \le 1$. Similarly, using the Cauchy-Schwarz inequality for integrals and the second bound in (3.12), it follows that

$$\int_T^{2T} \left| \int_X^{2X} E_2(x) x^{-s} \,\mathrm{d}x \right|^2 \,\mathrm{d}t \ll_\varepsilon X^{1-2\sigma+\varepsilon} \qquad (s = \sigma + it,\ \sigma > \tfrac{1}{2}). \tag{6.17}$$

Combining (6.16) and (6.17) we obtain then, similarly to the proof of (5.7) in [17],

$$\int_1^T |\mathcal{Z}_2(\sigma + it)|^2 \,\mathrm{d}t \ \ll_\varepsilon\ T^{4-4\sigma+\varepsilon} \qquad (\tfrac{1}{2} < \sigma \le 1),$$

and for $\sigma = 1$ we have (6.13). From (6.7) it follows that the integral in (6.13) is unconditionally bounded by $T^{3/5+\varepsilon}$, and any improvement of the exponent 3/5 would also result in the improvement of the exponent $\theta = 3/2$ in (6.12).

Suppose again that (6.12) holds with $\theta = 1$. Then the left-hand side of (6.14) is, for $T^\varepsilon \le G = G(T) \le T$,

$$\begin{aligned}
&\int_T^{2T}\left(\frac{1}{\sqrt{\pi}G}\int_{-\infty}^{\infty}|\zeta(\tfrac12+it+iu)|^4 \mathrm{e}^{-(u/G)^2}\,\mathrm{d}u\right)^2 \mathrm{d}t\\
\ll\ & 1+G^{-2}\int_T^{2T}\left(\int_{-G\log T}^{G\log T}|\zeta(\tfrac12+it+iu)|^4 \mathrm{e}^{-(u/G)^2}\,\mathrm{d}u\right)^2 \mathrm{d}t\\
\ll\ & 1+G^{-1}\int_{-G\log T}^{G\log T}\left(\int_T^{2T}|\zeta(\tfrac12+it+iu)|^8\,\mathrm{d}t\right)\mathrm{d}u\\
\ll_\varepsilon\ & G^{-1}\int_{-G\log T}^{G\log T} T^{1+\varepsilon}\,\mathrm{d}u\\
\ll_\varepsilon\ & T^{1+\varepsilon},
\end{aligned}$$

as asserted. We remark that (6.14) is trivial when $T^{2/3} \le G \le T$ (see e.g., [6, Chapter 5]). Finally, if (6.14) holds, then we use [15, Theorem 4], which in particular says that, for fixed $m \in \mathbb{N}$,

$$\int_T^{2T} I^m(t,G)\,\mathrm{d}t \ll_\varepsilon T^{1+\varepsilon} \quad (T^{\alpha_m+\varepsilon} \le G = G(T) \le T,\ 0 \le \alpha_m < 1) \tag{6.18}$$

implies that

$$\int_0^T |\zeta(\tfrac12+it)|^{4m}\,\mathrm{d}t \ll_\varepsilon T^{1+(m-1)\alpha_m+\varepsilon}.$$

Using this result with $m = 2, \alpha_2 = 0$, we obtain at once (6.12) with $\theta = 1$. This completes the proof of Theorem 7. So far (6.18) is known to hold unconditionally with $\alpha_2 = \frac12$ (see [14]), which yields another proof of (6.12) with $\theta = 3/2$.

The significance of (6.14) is that for $I(T,G)$ in (6.15) an explicit formula of Y. Motohashi (see [6], [26] or [28]) exists. It involves quantities from spectral theory, and thus the eighth moment problem is directly connected to this theory via Theorem 7. Namely

$$I(T,G) = \frac{\pi}{\sqrt{2T}}\sum_{j=1}^{\infty}\alpha_j H_j^3(\tfrac12)\kappa_j^{-\frac12}\sin\left(\kappa_j\log\frac{\kappa_j}{4\mathrm{e}T}\right)\times\exp\left(-\frac14\Big(\frac{G\kappa_j}{T}\Big)^2\right)+O(\log^{3D+9}T),$$

if $T^{1/2}\log^{-D}T \le G \le T/\log T$ for an arbitrary, fixed constant $D > 0$. We have the following new result, proved by the author in [16]:

THEOREM 8. *Let, for $m \in \mathbb{N}$ and $1 \ll K < K' \le 2K \ll T, T \le t \le 2T$,*

$$S_m(K;K',t) = \sum_{K<\kappa_j\le K'<2K}\alpha_j H_j^m(\tfrac12)\cos\Big(\kappa_j\log(\frac{4\mathrm{e}t}{\kappa_j})\Big).$$

Then, for $m = 1,2,3$,

$$\int_T^{2T}\left(S_m(K;K',t)\right)^2\mathrm{d}t \ll_\varepsilon T^{1+\varepsilon}K^3.$$

Corollary. *We have*

$$\int_0^T E_2^2(t)\,\mathrm{d}t \ \ll_\varepsilon \ T^{2+\varepsilon}, \quad \int_0^T |\zeta(\tfrac{1}{2}+it)|^{12}\,\mathrm{d}t \ \ll_\varepsilon \ T^{2+\varepsilon}. \tag{6.19}$$

The first bound (see [9], [19]), up to "ε", is best possible, the second is a consequence of the first and is a well-known result of D.R. Heath-Brown [4]. We shall just give a sketch of the first bound in (6.19). From [6, Lemma 5.1] we have

$$E_2(2T) - E_2(T) \le S(2T + \Delta\log T, \Delta) - S(T - \Delta\log T, \Delta) + O(\Delta\log^5 T) + O(T^{1/2}\log^C T)$$

with $T^{1/2} \le \Delta \le T^{1-\varepsilon}$ and

$$S(T,\Delta) := \pi\sqrt{\tfrac{1}{2}T}\sum_{j=1}^\infty \alpha_j H_j^3(\tfrac{1}{2})\kappa_j^{-\frac{3}{2}}\cos\Big(\kappa_j\log\frac{\kappa_j}{4\mathrm{e}T}\Big)\exp\Big(-\tfrac{1}{4}(\frac{\Delta\kappa_j}{T})^2\Big).$$

We can truncate the above series at $T\Delta^{-1}\log T$ with a negligible error and remove, by partial summation, the monotonic coefficients $\kappa_j^{-3/2}$ and $\exp\Big(-\frac{1}{4}(\frac{\Delta\kappa_j}{T})^2\Big)$.Then we obtain the sum $S_m(K;K',t)$ with $m=3$ and t replaced by $2t+\Delta\log T$ or $t-\Delta\log T$, which does not cause any trouble. We have

$$\int_T^{2T}(E_2(2t)-E_2(t))^2\,\mathrm{d}t \le \int_{T/2}^{5T/2}\varphi(t)(E_2(2t)-E_2(t))^2\,\mathrm{d}t, \tag{6.20}$$

where $\varphi(t)\ (\ge 0)$ is a smooth function suported in $[T/2,\,5T/2]$ which equals unity in $[T,\,2T]$. Hence the integral on the left-hand side of (6.20) is essentially majorized by $\ll_\varepsilon T^\varepsilon$ integrals of the type

$$T\int_{T/2}^{5T/2}\varphi(t)(K^{-3/2}S_m(K;K',t))^2\,\mathrm{d}t \ll_\varepsilon T^{2+\varepsilon},$$

with $m=3$ and (6.19) follows from Theorem 8 on replacing t by $t2^{-j}$ in the integrand in (6.20), and summing up the corresponding bounds over $j=1,2,\ldots$.

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