COMMENTARII MATHEMATICI
UNIVERSITATIS SANCTI PAULI
Vol. 60, No. 1, 2 2011

ed. RIKKYO UNIV/MATH
IKEBUKURO TOKYO
171–8501 JAPAN

An Ω-result for the Difference of The Coefficients of Two L-functions

by

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(Received March 7, 2011)

Dedicated to Professor Akio Fujii on the occasion of his retirement

We denote as usual by $\mathcal{S}^\sharp$ the extended Selberg class and recall that $F \in \mathcal{S}^\sharp$ if $(s-1)^m F(s)$ is entire of finite order for some non-negative integer m, $F(s)$ is representable for $\sigma > 1$ as an absolutely convergent Dirichlet series with coefficients $a_F(n)$ and satisfies a functional equation of type

$$\gamma(s)F(s) = \omega\bar{\gamma}(1-s)\bar{F}(1-s) \tag{1}$$

with $|\omega| = 1$ and

$$\gamma(s) = Q^s \prod_{j=1}^r \Gamma(\lambda_j s + \mu_j)\,,$$

where $Q > 0$, $\Re\mu_j \geq 0$ and $\lambda_j > 0$. Here $\bar{f}(s) = \overline{f(\bar{s})}$. We also write $d_F = 2\sum_{j=1}^r \lambda_j$ for the degree of $F(s)$, $\sigma_a(F)$ for the abscissa of absolute convergence and

$$A_F(x) = \sum_{n\leq x} a_F(n) = \text{res}_{s=1} F(s)\frac{x^s}{s} + R_F(x)\,,$$

say. Moreover, the Selberg class $\mathcal{S}$ is the subclass of $\mathcal{S}^\sharp$ of the L-functions satisfying in addition the Ramanujan conjecture $a_F(n) \ll n^\varepsilon$ and having a general Euler product of type

$$\log F(s) = \sum_{n=1}^\infty \frac{b_F(n)}{n^s}$$

with $b_F(n) = 0$ unless $n = p^m$ with $m \geq 1$ and $b_F(n) \ll n^\vartheta$ for some $\vartheta < 1/2$. We refer to Selberg [9], Conrey-Ghosh [1] and our survey papers [4], [3], [6], [7] and [8] for the basic properties of the classes $\mathcal{S}$ and $\mathcal{S}^\sharp$.

The following Ω-theorem for $R_F(x)$, for any $F \in \mathcal{S}^\sharp$, is due essentially to K. Chandrasekharan and R. Narasimhan

$$R_F(x) = \Omega(x^{1/2-1/2d_F});$$

see our paper [5] for a simple proof based on the properties of the standard twist of $F(s)$ (see below). According to the current expectation, this essentially settles the problem of Ω-results for L-functions. In this paper we consider Ω-results for the difference of the coefficients of two functions in $\mathcal{S}^\sharp$. Given $F, G \in \mathcal{S}^\sharp$ we define

$$\delta(F,G) = \limsup_{x\to 0^+} \frac{\log(1+\sum_{n=1}^\infty |a_F(n)-a_G(n)|e^{-nx})}{\log(1/x)}.$$

Based on the ideas in Kaczorowski [2], dealing with the Fourier coefficients of modular forms, we prove the following general result.

THEOREM. *For distinct $F, G \in \mathcal{S}^\sharp$ with $d_F, d_G > 0$ we have*

$$\delta(F,G) \geq \frac{1}{2} + \frac{1}{2}\min\left(\frac{1}{d_F}, \frac{1}{d_G}\right).$$

The Theorem has the following geometric interpretation. Let $\mathcal{A}$ denote the set of the arithmetic functions $f(n)$ with polynomial growth, and for $f, g \in \mathcal{A}$ let

$$\delta(f,g) = \limsup_{x\to 0^+} \frac{\log\left(1+\sum_{n=1}^\infty |f(n)-g(n)|e^{-nx}\right)}{\log(1/x)}.$$

Then δ is a pseudo ultrametric in $\mathcal{A}$, since for $f, g, h \in \mathcal{A}$ one can easily check that $\delta(f,f) = 0$, $\delta(f,g) = \delta(g,f)$ and $\delta(f,g) \leq \max(\delta(f,h), \delta(h,g))$. Hence the Theorem may be expressed by saying that the subset of $\mathcal{A}$ formed by the coefficients of the functions in $\mathcal{S}^\sharp$ with positive degree is a discrete subset of $\mathcal{A}$, in the topology induced by δ.

From the Theorem we deduce by a standard argument the following

COROLLARY 1. *For distinct $F, G \in \mathcal{S}^\sharp$ with $d_F, d_G > 0$ we have*

$$\sum_{n\leq x} |a_F(n)-a_G(n)| = \Omega\left(\left(\frac{x}{\log x}\right)^{\frac{1}{2}+\frac{1}{2}\min(\frac{1}{d_F},\frac{1}{d_G})}\right).$$

We believe that the "min" in the above results can be replaced by "max". Moreover, we believe that the following stronger Ω-result holds.

CONJECTURE 1. *For distinct $F, G \in \mathcal{S}^\sharp$ with $\max(d_F, d_G) > 0$ and $\varepsilon > 0$ we have*

$$\sum_{n\leq x} |a_F(n)-a_G(n)| = \Omega\left(x^{1-\varepsilon}\right).$$

In the case of L-functions in $\mathcal{S}$ we can say something more by elementary considerations, thanks to the following

LEMMA 1. *Let $f(n)$ and $g(n)$ be distinct multiplicative functions satisfying the Ramanujan conjecture and suppose that*

$$\sum_{n\leq x} |f(n)-g(n)| \ll x^{\theta+\varepsilon}$$

for some $\theta \leq 1$ and every $\varepsilon > 0$. Then for every $\varepsilon > 0$

$$\sum_{n\leq x} |f(n)| + \sum_{n\leq x} |g(n)| \ll x^{\theta+\varepsilon}.$$

Clearly, the opposite implication holds as well, without assumptions on $f(n)$ and $g(n)$.

Recalling that the only Dirichlet polynomial in $\mathcal{S}$ is the identically 1 function, an immediate consequence of Lemma 1 is

COROLLARY 2. *The Conjecture holds in $\mathcal{S}$ if and only if $\sigma_a(F) = 1$ for every $F \in \mathcal{S}$, $F \neq 1$.*

Actually, it is expected that $\sigma_a(F) = 1$ for every $F \in \mathcal{S}$, $F \neq 1$.

Given $f \in \mathcal{A}$ we consider the associated Dirichlet series

$$L(s, f) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

with (finite) abscissa of absolute convergence $\sigma_a(f)$. Then from Lemma 1 we immediately get

$$\max(0, \sigma_a(f - g)) = \max(0, \sigma_a(f), \sigma_a(g)) \tag{2}$$

if $f, g \in \mathcal{A}$ are multiplicative and satisfy the Ramanujan conjecture. Now we recall that for every $F \in \mathcal{S}^\sharp$

$$\sigma_a(F) \geq \frac{1}{2} + \frac{1}{2d_F} \qquad d_F > 0$$

since the standard twist

$$F_{1/d}(s, \alpha) = \sum_{n=1}^{\infty} \frac{a_F(n) e(-\alpha n^{1/d})}{n^s} \qquad d = d_F,\ e(x) = e^{2\pi i x}$$

has, for suitably chosen α's, a pole on the line $\sigma = 1/2 + 1/2d_F$. See [5], where the notation $F_d(s, \alpha)$ is used instead of $F_{1/d}(s, \alpha)$; see also below, at the beginning of the proof of the Theorem. Therefore from (2) and Lemma 2 below we have

COROLLARY 3. *For distinct $F, G \in \mathcal{S}$ with $d_F, d_G > 0$ we have*

$$\delta(F, G) \geq \frac{1}{2} + \frac{1}{2} \max\left(\frac{1}{d_F}, \frac{1}{d_G}\right).$$

Note that the lower bound in Corollary 3 is sharper than the lower bound in the Theorem.

The above results are special cases of the following general problem. Let $F_1, ..., F_m \in \mathcal{S}^\sharp$ have positive degree and let $L(x_1, ..., x_m)$ be a linear form such that $L(F_1(s), ..., F_m(s))$ does not vanish identically. Prove that there exists a $\theta = \theta(d_{F_1}, ..., d_{F_m}) > 0$ such that

$$\sum_{n \leq x} |L(a_{F_1}(n), ..., a_{F_m}(n))| = \Omega(x^\theta).$$

The supremum of such θ's may be called the measure of linear independence of $F_1(s), \ldots, F_m(s)$. Our results solve the problem for $n = 2$.

Proof of the Theorem. The proof is based on the properties of the nonlinear twists

$$F_\lambda(s, \alpha) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s} e(-\alpha n^\lambda),$$

with $0 < \lambda \leq 1/d_F$ and $\alpha > 0$, of the functions $F \in \mathcal{S}^\sharp$. Namely, $F_\lambda(s, \alpha)$ is an entire function for every α if $0 < \lambda < 1/d_F$, while if $\lambda = 1/d_F$ it has a simple pole s_0 on the line $\sigma = 1/2 + 1/2d_F$ for the α's such that $n_\alpha = q_F d_F^{-d_F} \alpha^{d_F}$ is an integer with $a_F(n_\alpha) \neq 0$ (if n_α is not an integer we let $a_F(n_\alpha) = 0$). Here $q_F = (2\pi)^{d_F} Q^2 \prod_{j=1}^{r} \lambda_j^{2\lambda_j}$ is the conductor of $F(s)$, and the value of the residue at s_0 is $c(F)\overline{a_F(n_\alpha)}$ with $c(F) \neq 0$. For a sketch of proof of the first assertion we refer to the remark after the proof of Lemma 4.1 in [5]. We will treat the general case of nonlinear twists of a given $F \in \mathcal{S}^\sharp$ with leading exponent $\lambda \leq 1/d_F$ in a future paper. For the second assertion we refer to Theorem 1 of [5].

Suppose first that $0 < d_G < d_F$ (or, analogously, $0 < d_F < d_G$), which is the simpler case, and consider the twist

$$L(s, \alpha, F - G) = \sum_{n=1}^{\infty} \frac{a_F(n) - a_G(n)}{n^s} e(-\alpha n^{1/d_F}) .$$

By the above results we have, choosing α appropriately, that $L(s, \alpha, F - G)$ has a simple pole on the line $\sigma = 1/2 + 1/2d_F$. Hence the abscissa of absolute convergence $\sigma_a(L)$ of $L(s, \alpha, F - G)$ satisfies

$$\sigma_a(L) \geq \frac{1}{2} + \frac{1}{2d_F} = \frac{1}{2} + \frac{1}{2} \min\left(\frac{1}{d_F}, \frac{1}{d_G}\right) .$$

Recalling the definition of $\mathcal{A}$ given above, the Theorem in this case follows then from the following

LEMMA 2. *Let $f \in \mathcal{A}$. Then*

$$\delta(f) = \max(0, \sigma_a(f)) .$$

Proof. This is Lemma 3 in Kaczorowski [2]. □

Now we turn to the more delicate case of $d_F = d_G$, and prove that also in this case the twist $L(s, \alpha, F - G)$ has a pole on the line $\sigma = 1/2 + 1/2d_F$ for a suitable $\alpha > 0$. The Theorem will then follow by the same argument as before.

Writing $d = d_F = d_G > 0$, we may consider without loss of generality only the point $s = 1/2 + 1/2d$. Suppose, by contradiction, that

$$\begin{aligned} 0 = r(\alpha) &= \mathrm{res}_{s=1/2+1/2d} L(s, \alpha, F - G) \\ &= \mathrm{res}_{s=1/2+1/2d} F_{1/d}(s, \alpha) - \mathrm{res}_{s=1/2+1/2d} G_{1/d}(s, \alpha) \end{aligned}$$

for every $\alpha > 0$ and choose $\alpha = \alpha_n = d q_F^{-1/d} n^{1/d}$ with any integer $n \geq 1$. Therefore

$$r(\alpha_n) = c(F)\overline{a_F(n)} - c(G)\overline{a_G(q_G n/q_F)}$$

and hence

$$a_F(n) = \gamma a_G(q_G n/q_F) \qquad \gamma \neq 0 . \tag{3}$$

This implies that $q_G/q_F \in \mathbb{Q}$, otherwise $a_F(n) = 0$ for every n, a contradiction. We write

$$q_G/q_F = a/q \qquad (a, q) = 1,\ a, q \in \mathbb{N} . \tag{4}$$

Hence from (3) we get that

$$a_F(n) \neq 0 \ \Rightarrow\ q|n , \tag{5}$$

and reversing the roles of $F(s)$ and $G(s)$ we also get

$$a_G(n) \neq 0 \ \Rightarrow\ a|n . \tag{6}$$

From (3) we further deduce that for every $n \in \mathbb{N}$

$$a_F(qn) = \gamma a_G(an) .$$

Thus, writing

$$H(s) = \sum_{n=1}^{\infty} \frac{a_F(qn)}{n^s} = \gamma \sum_{n=1}^{\infty} \frac{a_G(an)}{n^s}\,, \tag{7}$$

from (5) and (6) we deduce that

$$H(s) = q^s F(s) = \gamma a^s G(s)\,. \tag{8}$$

Since $F(s)$ satisfies a functional equation of type (1), thanks to (8) the function $H(s)$ satisfies

$$(Q/q)^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) H(s) = \omega (Q/q)^{1-s} \prod_{j=1}^{r} \Gamma(\lambda_j(1-s) + \bar{\mu}_j) \bar{H}(1-s)$$

and hence its conductor q_H equals q_F/q^2. Similarly, we may use (8) and the functional equation of $G(s)$ to compute the conductor of $H(s)$, thus getting $q_H = q_G/a^2$. Recalling that the conductor is an invariant we deduce

$$q_G/q_F = (a/q)^2\,. \tag{9}$$

Comparing (4) and (9) we conclude that $a = q = 1$ and therefore by (8)

$$F(s) = \gamma G(s)\,.$$

But then

$$L(s, \alpha, F - G) = (1 - \gamma) F_{1/d}(s, \alpha)$$

which has no poles only if $\gamma = 1$. Hence $F(s) = G(s)$ and the result follows. □

Proof of Lemma 1. Let $q_0 = p_0^{k_0}$ be a prime power such that $f(q_0) \neq g(q_0)$. Then for the integers $m \geq 1$ such that $p_0 \nmid m$ we have

$$f(q_0 m) - g(q_0 m) = f(q_0)(f(m) - g(m)) + g(m)(f(q_0) - g(q_0))\,,$$

hence

$$|g(m)| \ll |f(m) - g(m)| + |f(q_0 m) - g(q_0 m)|$$

and consequently

$$\sum_{\substack{m \leq x \\ p_0 \nmid m}} |g(m)| \ll \sum_{m \leq q_0 x} |f(m) - g(m)| \ll x^{\theta + \varepsilon}\,. \tag{10}$$

But

$$\sum_{n \leq x} |g(n)| = \sum_{0 \leq k \leq [\frac{\log x}{\log p_0}]} |g(p_0^k)| \sum_{\substack{m \leq x/p_0^k \\ p_0 \nmid m}} |g(m)| \ll x^{\varepsilon} \sum_{\substack{m \leq x \\ p_0 \nmid m}} |g(m)|\,, \tag{11}$$

hence from (10) and (11) we obtain

$$\sum_{n \leq x} |g(n)| \leq x^{\theta + \varepsilon}\,.$$

Analogously we have that

$$\sum_{n \leq x} |f(n)| \leq x^{\theta + \varepsilon}\,,$$

and the lemma follows. □

Acknowledgment. This research was partially supported by the Istituto Nazionale di Alta Matematica, by a grant PRIN2008 and by grant N N201 605940 of the National Science Centre.

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