LINEAR INDEPENDENCE IN THE SELBERG CLASS

by

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C. R. Math. Acad. Sci. Soc. R. Can., 21(1):28–32, 1999

Abstract. We prove that distinct functions in the Selberg class $\mathcal{S}$ are linearly independent over the ring $\mathcal{F}$ of p-finite Dirichlet series. As a consequence, $\mathcal{S}$ has unique factorization if and only if primitive functions of $\mathcal{S}$ are algebraically independent over $\mathcal{F}$.

Résumé. Nous montrons que les fonctions dans la classe de Selberg $\mathcal{S}$ sont linéairement indépendantes sur l'anneau $\mathcal{F}$ des séries de Dirichlet p-finies. Comme conséquence, $\mathcal{S}$ a une factorization unique si et seulement si les fonctions primitives de $\mathcal{S}$ sont algébriquement indépendantes sur $\mathcal{F}$.

AMS 1991 Mathematics Subject Classification: 11 M 41

Roughly speaking, the main conjecture about the Selberg class $\mathcal{S}$ (see the survey paper [4] for the basic definitions, conjectures and properties of $\mathcal{S}$) is that $\mathcal{S}$ can be essentially identified with the class of suitably normalized automorphic L-functions. Based on such conjecture, one may guess that $\mathcal{S}$ contains only countably many essentially different functions, although $\mathcal{S}$ itself is certainly not countable since it contains all *shifts* $F_\theta(s) = F(s + i\theta)$, $\theta \in \mathbb{R}$, of any entire function $F \in \mathcal{S}$. However, one may guess that $\mathcal{S}$ becomes a countable set provided shifts are not counted. This is the *countability problem* for $\mathcal{S}$. This problem was communicated to us by Peter Sarnak, and we now understand that it was independently raised also by Ram Murty and Eduard Wirsing.

In order to state the problem in a rigorous form, we consider the equivalence relation in $\mathcal{S}$

$$F(s) \approx G(s) \quad \Leftrightarrow \quad G(s) = F_\theta(s) \ \text{ for some } \theta \in \mathbb{R}.$$

Denoting by $\mathcal{P}$ the set of all *primitive* functions in $\mathcal{S}$, a formulation of Sarnak's problem is the *countability conjecture*

$$\mathcal{P}/\approx \ \text{ is countable.}$$

Observe that we need to consider $\mathcal{P}/\approx$, since $\mathcal{S}/\approx$ is still an *uncountable* set. In fact, for a given primitive character $\chi(\text{mod } q)$ with $q \geq 2$, the functions $\zeta(s)L_\theta(s,\chi)$, $\theta \in \mathbb{R}$, are all in $\mathcal{S}$ and non-equivalent under $\approx$. We refer to section 9 of [4] for another version of the countability conjecture, involving a suitable *normalization* of the whole class $\mathcal{S}$. Such normalization, and hence the statement of the other version of the conjecture, require the assumption of Selberg orthonormality conjecture (see section 4 of [4]). However, the two versions of the countability conjecture are equivalent under Selberg's conjecture.

An approach to the countability conjecture may run as follows. We believe that, modulo shifts, there are only countably many possibilities for the functional equations satisfied by the functions in $\mathcal{S}$, and in fact several conjectures (for example the degree and modulus conjectures, see section 9 of [4]) point to this direction. Moreover, we believe that the vector space formed by the meromorphic Dirichlet series satisfying a functional equation of such type and the Ramanujan hypothesis has at most countable dimension. This is in fact known when the degree d satisfies $0 \leq d \leq 1$, see sections 3 and 5 of [4], in which case the dimension is actually finite. We remark that Brian Conrey, David Farmer and Amit Ghosh have brought to our attention an interesting example, due to Hecke, of a vector space of meromorphic Dirichlet series with functional equation; see chapter 2 of [2]. Hecke proved that such a vector space is infinite dimensional, and it is not difficult to show that in fact it has an uncountable basis. However, we believe that such Dirichlet series do not satisfy Ramanujan hypothesis.

If the above assumptions are correct, the countability conjecture would then follow from a suitable linear independence property of functions in $\mathcal{S}$, ensuring that each functional equation has in fact at most countably many solutions in $\mathcal{S}$.

As a small step towards the countability conjecture, in this note we prove the required linear independence property. Although the proof is short and simple, we believe that the result is of some independent interest as well. In order to state our result in a general form, we need to define the *p-finite* Dirichlet series. A Dirichlet series

$$D(s) = \sum_{n=1}^{\infty} c(n) n^{-s},$$

absolutely convergent in some right half plane, is called p-finite if there exists a positive integer M such that $c(n) = 0$ whenever n has a prime factor p such that $p \nmid M$. In this case, the arithmetical function $c(n)$ is called p-finite as well. We denote by $\mathcal{F}$ both the ring of p-finite Dirichlet series and the ring of p-finite arithmetical functions. Observe that $\mathcal{F}$ contains all Dirichlet polynomials. We have

Theorem 1. *Distinct functions in $\mathcal{S}$ are linearly independent over $\mathcal{F}$.*

In particular, Theorem 1 implies

Corollary 1. *Distinct functions $F_1, ..., F_N \in \mathcal{S}$ are linearly independent over $\mathbb{C}$.*

We note that Selberg [6] and Bombieri-Hejhal [1] already observed the result in Corollary 1, under the assumption that $F_1(s), ..., F_N(s)$ are pairwise *orthogonal*, see section 7 of [4]. Moreover, special cases of Theorem 1 are already known in the literature, see for example Lemma 8.1 of [3] where Dirichlet L-functions are considered. However, in such special cases one usually exploits special properties of the involved functions. For instance, the proof of Lemma 8.1 of [3] is based on the fact that Dirichlet L-functions formed with distinct primitive characters are pairwise orthogonal. We also note that axiom *(iv)* of Selberg class, *i.e.*, Ramanujan hypothesis, plays no role in the proof of Theorem 1.

We remark that Theorem 1 is basically a result on multiplicative arithmetical functions. In fact, calling *equivalent* two multiplicative arithmetical functions $f(n)$ and $g(n)$ if $f(p^m) = g(p^m)$ for all integers $m \geq 1$ and all but finitely many primes p, we have

Theorem 2. *Pairwise non-equivalent multiplicative arithmetical functions are linearly independent over $\mathcal{F}$.*

Proof. By contradiction, assume that there exist pairwise non-equivalent multiplicative arithmetical functions $f_1(n), ..., f_N(n)$ and non-identically vanishing p-finite arithmetical functions $c_1(n), ..., c_N(n)$ such that

$$\sum_{j=1}^{N} (c_j * f_j)(n) = 0 \qquad \text{identically.} \tag{1}$$

Moreover, let N be the minimal value required in order (1) to be satisfied. Clearly, $N > 1$ since the ring of arithmetical functions has no zero-divisors.

Let M be such that $c_j(n)=0$, $j=1,...,N$, if n has a prime factor p with $p\nmid M$. Moreover, let $p_0>M$ and $k_0\geq 1$ be such that

$$f_1(p_0^{k_0})\neq f_j(p_0^{k_0}) \qquad \text{for } j=2,...,N.$$

Write

$$\mathbb{N}_0=\{m\in\mathbb{N}:(m,p_0)=1\}$$

and consider (1) with $n=p_0^{k_0}m$, $m\in\mathbb{N}_0$, thus getting

$$\sum_{j=1}^N f_j(p_0^{k_0})\sum_{d|m}c_j(d)f_j(\frac{m}{d})=0 \qquad \text{for every} \quad m\in\mathbb{N}_0. \tag{2}$$

Multiplying (1), with $n=m$, by $f_1(p_0^{k_0})$ and subtracting (2) we get

$$\sum_{j=2}^N \big(f_1(p_0^{k_0})-f_j(p_0^{k_0})\big)\sum_{d|m}c_j(d)f_j(\frac{m}{d})=0 \qquad \text{for every} \quad m\in\mathbb{N}_0. \tag{3}$$

For $j=2,...,N$ define

$$\tilde{c}_j(n)=\big(f_1(p_0^{k_0})-f_j(p_0^{k_0})\big)c_j(n)$$

and

$$\tilde{f}_j(n)=\begin{cases} f_j(n) & \text{if } n\in\mathbb{N}_0 \\ 0 & \text{otherwise.}\end{cases}$$

Clearly, the $\tilde{f}_j(n)$ are pairwise non-equivalent multiplicative arithmetical functions and the $\tilde{c}_j(n)$ are non-identically vanishing p-finite arithmetical functions. Moreover, in view of (3) such functions satisfy

$$\sum_{j=2}^N(\tilde{c}_j*\tilde{f}_j)(n)=0 \qquad \text{identically,}$$

which contradicts the minimality of N. Theorem 2 is therefore proved. $\square$

Theorem 1 follows immediately from Theorem 2 by means of the following result of Murty-Murty [5], see also section 2 of [4].

Proposition. (Murty-Murty [5]) *Let $F,G\in\mathcal{S}$. If $F_p(s)=G_p(s)$ for all but finitely many primes p, then $F=G$.*

Here $F_p(s)$ and $G_p(s)$ denote the p-th Euler factors of $F(s)$ and $G(s)$, respectively, *i.e.*,

$$F(s)=\prod_p F_p(s) \quad \text{and} \quad G(s)=\prod_p G_p(s), \quad \sigma>1.$$

Therefore, the Proposition asserts that the coefficients of distinct functions in $\mathcal{S}$ are pairwise non-equivalent, and hence Theorem 1 follows from Theorem 2 since such coefficients are multiplicative.

We further remark that it is not difficult to construct examples showing that the non-equivalence is a necessary assumption in Theorem 2.

We recall that a function $F\in\mathcal{S}$ is *primitive* if $F(s)=F_1(s)F_2(s)$ with $F_1,F_2\in\mathcal{S}$ implies that $F_1=1$ or $F_2=1$. Moreover, every function in $\mathcal{S}$ can be factored into primitive functions. It is conjectured that such a factorization is unique, and this would follow from Selberg orthonormality conjecture, see section 4 of [4]. Our final remark is that the unique factorization is equivalent to a suitable *algebraic* independence property in $\mathcal{S}$. In fact, from Theorem 1 we easily get

Corollary 2. *$\mathcal{S}$ has unique factorization if and only if distinct primitive functions are algebraically independent over $\mathcal{F}$.*

Acknowledgment. This research was partially supported by the CNR Visiting Professor Program and the KBN grant Nr.2 P03A 02809.

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