Int. Journal of Math. Analysis, Vol. 7, 2013, no. 12, 551 - 556

On Some Identities for k-Jacobsthal Numbers

Deepika Jhala, Kiran Sisodiya and G. P. S. Rathore [1]

School of Studies in Mathematics
Vikram University, Ujjain (India)
jhala.deepika28@gmail.com
[1]College of Horticulture, Mandsaur (M.P.)

Abstract

The aim of this paper is to obtain Binet formula for k-Jacobsthal numbers. And also with the help of Binet formula we obtain some properties for the k-Jacobsthal numbers.

Mathematics Subject Classification: 11B37, 05A15, 11B83

Keywords: Jacobsthal numbers, *k*-Jacobsthal numbers, Generating functions, Binet formula

1. Introduction

In recent year, Fibonacci numbers and their generalization have many interesting properties and application to almost every field of science and art. Koshy **[12]** has devoted nearly 700 pages to the properties of Fibonacci and Lucas number, with scarcely a mention of general two term recurrences. For further more links can be seen in **[13], [8], [11].**
In **[9]** Falcon and Plaza found general k-Fibonacci numbers and obtained many properties of these numbers directly from elementary matrix algebra. Also, In **[10]** Falcon and Plaza defined k-hyperbolic function. In **[5]** Bolat and Kőse obtain identities including generating function and divisibility properties for k- Fibonacci number. In **[6]** Koken and Bozkurt deduce some properties and Binet like formula for the Jacobsthal number by matrix method. In this paper, we present the k-Jacobsthal number in an explicit way, and many properties are proved by easy arguments for the k-Jacobsthal number.

2. The k-Jacobsthal Number and Properties

For any positive real number k, the k-Jacobsthal sequence say $\{J_{k,n}\}_{n\in N}$ is defined recurrently by

$$J_{k,n+1} = kJ_{k,n} + 2J_{k,n-1};\ for\ n \geq 1 \tag{2.1}$$

With initial condition $J_{k,0} = 0, \quad J_{k,1} = 1$ (2.2)

2.1 *Explicit formula for the general term of the k- Jacobsthal sequence*

Binet's formulas are well known in **[4,12].** In our case, Binet's formula allows us to express the k-Jacobsthal numbers in function of the roots r_1 and r_2 of the following characteristic equation, associated to the recurrence relation (2.1).

$$r^2 = kr + 2 \tag{2.3}$$

Proposition 2.1 ***(Binet's formula)***

The nth k-Jacobsthal number is given by

$$J_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2} \tag{2.4}$$

where r_1, r_2 *are the roots of the characteristic equation (2.3) and* $r_1 > r_2$

Proof: The roots of the characteristic equation (2.3) are $r_1 = \dfrac{k + \sqrt{k^2 + 8}}{2}$, $r_2 = \dfrac{k - \sqrt{k^2 + 8}}{2}$

Note that, since $k > 0$, then $r_2 < 0 < r_1$ and $|r_2| < |r_1|$

$r_1 + r_2 = k$, $r_1 r_2 = -2$ and $r_1 - r_2 = \sqrt{k^2 + 8}$

Therefore, the general term of the k-Jacobsthal sequence may be expressed in the form: $J_{k,n} = c_1 r_1^n + c_2 r_2^n$ for some coefficients c_1 and c_2. Giving to n the values n = 0 and n = 1 it is obtained $c_1 = \dfrac{1}{r_1 - r_2} = -c_2$, and therefore $J_{k,n} = \dfrac{r_1^n - r_2^n}{r_1 - r_2}$.

Proposition 2.2 ***(Catalan's identity)***

$$J_{k,n-r} J_{k,n+r} - J_{k,n}^2 = (-1)^{n+1-r} J_{k,r}^2 \, 2^{n-r} \tag{2.5}$$

Proof: By using Eq. (2.4) in the left hand side (LHS) of Eq. (2.5), and taking into account that $r_1 r_2 = -1$ it is obtained

$$\begin{aligned} J_{k,n-r} J_{k,n+r} - J_{k,n}^2 &= \frac{r_1^{n-r} - r_2^{n-r}}{r_1 - r_2} \frac{r_1^{n+r} - r_2^{n+r}}{r_1 - r_2} - \left(\frac{r_1^n - r_2^n}{r_1 - r_2} \right)^2 \\ &= \frac{(-1)^{n+1} (2)^n}{(r_1 - r_2)^2} \left(\frac{r_2^{2r} + r_1^{2r}}{(r_1 r_2)^r} - 2 \right) \\ &= (-1)^{n+1-r} (2)^{n-r} J_{k,r}^2 \end{aligned}$$

Note that for r = 1, Eq. (2.5) gives Cassini's identity for the k-Jacobsthal sequence

$$J_{k,n-1}J_{k,n+1}-J_{k,n}^2= (-1)^n (2)^{n-r} \tag{2.6}$$

Proposition 2.3 ***(D'ocagne's identity)***

If m > n then $\quad J_{k,m}J_{k,n+1}-J_{k,m+1}J_{k,n} =(-2)^n J_{k,m-n}$ (2.7)

Proof: By using Eq. (2.4)

$$\begin{aligned}J_{k,m}J_{k,n+1}-J_{k,m+1}J_{k,n} &= \frac{r_1^m-r_2^m}{r_1-r_2}\frac{r_1^{n+1}-r_2^{n+1}}{r_1-r_2}-\frac{r_1^{m+1}-r_2^{m+1}}{r_1-r_2}\frac{r_1^n-r_2^n}{r_1-r_2}\\ &= (r_1r_2)^n\left(\frac{r_1^{m-n}-r_2^{m-n}}{r_1-r_2}\right)\\ &= (-2)^n J_{k,m-n}\end{aligned}$$

2.2 *Another explicit expression for calculating the general term of the k-Jacobsthal sequence is given by the following preposition-*

Proposition 2.4 $\quad J_{k,n}=\dfrac{1}{2^{n-1}}\displaystyle\sum_{i=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\binom{n}{2i+1} k^{n-1-2i}\left(k^2+8\right)$ (2.8)

where $\lfloor a \rfloor$ is the floor function of a, that is $\lfloor a \rfloor=\sup\{n\in N; n\le a\}$ and says the integer part of a, for $a\ge 0$.

Proof: By using the values of r_1 and r_2 obtained in Eq. (2.4), we get

$$J_{k,n}=\frac{r_1^n-r_2^n}{r_1-r_2} \quad =\frac{1}{\sqrt{k^2+8}}\left[\left(\frac{k+\sqrt{k^2+8}}{2}\right)^n-\left(\frac{k-\sqrt{k^2+8}}{2}\right)^n\right]$$

From where, by developing the nth powers, it follows:

$$=\frac{1}{\sqrt{k^2+8}}\left\{\frac{k^n}{2^{n-1}}\left[\binom{n}{1}\frac{\sqrt{k^2+8}}{k}+\binom{n}{2}\frac{\left(\sqrt{k^2+8}\right)^3}{k^3}+\ldots\right]\right\}$$

$$=\frac{1}{2^{n-1}}\sum_{i=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\binom{n}{2i+1} k^{n-1-2i}\left(k^2+8\right)^i$$

2.3 Limit *of the quotient of two consecutive terms is equal to the positive root of the corresponding characteristic equation*

Proposition 2.5

$$\lim_{n\to\omega}\frac{J_{k,n}}{J_{k,n-1}}=r_1 \tag{2.9}$$

Proof. By using Eq. (2.4)

$$\lim_{n\to\omega}\frac{J_{k,n}}{J_{k,n-1}}=\lim_{n\to\omega}\frac{r_1^n-r_2^n}{r_1^{n-1}-r_2^{n-1}}=\lim_{n\to\omega}\frac{1-\left({}^{r_2}\!/_{r_1}\right)^n}{{}^{1}\!/_{r_1}-\left({}^{r_2}\!/_{r_1}\right)^n\frac{1}{r_2}}$$

and taking into account that $\lim_{n\to\omega}\left(\frac{r_2}{r_1}\right)^n=0$

since $|r_2|<1$, Eq. (2.9) is obtained.

3. Generating functions for the k-Jacobsthal sequences

In this section, the generating functions for the k-*Jacobsthal* sequences are given. As a result, k-*Jacobsthal* sequences are seen as the coefficients of the power series of the corresponding generating function.

Let us suppose that the *Jacobsthal* numbers of order k are the coefficients of a potential series centered at the origin, and let us consider the corresponding analytic function $j_k(x)$ the function defined in such a way is called the generating function of the *k-Jacobsthal* numbers.

So,

$$j_k(x)=J_{k,0}+J_{k,1}x+J_{k,2}x^2+\ldots+J_{k,n}x^n$$

And then,

$$kxj_k(x)=kJ_{k,0}x+kJ_{k,1}x^2+kJ_{k,2}x^3+\ldots+kJ_{k,n}x^{n+1}$$

$$2x^2j_k(x)=2J_{k,0}x^2+2J_{k,1}x^3+2J_{k,2}x^4+\ldots+2J_{k,n}x^{n+2}$$

From where, since $J_{k,n+1} = kJ_{k,n} + 2J_{k,n-1}$, $J_{k,0} = 0$ *and* $J_{k,1} = 1$, it is obtained

$$\left(1 - kx - 2x^2\right) j_k(x) = x$$

So the generating function for k-Jacobsthal sequence $\left\{J_{k,n}\right\}_{n=0}^{\infty}$ is $j_k(x) = \dfrac{x}{1 - kx - 2x^2}$

References

1. A .F. Horadam, "*Jacobsthal and pell curves*", The Fibonacci Quarterly, 1988; 26(1), 79-83.

2. A. F. Horadam, "*Associated sequences of general order*", The Fibonacci Quarterly, 1993; 31(2), 166-172.

3. A. F., Horadam, "*Jacobsthal Representation Number*", The Fibonacci Quarterly, 1996; 34(1), 40-54.

4. A. Stakhov, B. Rozin, "*Theory of Binet formulas for Fibonacci and Lucas p-numbers*", Chaos. Solitons & Fractals 2006; 27(5), 1162–1177.

5. C. Bolat, H. Kőse, "*On the Properties of k-Fibonacci Numbers*", Int. J. Contemp. Math. Sciences, 2010; 22(5), 1097-1105.

6. F. Koken, D. Bozkurt, "*On the Jacobsthal numbers by matrix methods*", Int. J. Contemp. Math. Sciences, 2008; 30(3), 605-614.

7. F. Yilmaz, D. Bozkurt, "*The Generalized Order-k Jacobsthal Numbers*", Int. J. Contemp. Math. Sciences, (2009); 34(4), 1685-1694.

8. R. Honsberger, "*Mathematical germs III* ", Mathematical Association of America, Washington, DC, 1985.

9. S. Falcőn, À. Plaza, "*On the Fibonacci k-numbers*", Chaos, Solitons & Fractals 2007; 32(5), 1615-1624.

10. S. Falcőn, À. Plaza, "*The k-Fibonacci hyperbolic functions*", Chaos, Solitons & Fractals 2008; 38(2), 409-420.

11. S. Vajda, "*Fibonacci & Lucas numbers, and the golden section.*" Theory and Applications Ellis Horwood Limited; 1989.

12. T. Koshy, "*Fibonacci and Lucas number with application*", wiley, New York, 2001.

13. V. E. Hoggat, Jr., "*Fibonacci and Lucas number*", Santa clora, Calif, The Fibonacci Association, 1973.

Received: October, 2012