

# IDENTITIES INVOLVING GENERALIZED FIBONACCI NUMBERS

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## I. INTRODUCTION

K. Subba Rao [4], and more recently V. C. Harris [1] have obtained some identities involving Fibonacci Numbers  $F_n$  defined by

$$F_1 = 1, \quad F_2 = 1, \quad F_n = F_{n-1} + F_{n-2} \quad n \geq 3 .$$

Our object in this paper is to obtain similar results for the generalized Fibonacci Numbers  $H_n$  as defined by A. F. Horadam [2],

$$H_1 = p, \quad H_2 = p + q$$

and

$$H_n = H_{n-1} + H_{n-2} \quad n \geq 3.$$

The numbers  $p$  and  $q$  are arbitrary. By solving the difference equation for  $H_n$  by the usual procedure it is easy to see that

$$H_n = \frac{1}{2\sqrt{5}} [la^n - mb^n] \quad [3]$$

where

$$l = 2(p - qb), \quad m = 2(p - qa)$$

and  $a$  and  $b$  are the roots of the quadratic equation  $x^2 - x - 1 = 0$ . We call

$$a = \frac{1 + \sqrt{5}}{2} ; \quad b = \frac{1 - \sqrt{5}}{2}$$

so that

$$a + b = 1, \quad ab = -1, \quad a - b = \sqrt{5} .$$

By making use of these results we get

$$1 + m = 2(2p - q), \quad 1 - m = 2q \sqrt{5} ,$$

$$\frac{1}{4} 4m = p^2 - pq - q^2 = e \text{ (say).}$$

It is also easy to see that  $H_n = pF_n + qF_{n-1}$  where  $F_n$  is the  $n^{\text{th}}$  Fibonacci number given by

$$\frac{a^n - b^n}{\sqrt{5}} .$$

SECTION 2

In this section we obtain certain identities for the generalized Fibonacci numbers. From result (9) of [2] we have the identity

$$H_{r-1}^2 + H_r^2 = (2p - q) H_{2r-1} - e F_{2r-1} .$$

In this relation putting  $r = 2, 3, \dots, n$  in succession, adding and simplifying, we arrive at the result

$$(1) \quad \sum_{r=1}^n H_r^2 = F_n [(p + 2q)H_n + eF_{n-1}] + pq [(-1)^n - 1] .$$

Consider now  $H_{2r-1} = pF_{2r-1} + qF_{2r-2}$  so that

$$\sum_{r=1}^n H_{2r-1} = p \sum_{r=1}^n F_{2r-1} + q \sum_{r=1}^n F_{2r-2} .$$

From the formula for  $F_n$  this sum reduces to

$$(2) \quad \sum_{r=1}^n H_{2r-1} = H_{2n} - H_2 + H_1$$

$$(3) \quad \sum_{r=1}^n H_{2r} = H_{2n+1} - H_1$$

On the same lines we get the following identities

$$(4) \quad \sum_{r=1}^n H_{3r-2} = \frac{1}{2} [H_{3n} - H_2 + H_1]$$

$$(5) \quad \sum_{r=1}^n H_{3r-1} = \frac{1}{2} [H_{3n+1} - H_1]$$

$$(6) \quad \sum_{r=1}^n H_{3r} = \frac{1}{2} [H_{3n+2} - H_2]$$

$$(7) \quad \sum_{r=1}^n H_{4r-3} = F_{2n-1} H_{2n} - H_2 + H_1$$

$$(8) \quad \sum_{r=1}^n H_{4r-2} = F_{2n} H_{2n}$$

$$(9) \quad \sum_{r=1}^n H_{4r-1} = F_{2n} H_{2n+1}$$

$$(10) \quad \sum_{r=1}^n H_{4r} = F_{2n+1} H_{2n+1} - H_1$$

$$(11) \quad \sum_{r=1}^n H_{2r-1}^2 = \frac{1}{5} [H_{2n} (H_{2n-1} + H_{2n+1}) + 2ne + q(q - 2p)]$$

$$(12) \quad \sum_{r=1}^n H_{2r}^2 = \frac{1}{5} [H_{2n+1} (H_{2n} + H_{2n+2}) - 2ne - p(p + 2q)]$$

Let us now consider product terms as follows:

$$(13) \quad \sum_{r=1}^n H_{2r-2} H_{2r-1} = \frac{1}{5} [H_{2n-1}^2 + H_{2n}^2 - ne - (p + q)(p + 2q)]$$

$$(14) \quad \sum_{r=1}^n H_{2r-1} H_{2r} = \frac{1}{5} [H_{2n}^2 + H_{2n+1}^2 + ne - (p^2 + q^2)]$$

$$(15) \quad \sum_{r=1}^n H_{2r-1} H_{2r+1} = \frac{1}{5} [H_{2n+1} (H_{2n} + H_{2n+2}) + 3ne - p(p + 2q)]$$

$$(16) \quad \sum_{r=1}^n H_{2r} H_{2r+2} = \frac{1}{5} [H_{2n+2} (H_{2n+1} + H_{2n+3}) - 3ne - (p + q)(3p + q)]$$

Corresponding to the identity

$$F_r^2 - F_{r-k} F_{r+k} = (-1)^{r-k} F_k^2$$

for the generalized Fibonacci numbers we get in the generalized Fibonacci numbers the identity

$$(17) \quad H_r^2 - H_{r-k}H_{r+k} = (-1)^{r-k}eF_k^2$$

Now consider the sums

$$(18) \quad \sum_{r=1}^n H_{2r-2}H_{2r+2} = \frac{1}{5} [H_{2n+1}(H_{2n} + H_{2n+2}) - 7ne - (p^2 + 2pq + 10q^2)]$$

$$(19) \quad \sum_{r=1}^n H_{2r-1}H_{2r+3} = \frac{1}{5} [H_{2n+2}(H_{2n+1} + H_{2n+3}) + 7ne - (p+q)(3p+q)]$$

Evaluating the quantity  $H_k H_{k+1} H_{k+2}$  we get

$$(20) \quad H_k H_{k+1} H_{k+2} = H_{k+1}^3 + (-1)^{k-1} eH_{k+1}$$

Therefore

$$H_{2r-1} H_{2r} H_{2r+1} = H_{2r}^3 + eH_{2r}$$

Hence

$$\sum_{r=1}^n H_{2r-1} H_{2r} H_{2r+1} = \sum_{r=1}^n H_{2r}^3 + e \sum_{r=1}^n H_{2r}$$

After simplification this becomes,

$$(21) \quad \sum_{r=1}^n H_{2r-1} H_{2r} H_{2r+1} = \frac{1}{4} [(H_{2n+1}^3 - H_1^3) + e(H_{2n+1} - H_1)]$$

$$(22) \quad \sum_{r=1}^n H_r^3 = \frac{1}{2} [(H_n H_{n+1}^2 - q^2 H_2) + e\{(p-2q)(-1)^n H_{n-1}\}]$$

Now

$$H_{2r}^3 = (pF_{2r} + qF_{2r-1})^3 .$$

On expanding the right side, taking the sum from  $r = 1$  to  $n$  and simplifying we get the relation

$$(23) \quad \sum_{r=1}^n H_{2r}^3 = \frac{1}{4} [(H_{2n+1}^3 - H_1^3) - 3e(H_{2n+1} - H_1)]$$

$$(24) \quad \sum_{r=1}^n H_{2r}^2 H_{2r-1} = \frac{1}{4} [(H_{2n} H_{2n+1}^2 - q^2 H_2) + e(H_{2n-1} - H_1)]$$

$$(25) \quad \sum_{r=1}^n H_{2r} H_{2r-1}^2 = \frac{1}{4} [(H_{2n-1} H_{2n+1}^2 - H_1 q^2) + e(H_{2n} - H_2)]$$

$$(26) \quad \sum_{r=1}^n H_{2r-1}^2 = \frac{1}{4} [(H_{2n}^3 - q^3) + 3e(H_{2n} - q)]$$

From the formula for  $H_r$  we can find the sums of the following:

$$(27) \quad \sum_{r=0}^n rH_r = nH_{n+2} - H_{n+3} + H_3$$

$$(28) \quad \sum_{r=0}^n (-1)^r rH_r = [(-1)^n [(n+1)H_{n-1} - H_{n-2}] + (3q - 2p)]$$

$$(29) \quad \sum_{r=0}^n (-1)^r H_{2r} = \frac{1}{5} [(-1)^{n+1} (H_{2n} + H_{2n+2}) - (p + 2q)]$$

$$(30) \quad \sum_{r=0}^n (-1)^r H_{2r+1} = \frac{1}{5} \left[ (-1)^n (H_{2n+1} + H_{2n+3}) + (2p - q) \right]$$

$$(31) \quad \sum_{r=0}^n r H_{2r} = \left[ n H_{2n+1} - H_1 \right] - \left[ H_{2n} - H_2 \right]$$

$$(32) \quad \sum_{r=0}^n r H_{2r+1} = n H_{2n+2} - \left[ H_{2n+1} - H_1 \right]$$

$$(33) \quad \sum_{r=0}^n (-1)^r r H_{2r} = \frac{1}{5} \left[ (-1)^n ((n+1)H_{2n} + nH_{2n+2}) - (H_2 - H_1) \right]$$

$$(34) \quad \sum_{r=0}^n (-1)^r r H_{2r+1} = \frac{1}{5} \left[ (-1)^n ((n+1)H_{2n+1} + nH_{2n+3}) - H_1 \right]$$

It is easy to see that the list of identities given by K. Subba Rao can be extended to Fibonacci Quaternions defined by

$$Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3}.$$

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